Binomial Trees

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OUTLINE

1. REVIEW

2. BINOMIAL TREES
   - Theory
   - Valuing an Option
   - American Options
   - Dividends
1 REVIEW

2 BINOMIAL TREES
   • Theory
   • Valuing an Option
   • American Options
   • Dividends

3 SUMMARY
MONTE CARLO METHODS

- Simple to program and to understand
- Convergence is slow, extrapolation impossible.
- Forward looking method ideal for path dependent derivatives
- Computational effort increases linearly in multiple sources of uncertainty
THE FUNDAMENTAL THEOREM OF ASSET PRICING

If there are no arbitrage opportunities and markets are complete then there exists a unique, risk-neutral, pricing measure.

As such we can write the value of an option at time $t$, $V_t$, as

$$V_0 = e^{-rT}E^Q[V_T]$$

where $r$ is the continuously compounded interest rate.
**The Fundamental Theorem of Asset Pricing**

If there are no arbitrage opportunities and markets are complete then there exists a unique, risk-neutral, pricing measure.

- As such we can write the value of an option at time $t$, $V_t$, as

\[ V_0 = e^{-rT} \mathbb{E}^Q[V_T] \]

where $r$ is the continuously compounded interest rate.

- We can apply the same argument for multiple time steps.
A ONE STEP TREE

- Assume a three asset world with:
  - a Bond, $B_t$,
  - a Stock, $S_t$
  - and a call option $C_t$,

where interest rates are continuously compounded

- risk neutral probability of the up and down states occurring are $q$ and $(1 - q)$ then we have
Basic binomial set up

- If we have a three asset world with a Bond, $B_t$, a Stock, $S_t$ and a call option $C_t$, where interest rates are continuously compounded and the risk neutral probability of the up and down states occurring are $q$ and $(1-q)$ then we have

\[
\begin{align*}
B_0 &= 1 \\
B_T &= e^{rT} \\
S_0 &= s \\
S_T &= us \\
C_0 &= ? \\
C_T &= \begin{cases} 
us - X, & \text{if up state occurs} \\
0, & \text{otherwise} 
\end{cases} \\
C_T &= \begin{cases} 
ds - X, & \text{if down state occurs} \\
0, & \text{otherwise} 
\end{cases}
\end{align*}
\]
How to find the option value?

- From the fundamental theory of finance the price of the call option is

\[ C_0 = e^{-rT} \left[ q \max(uS - X, 0) + (1 - q) \max(dS - X, 0) \right] \]

- So in order to find \( C_0 \) we need to find \( q, u \) and \( d \)
HOW TO FIND THE OPTION VALUE?

From the fundamental theory of finance the price of the call option is

\[ C_0 = e^{-rT}[q \max(uS - X, 0) + (1 - q) \max(dS - X, 0)] \]

- So in order to find \( C_0 \) we need to find \( q, u \) and \( d \)
- We will do this by first matching the return on the stock to the return on a bond
- and then match the variance of the stock to that from data.
Determining \( q, u \) and \( d \)

As the probabilities are risk neutral, match return on the stock to that of the bond,

\[
qsu + (1 - q)sd = se^{rT}
\]

\[
qu + (1 - q)d = e^{rT}
\]
Determining $q, u$ and $d$

- As the probabilities are risk neutral, match return on the stock to that of the bond,

$$qsu + (1 - q)sd = se^{rT}$$
$$qu + (1 - q)d = e^{rT}$$

- Next, match the variance of our returns to the data.
- Under the continuous risk-neutral model we have

$$dS = rSdt + \sigma SdX$$
$$S_T = s \exp[(r - \frac{1}{2}\sigma^2)T + \sigma\sqrt{T} \phi]$$
We can show that the variance for the continuous case is:

\[ \text{var}[S_T] = s^2 \exp[(2r + \sigma^2)T] - s^2 e^{2rT} \]

and in the binomial case:

\[ \text{var}[S_T] = s^2 (qu^2 + (1-q)d^2) - (qsu + (1-q)sd)^2 \]

So the second equation to satisfy is:

\[ e^{(2r+\sigma^2)T} = qu^2 + (1-q)d^2 \]
Cox, Ross and Rubinstein (1979) set:

\[ ud = 1 \]

thus

\[ u = e^{\sigma \sqrt{T}}, \quad d = e^{-\sigma \sqrt{T}}, \quad q = \frac{e^{rd} - d}{u - d} \]

Rendleman and Bartter (1979) choose:

\[ q = \frac{1}{2} \]

and so

\[ u = e^{\left(r - \frac{1}{2} \sigma^2\right)T + \sigma \sqrt{T}}, \quad d = e^{\left(r - \frac{1}{2} \sigma^2\right)T - \sigma \sqrt{T}}. \]
Valuing a European option

So now we have expressions for \( u, d \) and \( q \) which approximate the continuous lognormal distribution.

