Binomial Trees

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2013

Outline

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Monte Carlo Methods

- Simple to program and to understand
- Convergence is slow, extrapolation impossible.
- Forward looking method ideal for path dependent derivatives
- Computational effort increases linearly in multiple sources of uncertainty

THE FUNDAMENTAL THEOREM OF ASSET PRICING

If there are no arbitrage opportunities and markets are complete then there exists a unique, risk-neutral, pricing measure.

- As such we can write the value of an option at time t, V_t, as

$$V_0 = e^{-rT} E^{Q}[V_T]$$

where r is the continuously compounded interest rate.

The fundamental theorem of asset pricing

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- As such we can write the value of an option at time t, V_t, as

$$V_0 = e^{-rT} E^Q [V_T]$$

where r is the continuously compounded interest rate.

- We can apply the same argument for multiple time steps

A ONE STEP TREE

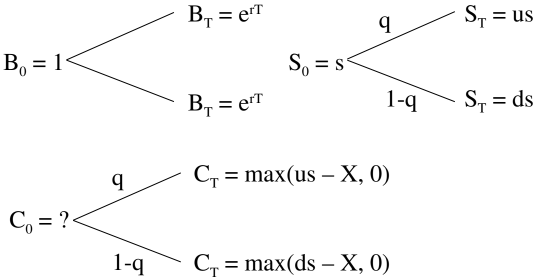
- Assume a three asset world with:
  - a Bond, B_t,
  - a Stock, S_t
  - and a call option C_t,

  where interest rates are continuously compounded
- risk neutral probability of the up and down states occurring are q and $(1-q)$ then we have

A ONE STEP TREE

Basic binomial set up

- If we have a three asset world with a Bond, B_t, a Stock, S_t and a call option C_t, where interest rates are continuously compounded and the risk neutral probability of the up and down states occurring are q and (1-q) then we have



How to find the option value?

- From the fundemental theory of finance the price of the call option is

$$C_0 = e^{-rT}[q \max(uS - X, 0) + (1-q)\max(dS - X, 0)]$$

- So in order to find C_0 we need to find q, u and d

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- So in order to find C_0 we need to find q, u and d
- We will do this by first matching the return on the stock to the return on a bond
- and then match the variance of the stock to that from data.

DETERMINING q, u AND d

- As the probabilities are risk neutral, match return on the stock to that of the bond,

$$qsu + (1-q)sd = se^{rT}$$
$$qu + (1-q)d = e^{rT}$$

DETERMINING q, u AND d

- As the probabilities are risk neutral, match return on the stock to that of the bond,

$$qsu + (1-q)sd = se^{rT}$$
$$qu + (1-q)d = e^{rT}$$

- Next, match the variance of our returns to the data.
- Under the continuous risk-neutral model we have

$$dS = rSdt + \sigma SdX$$
$$S_T = s\exp[(r - \tfrac{1}{2}\sigma^2)T + \sigma\sqrt{T}\phi]$$

DETERMINING q, u AND d

- We can show that the variance for the continuous case is:

$$var[S_T] = s^2 \exp[(2r + \sigma^2)T] - s^2 e^{2rT}$$

and in the binomial case

$$var[S_T] = s^2(qu^2 + (1-q)d^2) - (qsu + (1-q)sd)^2$$

- So the second equation to satisfy is

$$e^{(2r+\sigma^2)T} = qu^2 + (1-q)d^2$$

Popular Models

- Cox, Ross and Rubinstein (1979) set:

$$ud = 1$$

thus

$$u = e^{\sigma\sqrt{T}}, \quad d = e^{-\sigma\sqrt{T}}, \quad q = \frac{e^{rd} - d}{u - d}$$

- Rendleman and Bartter (1979) choose:

$$q = \frac{1}{2}$$

and so

$$u = e^{(r-\frac{1}{2}\sigma^2)T+\sigma\sqrt{T}}, \quad d = e^{(r-\frac{1}{2}\sigma^2)T-\sigma\sqrt{T}}.$$

Valuing a European option

- So now we have expressions for u, d and q which approximate the continuous lognormal distribution.
- There are other ways to construct the tree if the underlying asset follows a different stochastic process.
- Now we turn our attention to valuing a European call option.
  - Assume initial asset price S_0,
    the time to expiry is T with N time steps,
    the size of the time-step $\Delta t = T/N$
    the risk-free rate is r and the volatility is σ,
    the exercise price of the option is X

- If we denote the value of the underlying asset after timestep i and upstate j by S_{ij} and the option price by V_{ij} then we have that:

$$\begin{aligned} S_{ij} &= S_0 u^j d^{i-j} \\ V_{Nj} &= \max(S_0 u^j d^{N-j} - X, 0) \\ V_{ij} &= e^{-r\Delta t}(qV_{i+1,j+1} + (1-q)V_{i+1,j}) \quad \text{for} \quad i < N \end{aligned}$$

where q, u and d are selected according to your preferred model (CRR or alternative).

