Exercise Sheet 6: Solutions

1. (a) We have
\[ \frac{\partial C}{\partial t} = 0, \quad \frac{\partial C}{\partial S} = A, \quad \frac{\partial^2 C}{\partial S^2} = 0. \]
When these are substituted into the Black-Scholes equation, we obtain
\[ rSA - rAS \equiv 0. \]

(b) We have
\[ \frac{\partial C}{\partial t} = -rKe^{-r(T-t)}, \quad \frac{\partial C}{\partial S} = 1, \quad \frac{\partial^2 C}{\partial S^2} = 0. \]
When these are substituted into the Black-Scholes equation, we obtain
\[ -rKe^{-r(T-t)} + rS - r \left( S - Ke^{-r(T-t)} \right) \equiv 0. \]

2. Substitution of \( S^\alpha e^{-r(T-t)} \) into the Black-Scholes equation gives
\[ \frac{1}{2} \sigma^2 \alpha (\alpha - 1) + r \alpha = 0. \]
Factorizing gives \( \alpha \left( \frac{1}{2} \sigma^2 (\alpha - 1) + r \right) = 0. \) Therefore \( \alpha_1 = 0 \) and \( \alpha_2 = 1 - \frac{2r}{\sigma^2}. \)

3. In this case \( S_0 = 80, E = 60, T = 0.25, \sigma = 0.3 \) and \( r = 0.1. \) First, we compute the values of \( d_1 \) and \( d_2 \)
\[ d_1 = \frac{\ln (S/E) + (r + \sigma^2/2)(T - t)}{\sigma \sqrt{T - t}} = \frac{\ln \left( \frac{80}{60} \right) + (0.1 + (0.3)^2 \times 0.5)}{0.3 \times \sqrt{0.25}} \approx 2.1595 \]
\[ d_2 = d_1 - \sigma \sqrt{T - t} = 2.1595 - 0.3 \times \sqrt{0.25} \approx 2.0095 \]
The value of call option is
\[ C_0 = S_0 N(d_1) - E e^{-rT} N(d_2). \]
Since
\[ N(2.1595) \approx 0.9846, \quad N(2.0095) \approx 0.9778, \]
then
\[ C_0 \approx 80 \times 0.9846 - 60 \times e^{-0.1 \times 0.25} \times 0.9778 = 21.549 \]
4. Let us find the limit
\[ \lim_{{\sigma \to 0}} C(S,t). \]

We know that
\[ C(S,t) = SN(d_1) - Ee^{-r(T-t)}N(d_2), \]

and
\[ d_1 = \frac{\ln(S/E) + (r + \sigma^2/2)(T-t)}{\sigma \sqrt{T-t}}, \]
\[ d_2 = d_1 - \sigma \sqrt{T-t}. \]

We can see that the denominator here will go to zero for both \( d_1 \) and \( d_2 \) and hence there value to either \( +\infty \) or \( -\infty \) depending on the sign of the numerator. This gives
\[ \lim_{{\sigma \to 0}} d_1 = \begin{cases} 
-\infty & \text{if } \ln(S/E) + r(T-t) < 0 \\
\infty & \text{if } \ln(S/E) + r(T-t) > 0 
\end{cases} \]

and as a consequence
\[ \lim_{{\sigma \to 0}} N(d_1) = \begin{cases} 
0 & \text{if } \ln(S/E) + r(T-t) < 0 \\
1 & \text{if } \ln(S/E) + r(T-t) > 0 
\end{cases} \]

Similar results can be obtained for \( N(d_2) \).

Therefore
\[ \lim_{{\sigma \to 0}} C(S,t) = \begin{cases} 
S - Ee^{-r(T-t)} & \text{if } S > Ee^{-r(T-t)}, \\
0 & \text{otherwise.} 
\end{cases} \]