There are other ways to construct the tree if the underlying asset follows a different stochastic process.

Now we turn our attention to valuing a European call option.

- Assume initial asset price \( S_0 \),
  - the time to expiry is \( T \) with \( N \) time steps,
  - the size of the time-step \( \Delta t = \frac{T}{N} \)
  - the risk-free rate is \( r \) and the volatility is \( \sigma \),
  - the exercise price of the option is \( X \)
If we denote the value of the underlying asset after timestep $i$ and upstate $j$ by $S_{ij}$ and the option price by $V_{ij}$ then we have that:

$$S_{ij} = S_0 u^j d^{i-j}$$

$$V_{Nj} = \max(S_0 u^j d^{N-j} - X, 0)$$

$$V_{ij} = e^{-r \Delta t} (q V_{i+1,j+1} + (1-q) V_{i+1,j}) \quad \text{for} \quad i < N$$

where $q$, $u$, and $d$ are selected according to your preferred model (CRR or alternative).
**Example - A Three Step Tree**

Consider a European call option where $S_0 = 100$, $X = 100$, $T = 1$, $r = 0.06$, $\sigma = 0.2$. 
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Choosing the CRR tree we have

\[ u = e^{\sigma \sqrt{\Delta t}} = 1.1224 \]
\[ d = e^{-\sigma \sqrt{\Delta t}} = 0.8909 \]
\[ q = \frac{e^{r\Delta t} - d}{u - d} = 0.5584 \]

Next we show the calculation of the European call option price using 3 time steps, where we end up with an option value of $11.55$. 
**EXAMPLE - A THREE STEP TREE**

<table>
<thead>
<tr>
<th>Time</th>
<th>Asset Price</th>
<th>Option Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t=0$</td>
<td>$S_{0,0} = 100.00$</td>
<td>$V_{0,0} = 11.552$</td>
</tr>
<tr>
<td>$t=1/3$</td>
<td>$S_{1,0} = 89.095$</td>
<td>$V_{1,0} = 3.6675$</td>
</tr>
<tr>
<td>$t=2/3$</td>
<td>$S_{2,0} = 79.379$</td>
<td>$V_{2,0} = 0$</td>
</tr>
<tr>
<td>$t=1$</td>
<td>$S_{3,0} = 70.722$</td>
<td>$V_{3,0} = 0$</td>
</tr>
</tbody>
</table>

At $t=0$, the initial asset price is $S_{0,0} = 100.00$, and the initial option value is $V_{0,0} = 11.552$. As time progresses, the asset price and option value are updated based on the binomial model. The diagram illustrates the progression of these values over three time steps.
American options are call (put) options where it is possible to exercise early.

The problem is a free boundary or optimal stopping problem where the current option value $V_t$ is given by

$$V_t = \max_{\tau} E_t^Q \left[ e^{-r(\tau-t)} \text{Payoff} (S_\tau) \right]$$

where $\tau$ denotes the continuum of possible stopping times.

NOTE: This representation is not particularly useful when attempting to value the option.
Obviously any rational investor would only exercise if the value from exercising is greater than the value from not exercising, i.e. holding the option for one more period.

Binomial lattices are calculated backwards from expiry so
- we already know the value of holding the option until the next period (the continuation value)
- we know the early exercise value (the payoff from the option).
The option to hold

- Obviously any rational investor would only exercise if the value from exercising is greater than the value from not exercising, i.e. holding the option for one more period.
- Binomial lattices are calculated backwards from expiry so
  - we already know the value of holding the option until the next period (the continuation value)
  - we know the early exercise value (the payoff from the option).
- American options are a straightforward adaptation of European
THE ALGORITHM

- Let us write the continuation (or hold) value, as $V_{hij}$
- the early exercise value as $V_{xij}$
Let us write the continuation (or hold) value, as $V_{hij}$

the early exercise value as $V_{xij}$

then the algorithm to determine $V_{ij}$ is

$$V_{ij} = \text{Payoff}(S_{Nj}) \quad \text{for} \quad t = T$$

$$V_{hij} = e^{-r\Delta t}(qV_{i+1,j+1} + (1-q)V_{i+1,j}) \quad \text{for} \quad t < T$$

$$V_{xij} = \text{Payoff}(S_{ij}) \quad \text{for} \quad t < T$$

$$V_{ij} = \max(V_{hij}, V_{xij}) \quad \text{for} \quad t < T$$
Consider an American put option where \( S_0 = 100, X = 100, T = 1, r = 0.06, \sigma = 0.2 \).

