1. Two-step binomial tree. In this case \( u = 1.1 \), \( d = 0.9 \), \( r = 0.12 \), \( E = 42 \). There are two time steps of \( \Delta t = \frac{1}{4} \)

The value of the risk-neutral probability.

\[
p = \frac{\exp (0.12 \times 0.25) - 0.9}{1.1 - 0.9} = 0.6523.
\]

The possible final asset prices are \( 40 \times (1.1)^2 = 48.4 \), \( 40 \times 1.1 \times 0.9 = 39.6 \) and \( 40 \times (0.9)^2 = 32.4 \). The corresponding payoff: \( p_{uu} = 0 \), \( p_{ud} = 42 - 39.6 = 2.4 \), \( p_{dd} = 42 - 32.4 = 9.6 \).

Then for the European put option

\[
p_u = \exp (-0.12 \times 0.25) \times [0.6523 \times 0 + (1 - 0.6523) \times 2.4] = 0.8098
\]
\[
p_d = \exp (-0.12 \times 0.25) \times [0.6523 \times 2.4 + (1 - 0.6523) \times 9.6] = 4.7585
\]

Finally, the value of a 6-month European put option is

\[
p_0 = \exp (-0.12 \times 0.25) \times [0.6523 \times 0.8098 + (1 - 0.6523) \times 4.7585] = 2.1183
\]

For the American put option:

the payoff from early exercise at time \( \Delta t \) is \( 42 - 40 \times 1.1 = -2.0 \) and \( 42 - 40 \times 0.9 = 6.0 \). In the later case early exercise is optimal (\( 6.0 > 4.7585 \)), the value of the American put option should be 6 rather than 4.7585.

Therefore the value at \( t = 0 \) is

\[
p_0 = \exp (-0.12 \times 0.25) \times [0.6523 \times 0.8098 + (1 - 0.6523) \times 6] = 2.5372
\]

The payoff at \( t = 0 \) is \( 42 - 40 = 2 \). It is not optimal.

2. Initial stock price is \( S_0 \). Stock price can either move up to \( S_0u \) or down to \( S_0d \). Consider a portfolio consisting of a long position in \( \Delta \) shares and a short position in \( N \) numbers of bonds: \( \Pi = \Delta S - NB \).

At maturity

\[
\Pi (T) = \begin{cases} 
\Delta S_0u - NB_0e^{rT}, \\
\Delta S_0d - NB_0e^{rT},
\end{cases}
\]

Let us find \( \Delta \) and \( NB_0 \) when

\[
\Delta S_0u - NB_0e^{rT} = C_u,
\]
\[
\Delta S_0d - NB_0e^{rT} = C_d.
\]
One can find that
\[
\Delta = \frac{C_u - C_d}{S_0u - S_0d}
\]
and
\[
NB_0 = \frac{C_u - C_d}{u - d} e^{-rT}.
\]
The cost of setting up this portfolio is \(S_0\Delta - NB_0\). Therefore
\[
C_0 = S_0\Delta - NB_0 = \frac{C_u - C_d}{u - d} - \frac{C_u - C_d}{u - d} e^{-rT}
\]
or
\[
C_0 = e^{-rT} E[C_T] = e^{-rT} [pC_u + (1 - p)C_d],
\]
where
\[
p = \frac{e^{rT} - d}{u - d}.
\]

3. Since the put-call parity is violated, \(7.78 < 5.09 - 20.37 + 24 e^{-0.0748\times0.5} = 7.8390\), there exist an arbitrage opportunity.

We set up the portfolio \(\Pi = S + P - C - B\) such that \(\Pi_0 = 20.37 + 7.78 - 5.09 - 23.06 = 0\). The balance is zero!

After six months:
\[
\Pi_T = E - 23.06 \times e^{0.0748\times0.5} = 24 - 23.06 \times e^{0.0748\times0.5} \approx 0.06.
\]
This is an risk-free profit. Note that \(S + P - C = E\) at \(t = T\).