Example - A three step tree

- Consider a European call option where $S_0 = 100$, $X = 100$, $T = 1$, $r = 0.06$, $\sigma = 0.2$.

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- Choosing the CRR tree we have

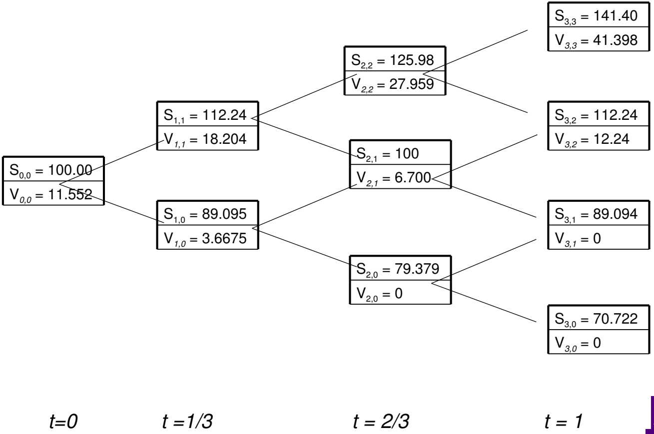
$$u = e^{\sigma\sqrt{\Delta t}} = 1.1224$$

$$d = e^{-\sigma\sqrt{\Delta t}} = 0.8909$$

$$q = \frac{e^{r\Delta t} - d}{u - d} = 0.5584$$

- Next we show the calculation of the European call option price using 3 time steps, where we end up with an option value of \$11.55.

Example - A three step tree



American Option

- American options are call (put) options where it is possible to exercise early
- The problem is a free boundary or optimal stopping problem where the current option value V_t is given by

$$V_t = \max_{\tau} E_t^Q[e^{-r(\tau-t)} Payoff(S_\tau)]$$

where τ denotes the continuum of possible stopping times.
- NOTE: This representation is not particularly useful when attempting to value the option.

The option to hold

- Obviously any rational investor would only exercise if the value from exercising is greater than the value from not exercising, i.e. holding the option for one more period.
- Binomial lattices are calculated backwards from expiry so
  - we already know the value of holding the option until the next period (the continuation value)
  - we know the early exercise value (the payoff from the option).

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- Binomial lattices are calculated backwards from expiry so
  - we already know the value of holding the option until the next period (the continuation value)
  - we know the early exercise value (the payoff from the option).
- American options are a straight forward adaptation of European

The algorithm

- Let us write the the continuation (or hold) value, as V_{hij}
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THE ALGORITHM

- Let us write the the continuation (or hold) value, as V_{hij}
- the early exercise value as V_{xij}
- then the algorithm to determine V_{ij} is

$$V_{ij} = Payoff(S_{Nj}) \quad \text{for} \quad t = T$$

$$V_{hij} = e^{-r\Delta t}(qV_{i+1,j+1} + (1-q)V_{i+1,j}) \quad \text{for} \quad t < T$$

$$V_{xij} = Payoff(S_{ij}) \quad \text{for} \quad t < T$$

$$V_{ij} = \max(V_{hij}, V_{xij}) \quad \text{for} \quad t < T$$

Example - 3 step American tree

- Consider an American put option where $S_0 = 100$, $X = 100$, $T = 1$, $r = 0.06$, $\sigma = 0.2$.
- Choosing the CRR tree we have