Choosing the CRR tree we have

\[
\begin{align*}
    u &= e^{\sigma \sqrt{\Delta t}} = 1.1224 \\
    d &= e^{-\sigma \sqrt{\Delta t}} = 0.8909 \\
    q &= \frac{e^{r \Delta t} - d}{u - d} = 0.5584
\end{align*}
\]

Next we show the calculation of the American put.

The nodes in red denote that the holder of the option exercised early.
**Example - 3 Step American Tree**

- **Initial State**:
  - \( S_{0,0} = 100.00 \)
  - \( V_{h,0,0} = 6.099 \)
  - \( V_{x,0,0} = 0.00 \)
  - \( V_{f,0} = 6.099 \)

- **Time t = 0**:
  - \( S_{1,0} = 89.094 \)
  - \( V_{h,1,0} = 11.51 \)
  - \( V_{x,1,0} = 10.91 \)
  - \( V_{f,1,0} = 11.509 \)

- **Time t = 1/3**:
  - \( S_{2,0} = 79.38 \)
  - \( V_{h,2,0} = 18.64 \)
  - \( V_{x,2,0} = 20.62 \)
  - \( V_{f,2,0} = 20.62 \)

- **Time t = 2/3**:
  - \( S_{2,2} = 125.98 \)
  - \( V_{h,2,2} = 0.00 \)
  - \( V_{x,2,2} = -26.0 \)
  - \( V_{f,2} = 0.00 \)

- **Time t = 1**:
  - \( S_{3,3} = 141.40 \)
  - \( V_{h,3,3} = 0.00 \)
  - \( V_{x,3,3} = -41.4 \)
  - \( V_{f,3} = 0.00 \)

- **Final State**:
  - \( S_{3,0} = 70.72 \)
  - \( V_{h,3,0} = 0.00 \)
  - \( V_{x,3,0} = 29.28 \)
  - \( V_{f,3} = 29.28 \)
For American call options with no dividends it is never optimal to exercise early.

From the lattice we can determine the early exercise region, as the set of points $S$ and $t$ for which you would exercise early.
For American call options with no dividends it is never optimal to exercise early.

From the lattice we can determine the early exercise region, as the set of points $S$ and $t$ for which you would exercise early.

Technically what we have evaluated here is a Bermudan option, which is an American option that can only be exercised on certain specified dates.

In the limit of more and more observation dates we will approach the American option price.
The fundamental theorem of finance does not directly apply to dividend paying assets.

If $S_t$ is the value of an asset at time $t$ which pays out a continuously compounded dividend yield, $\delta$, then consider a new asset $X$ which is defined as

$$ X_0 = e^{-\delta t} S_0 $$

at time $t$ the $S_0$ will have grown to $S_t e^{\delta t}$ and so $X_t = S_t$. Thus it will be possible to replicate the value of an option expiring at time $t$ by holding $e^{-\delta t}$ of the underlying asset.
Thus by the fundamental theorem of finance it will be $X$ which is priced under the risk-neutral measure given a known future asset price $S_t$ thus:

$$X_0 = e^{-rt} E^Q_0 [S_t]$$

and so

$$S_0 = e^{-(r-\delta)t} E^Q_0 [S_t]$$
We need new values of $u$ and $d$ where

\[
E[S_T] = s \exp[(r - \delta)T]
\]

\[
E[(S_T)^2] = s^2 \exp[(2(r - \delta) + \sigma^2)T]
\]

and using CRR gives:

\[
u = e^{\sigma \sqrt{\Delta t}}, \quad d = e^{-\sigma \sqrt{\Delta t}}, \quad q = \frac{e^{(r-\delta)\Delta t} - d}{u - d}
\]

Thus if we consider the American put option only now when $\delta = 0.03$ we see that the theoretical price is now $7.32$
Review
Binomial Trees
Summary

Theory
Valuing an Option
American Options
Dividends

\[ t=0 \]
\[ t =1/3 \]
\[ t = 2/3 \]
\[ t = 1 \]

\[ V_{3,3} = 0.000 \]
\[ V_{x,3,3} = -41.4 \]
\[ V_{3,3} = 0.000 \]

\[ S_{3,3} = 141.40 \]
\[ V_{h,3,3} = 0.000 \]
\[ V_{3,3} = 0.000 \]

\[ S_{3,2} = 112.24 \]
\[ V_{h,3,2} = 0.000 \]
\[ V_{x,3,2} = -12.2 \]
\[ V_{3,2} = 0.000 \]

\[ S_{3,1} = 89.09 \]
\[ V_{h,3,1} = 0.000 \]
\[ V_{x,3,1} = 10.90 \]
\[ V_{3,1} = 10.90 \]

