1. In this case \( S_0 = 100, \ u = 1.1, \ d = 0.9, \ r = 0.05, \ T = 0.25, \ C_u = 15, \ C_d = 0. \)

*No-arbitrage* arguments: the number of shares

\[ \Delta = \frac{C_u - C_d}{S_0u - S_0d} = \frac{15 - 0}{100 \times 1.1 - 100 \times 0.9} = 0.75 \]

and the value of call option

\[ C_0 = S_0\Delta - (S_0u\Delta - C_u) e^{-rT} = 100 \times 0.75 - (100 \times 1.1 \times 0.75 - 15) \times e^{-0.05 \times 0.25} = 8.3385 \]

*Risk-neutral valuation*: one can find the probability \( p \)

\[ p = \frac{e^{rT} - d}{u - d} = \frac{e^{0.05 \times 0.25} - 0.9}{1.1 - 0.9} = 0.56289\]

and the value of call option

\[ C_0 = e^{-rT} \left[ pC_u + (1 - p)C_d \right] = e^{-0.05 \times 0.25} \left[ 0.56289 \times 15 + 0 \right] = 8.3385 \]

2. In this case \( u = \frac{85}{80}, \ d = \frac{75}{80}, \ r = 0.05, \ T = \frac{1}{4}, \ P_u = 0, \ P_d = 5. \)

*Risk-neutral valuation*: one can find the probability \( p \)

\[ p = \frac{e^{rT} - d}{u - d} = \frac{e^{0.05 \times \frac{1}{4}} - \frac{75}{80}}{\frac{85}{80} - \frac{75}{80}} = 0.63445 \]

and the value of put option

\[ P_0 = e^{-rT} \left[ pP_u + (1 - p)P_d \right] = e^{-0.05 \times \frac{1}{4}} \left[ 0 + (1 - 0.63445) \times 5 \right] = 1.7975 \]

3. In this case \( u = 1.1, \ d = 0.9, \ r = 0.12, \ \Delta t = 0.25, \ T = 2\Delta t. \)

*Risk-neutral valuation*: one can find the probability \( p \)

\[ p = \frac{e^{r\Delta t} - d}{u - d} = \frac{e^{0.12 \times 0.25} - 0.9}{1.1 - 0.9} = 0.65227. \]

The value of call option \( C_0 \) is

\[ C_0 = e^{-r\Delta t} \left[ pC_u + (1 - p)C_d \right] = e^{-0.12 \times 0.25} \left[ 0.6523 \times 2.0256 + 0 \right] = 1.2822 \]

where

\[ C_u = e^{-r\Delta t} \left[ pC_uu + (1 - p)C_ud \right] = e^{-0.12 \times 0.25} \left[ 0.65 \times 3.2 + 0 \right] = 2.0256 \]

\[ C_d = e^{-r\Delta t} \left[ pC_ud + (1 - p)C_dd \right] = 0 \]

(see two-step tree).
4. $\Delta t = 0.25$

The risk-neutral probability $p$ is $\frac{e^{0.05\times 0.25} - 0.9}{1.2 - 0.9} = 0.3753$. So with probability $p^2 = 0.141$, $S_0 u^2 = 40 \times (1.2)^2 = 57.6$, with probability $2p(1-p) = 2 \times 0.3753 \times (1-0.3753) = 0.4689$, $S_0 ud = 40 \times 1.2 \times 0.9 = 43.2$ and with probability $(1-p)^2 = (1-0.3753)^2 = 0.390$, $S_0 d^2 = 40 \times (0.9)^2 = 32.4$

5. We have three equations

$$qu + (1-q)d = e^{\mu \Delta t}, \quad qu^2 + (1-q)d^2 - (qu + (1-q)d)^2 = \sigma^2 \Delta t,$$

for three unknown parameters $u$, $d$ and $q$. From the first equation we find

$$q = \frac{e^{\mu \Delta t} - d}{u - d}$$

Substituting from the last equation into the second equation, we get

$$e^{\mu \Delta t}(u + d) - ud - e^{2\mu \Delta t} = \sigma^2 \Delta t.$$

By using series expansions $e^{\mu \Delta t} \approx 1 + \mu \Delta t$, $e^{2\mu \Delta t} \approx 1 + 2\mu \Delta t$ and $d = u^{-1}$, we get an equation for $u$

$$(1 + \mu \Delta t)(u + u^{-1}) - 2 - 2\mu \Delta t = \sigma^2 \Delta t$$

or

$$u^2 + 1 = u \frac{2 + 2\mu \Delta t + \sigma^2 \Delta t}{1 + \mu \Delta t} = 2u + u \frac{\sigma^2 \Delta t}{1 + \mu \Delta t} = u \left(2 + \sigma^2 \Delta t\right) + o(\Delta t).$$

The purpose now is to find $u$ as a function of $\sqrt{\Delta t}$. Last equation can be rewritten as quadratic equation: $u^2 - 2ku + 1 = 0$ with $k = 1 + \frac{\sigma^2 \Delta t}{2}$. The solution is $u = k + \sqrt{(k^2 - 1)} > 1$. Substituting $k = 1 + \frac{\sigma^2 \Delta t}{2}$ into $u = k + \sqrt{(k^2 - 1)}$, we get

$$u = 1 + \frac{\sigma^2 \Delta t}{2} + \sqrt{\left(1 + \frac{\sigma^2 \Delta t}{2}\right)^2 - 1} = 1 + \frac{\sigma^2 \Delta t}{2} + \sqrt{\sigma^2 \Delta t + \left(\frac{\sigma^2 \Delta t}{2}\right)^2}.$$

If we expand this in a Taylor series in terms of $\sqrt{\Delta t}$, we obtain

$$u \approx 1 + \sigma \sqrt{\Delta t}.$$

Since $\sqrt{\Delta t}$ is small, we can write

$$u \approx e^{\sigma \sqrt{\Delta t}} \approx 1 + \sigma \sqrt{\Delta t} \quad \text{Cox, Ross, Rubinstein (1979)}$$