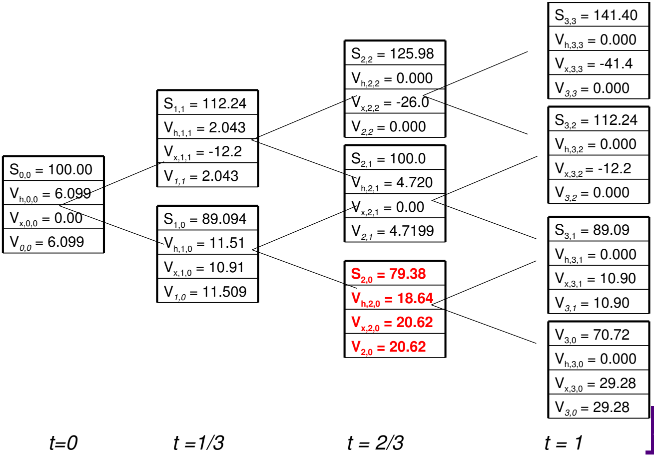
$$u = e^{\sigma\sqrt{\Delta t}} = 1.1224$$

$$d = e^{-\sigma\sqrt{\Delta t}} = 0.8909$$

$$q = \frac{e^{r\Delta t} - d}{u - d} = 0.5584$$

- Next we show the calculation of the American put.
- The nodes in red denote that the holder of the option exercised early.

Example - 3 step American tree



Notes on the American option

- For American call options with no dividends it is never optimal to exercise early.
- From the lattice we can determine the early exercise region, as the

set of points S and t for which you would exercise early

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- From the lattice we can determine the early exercise region, as the

set of points S and t for which you would exercise early

- Technically what we have evaluated here is a Bermudan option, which is an American option that can only be exercised on certain specified dates.
- In the limit of more and more observation dates we will approach the American option price.

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DIVIDEND PAYING ASSETS

- The fundamental theorem of finance does not directly apply to dividend paying assets.
- If S_t is the value of an asset at time t which pays out a continuously compounded dividend yield, δ, then consider a new asset X which is defined as

$$X_0 = e^{-\delta t} S_0$$

at time t the S_0 will have grown to $S_t e^{\delta t}$ and so $X_t = S_t$. Thus it will be possible to replicate the value of an option expiring at time t by holding $e^{-\delta t}$ of the underlying asset.

Dividend paying assets

- Thus by the fundamental theorem of finance it will be X which is priced under the risk-neutral measure given a known future asset price S_t thus:

$$X_0 = e^{-rt} E_0^{\tilde{Q}}[S_t]$$

and so

$$S_0 = e^{-(r-\delta)t} E_0^{\tilde{Q}}[S_t]$$

EXAMPLE - CONTINUOUS DIVIDENDS

- We need new values of u and d where

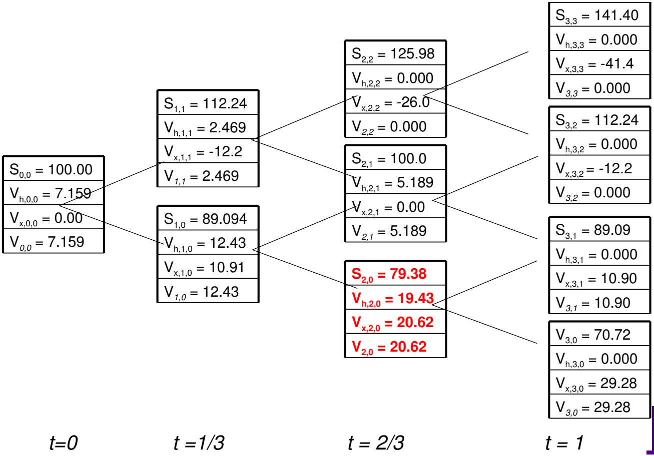
$$E[S_T] = s \exp[(r-\delta)T]$$

$$E[(S_T)^2] = s^2 \exp[(2(r-\delta)+\sigma^2)T]$$

and using CRR gives:

$$u = e^{\sigma\sqrt{\Delta t}}, \quad d = e^{-\sigma\sqrt{\Delta t}}, \quad q = \frac{e^{(r-\delta)\Delta t} - d}{u-d}$$

- Thus if we consider the American put option only now when $\delta = 0.03$ we see that the theoretical price is now \$7.32