\[ S_{3,0} = 79.38 \]
\[ V_{h,3,0} = 19.43 \]
\[ V_{x,3,0} = 20.62 \]
\[ V_{3,0} = 20.62 \]

\[ S_{2,2} = 125.98 \]
\[ V_{h,2,2} = 0.000 \]
\[ V_{x,2,2} = -26.0 \]
\[ V_{2,2} = 0.000 \]

\[ S_{2,1} = 100.0 \]
\[ V_{h,2,1} = 5.189 \]
\[ V_{x,2,1} = 0.00 \]
\[ V_{2,1} = 5.189 \]

\[ S_{2,0} = 79.38 \]
\[ V_{h,2,0} = 19.43 \]
\[ V_{x,2,0} = 20.62 \]
\[ V_{2,0} = 20.62 \]

\[ S_{1,1} = 112.24 \]
\[ V_{h,1,1} = 2.469 \]
\[ V_{x,1,1} = -12.2 \]
\[ V_{1,1} = 2.469 \]

\[ S_{1,0} = 89.094 \]
\[ V_{h,1,0} = 12.43 \]
\[ V_{x,1,0} = 10.91 \]
\[ V_{1,0} = 12.43 \]

\[ S_{0,0} = 100.00 \]
\[ V_{h,0,0} = 7.159 \]
\[ V_{x,0,0} = 0.00 \]
\[ V_{0,0} = 7.159 \]
Perhaps a more realistic case is when there is a known discrete dividend payment at a certain point in time. In our example, imagine there is a known dividend, payable after 2/3 of a year which is 3% of the share price.

Here the fundamental theorem will hold from period to period and our values of $u$, $d$ and $q$ will remain the same as for the no dividend case but at $t = 2/3$, $S_{2j} \rightarrow 0.97 \times S_{2j}$

This is depicted in the worked example next:
### Review

**Binomial Trees**

#### Summary

1. **Theory**
   - Valuing an Option
   - American Options
   - Dividends

#### Prices and Values at Various Times

<table>
<thead>
<tr>
<th>Time</th>
<th>$S$</th>
<th>$V$</th>
</tr>
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<tbody>
<tr>
<td>$t=0$</td>
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</tr>
<tr>
<td>$t=0$</td>
<td>$V_{h,0,0} = 7.094$</td>
<td></td>
</tr>
<tr>
<td>$t=0$</td>
<td>$V_{x,0,0} = 0.00$</td>
<td></td>
</tr>
<tr>
<td>$t=0$</td>
<td>$S_{1,0} = 89.09$</td>
<td>$V_{1,0} = 13.17$</td>
</tr>
<tr>
<td>$t=1/3$</td>
<td>$S_{1.1} = 112.24$</td>
<td>$V_{h,1.1} = 2.544$</td>
</tr>
<tr>
<td>$t=2/3$</td>
<td>$S_{2.0} = 79.38$</td>
<td>$V_{h,2.0} = 23.00$</td>
</tr>
<tr>
<td>$t=2/3$</td>
<td>$V_{h,2.0} = 0.000$</td>
<td>$V_{x,2.0} = -26.0$</td>
</tr>
<tr>
<td>$t=1$</td>
<td>$S_{3.0} = 68.60$</td>
<td>$V_{h,3.0} = 0.000$</td>
</tr>
<tr>
<td>$t=1$</td>
<td>$S_{3.1} = 86.42$</td>
<td>$V_{h,3.1} = 0.000$</td>
</tr>
<tr>
<td>$t=1$</td>
<td>$S_{3.2} = 108.87$</td>
<td>$V_{h,3.2} = 0.000$</td>
</tr>
<tr>
<td>$t=1$</td>
<td>$S_{3.3} = 137.16$</td>
<td>$V_{h,3.3} = 0.000$</td>
</tr>
</tbody>
</table>

#### Valuing an Option

- **American Options**
- **Dividends**

**Bold** figures denote value if held until after the dividend date.

$t=0$  
$t=1/3$  
$t=2/3$ Pre Div  
$t=2/3$ Post Div  
$t=1$
**SUMMING UP DIVIDENDS**

- Non-proportional cash dividends can be problematic as this leads to a non-recombining tree.
- This leads to a large increase in the computational effort.
- There is an adjustment for European options but this is not of great practical use as Black-Scholes can be used quite simply in the European case.
- In American option cases the simplest approach is to use interpolation.
We have developed a multistep binomial lattice which will approximate the value of a European or American call option when the underlying asset pays out dividends.

The construction comes from an extension to the fundamental theorem of finance and you have a choice of parameters which are typically chosen to fit the binomial distribution to the Black-Scholes lognormal distribution.

The most useful outcome is the ability to price American options easily.