$S_{0,0}$ = 100.00
$V_{h,0,0}$ = 7.159
$V_{x,0,0}$ = 0.00
$V_{0,0}$ = 7.159
$S_{1,1}$ = 112.24
$V_{h,1,1}$ = 2.469
$V_{x,1,1}$ = -12.2
$V_{1,1}$ = 2.469
$S_{1,0}$ = 89.094
$V_{h,1,0}$ = 12.43
$V_{x,1,0}$ = 10.91
$V_{1,0}$ = 12.43
$S_{2,2}$ = 125.98
$V_{h,2,2}$ = 0.000
$V_{x,2,2}$ = -26.0
$V_{2,2}$ = 0.000
$S_{2,1}$ = 100.0
$V_{h,2,1}$ = 5.189
$V_{x,2,1}$ = 0.00
$V_{2,1}$ = 5.189
$S_{2,0}$ = 79.38
$V_{h,2,0}$ = 19.43
$V_{x,2,0}$ = 20.62
$V_{2,0}$ = 20.62
$S_{3,3}$ = 141.40
$V_{h,3,3}$ = 0.000
$V_{x,3,3}$ = -41.4
$V_{3,3}$ = 0.000
$S_{3,2}$ = 112.24
$V_{h,3,2}$ = 0.000
$V_{x,3,2}$ = -12.2
$V_{3,2}$ = 0.000
$S_{3,1}$ = 89.09
$V_{h,3,1}$ = 0.000
$V_{x,3,1}$ = 10.90
$V_{3,1}$ = 10.90
$V_{3,0}$ = 70.72
$V_{h,3,0}$ = 0.000
$V_{x,3,0}$ = 29.28
$V_{3,0}$ = 29.28
$t=0$
$t=1/3$
$t=2/3$
$t=1$

Example - Discrete Dividends

- Perhaps a more realistic case is when there is a known discrete dividend payment at a certain point in time. In our example, imagine there is a known dividend, payable after 2/3 of a year which is 3% of the share price.
- Here the fundamental theorem will hold from period to period and our values of u, d and q will remain the same as for the no dividend case but at $t = 2/3$, $S_{2j} \rightarrow 0.97 \times S_{2j}$
- This is depicted in the worked example next:

Node	$t=0$	$t=1/3$	$t=2/3$ Pre Div	$t=2/3$ Post Div	$t=1$
3					$S_{3,3} = 137.16$ $V_{h,3,3} = 0.000$ $V_{x,3,3} = -37.2$ $V_{3,3} = 0.000$
2			$S_{2,2} = 125.98$ $V_{h,2,2} =$ **0.000** $V_{x,2,2} = -26.0$ $V_{2,2} = 0.000$	$S_{2,2} = 122.20$ $V_{h,2,2} = 0.000$ $V_{x,2,2} = -22.2$ $V_{2,2} = 0.000$	$S_{3,2} = 108.87$ $V_{h,3,2} = 0.000$ $V_{x,3,2} = -8.87$ $V_{3,2} = 0.000$
1		$S_{1,1} = 112.24$ $V_{h,1,1} = 2.544$ $V_{x,1,1} = -12.2$ $V_{1,1} = 2.544$	$S_{2,1} = 100.0$ $V_{h,2,1} =$ **5.877** $V_{x,2,1} = 0.00$ $V_{2,1} = 5.877$	$S_{2,1} = 97.0$ $V_{h,2,1} = 5.877$ $V_{x,2,1} = 3$ $V_{2,1} = 5.877$	$S_{3,1} = 86.42$ $V_{h,3,1} = 0.000$ $V_{x,3,1} = 13.58$ $V_{3,1} = 13.58$
0	$S_{0,0} = 100.00$ $V_{h,0,0} = 7.094$ $V_{x,0,0} = 0.00$ $V_{0,0} = 7.094$	$S_{1,0} = 89.09$ $V_{h,1,0} = 13.17$ $V_{x,1,0} = 10.91$ $V_{1,0} = 13.172$	$S_{2,0} = 79.38$ $V_{h,2,0} =$ **23.00** $V_{x,2,0} = 20.62$ $V_{2,0} = 23.00$	**$S_{2,0} = 77.00$** **$V_{h,2,0} = 21.02$** **$V_{x,2,0} = 23.00$** **$V_{2,0} = 23.00$**	$S_{3,0} = 68.60$ $V_{h,3,0} = 0.000$ $V_{x,3,0} = 31.40$ $V_{3,0} = 31.40$

Bold figures denote value if held until after the dividend date

t=0 *t =1/3* *t = 2/3 Pre Div* *t = 2/3 Post Div* *t=1*

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Summing up dividends

- Non-proportional cash dividends can be problematic as this leads to a non-recombining tree .
- This leads to a large increase in the computational effort.
- There is an adjustment for European options but this is not of great practical use as Black-Scholes can be used quite simply in the European case.
- In American option cases the simplest approach is to use interpolation.

SUMMARY

- We have developed a multistep binomial lattice which will approximate the value of a European or American call option when the underlying asset pays out dividends.
- The construction comes from an extension to the fundamental theorem of finance and you have a choice of parameters which are typically chosen to fit the binomial distribution to the Black-Scholes lognormal distribution.
- The most useful outcome is the ability to price American options easily.