INTRODUCTION TO FINANCIAL MATHEMATICS

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Lecture 1

1 Introduction
   - Elementary economics background
   - What is financial mathematics?
   - The role of SDE’s and PDE’s

2 Time Value of Money

3 Continuous Model for Stock Price
MATH20912 Introduction to Financial Mathematics

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Assessment:

Test in week 7, 9am Monday 9th March: 20%
2 hours examination: 80%
This course is concerned with mathematical models for financial markets:

- **Stock Markets**, such as London Stock Exchange, etc.
- **Bond Markets**, where participants buy and sell debt securities.
- **Futures and Option Markets**, where derivatives are traded.

Example: **European call option** gives the holder the right (not obligation) to buy underlying asset at a prescribed time $T$ for a specified price $E$.

Option market is massive! More money is invested in options than in the underlying securities. The main purpose of this course is to determine the fair price of options.
Elementary Economics Background

Shares have been around for hundreds of years
as have bonds
and are still being issued today.
Why do we need stochastic differential equations (SDE's) and partial differential equations (PDE's) to value these contracts?
What is value?

- Market or fair value of a contract - a price everyone agrees on
- We can calculate the value of holding a contract (also known as going long)
- or the value of selling a contract (also known as going short)
- The value of a contract is the amount another investor would pay in exchange for the contract.
How do we value money invested in a bank? There are several ways...

**Definition: Simple interest rate**

For the interest rate $r$ the value $V(T)$ at time $T$ of holding $P$ units of currency starting at time $t = 0$:

$$V(T) = (1 + rT)P$$  \hspace{1cm} (1)

where $T$ is expressed in years.

**Definition: Compound interest rate**

$$V(T) = \left(1 + \frac{r}{m}\right)^{mT}P$$  \hspace{1cm} (2)

where $m$ is the number interest payments made per annum.
**Definition: Continuous compounding**

For a constant interest rate $r$ the time value of money under continuous compounding is given by:

$$V(T) = e^{rT}P.$$  \hfill (3)

In the limit $m \to \infty$, we obtain the results above since

$$e = \lim_{z \to \infty} \left(1 + \frac{1}{z}\right)^z.$$

Throughout this course we assume that the interest rate $r$ will be continuously compounded.
Investors are primarily interested in the return on investment.

**Definition: Return**

\[
\text{return} = \frac{\text{change in value over a period of time}}{\text{initial investment}} \quad (4)
\]
Let $S(t)$ represent the stock price at time $t$. How to write a simple model for this quantity?

• Return:

$$\frac{\Delta S}{S}$$

(5)

where $\Delta S = S(t + \delta t) - S(t)$

In the limit $\delta t \to 0$:

$$\frac{dS}{S}$$

(6)

• Can we model the return?

Let us decompose the return into two parts: deterministic and stochastic
Modelling Return

Return:

\[
\frac{dS}{S} = \mu dt + \sigma dW
\]  \hspace{1cm} (7)

- $\mu dt$ is a measure of the deterministic expected rate of growth of the stock price. In general, $\mu = \mu(S, t)$. In simple models $\mu$ is taken to be constant ($\mu = 0.1 \ yr^{-1} = 10 \ %yr^{-1}$).

- $\sigma dW$ describes the stochastic change in the stock price, where $dW$ stands for

\[
\Delta W = W(t + \Delta t) - W(t)
\]

as $\Delta t \to 0$

- $W(t)$ is a Wiener process

- $\sigma$ is the volatility ($\sigma = 0.2 \ yr^{-\frac{1}{2}} = 20 \ %yr^{-\frac{1}{2}}$)
On the left we show computer simulations with $\mu = 0.08$ and $\sigma = 0.1$, on the right is the real data from yahoo finance.

\[ dS = \mu S dt + \sigma S dW \]
The standard Wiener process $W(t)$ is a Gaussian process such that:

- $W(t)$ has independent increments: if $u \leq v \leq s \leq t$, then $W(t) - W(s)$ and $W(v) - W(u)$ are independent.
- $W(s + t) - W(s)$ is $N(0, t)$ and $W(0) = 0$
Clearly

- $\mathbb{E}[W(t)] = 0$ and $\mathbb{E}[W^2] = t$, where $\mathbb{E}$ is the expectation operator.

- The increment $\Delta W = W(t + \Delta t) - W(t)$ can be written as $\Delta W = X (\Delta t)^{\frac{1}{2}}$, where $X$ is a random variable with normal distribution with zero mean and unit variance:

  $$X \sim N(0, 1)$$

- $\mathbb{E}[\Delta W] = 0$ and $\mathbb{E}[(\Delta W)^2] = \Delta t$. 
Lecture 2

1. Properties of Wiener Process
2. Approximation for Stock Price Equation
3. Itô’s Lemma
The probability density function for $W(t)$ is

$$p(y, t) = \frac{1}{\sqrt{2\pi t}} \exp \left( -\frac{y^2}{2t} \right)$$

and $\mathbb{P} (a \leq W(t) \leq b) = \int_a^b p(y, t) \, dt$

- Simulations of a Wiener process:
Approximation of SDE for small $\Delta t$

- The increment $\Delta W = W(t + \Delta t) - W(t)$ can be written as $\Delta W = X (\Delta t)^{\frac{1}{2}}$, where $X$ is a random variable with normal distribution with zero mean and unit variance: $X \sim N(0, 1)$

- $E\Delta W = 0$ and $E(\Delta W)^2 = \Delta t$.

Recall: equation for the stock price is

$$dS = \mu S dt + \sigma S dW,$$

then

$$\Delta S \approx \mu S \Delta t + \sigma S X (\Delta t)^{\frac{1}{2}}$$

It means $\Delta S \sim N(\mu S \Delta t, \sigma^2 S^2 \Delta t)$
Example 1. Consider a stock that has volatility 30% and provides expected return of 15% p.a. Find the increase in stock price for one week if the initial stock price is 100.

Answer: $\Delta S = 0.288 + 4.16X$

Note: 4.16 is the standard deviation per week
Example 2. Show that the return $\frac{\Delta S}{S}$ is normally distributed with mean $\mu \Delta t$ and variance $\sigma^2 \Delta t$
We assume that $f(S, t)$ is a smooth function of $S$ and $t$.

Find $df$ if $dS = \mu S dt + \sigma S dW$

- Volatility $\sigma = 0$

$$df = \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial S} dS = \left( \frac{\partial f}{\partial t} + \mu S \frac{\partial f}{\partial S} \right) dt$$

- Volatility $\sigma \neq 0$

**Itô’s Lemma:**

$$df = \left( \frac{\partial f}{\partial t} + \mu S \frac{\partial f}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} \right) dt + \sigma S \frac{\partial f}{\partial S} dW$$
Example 3. Find the SDE satisfied by $f = S^2$. 
1 Distribution for $\ln S(t)$

2 Solution to Stochastic Differential Equation for Stock Price

3 Examples
Example 1. Find the stochastic differential equation (SDE) for

\[ f = \ln S \]

by using Itô’s Lemma:

\[ df = \left( \frac{\partial f}{\partial t} + \mu S \frac{\partial f}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} \right) dt + \sigma S \frac{\partial f}{\partial S} dW. \]
We obtain

\[ df = \left( \mu - \frac{\sigma^2}{2} \right) dt + \sigma dW \]

and this is a constant coefficient SDE.

Therefore integration from 0 to \( t \) gives

\[ f - f_0 = \left( \mu - \frac{\sigma^2}{2} \right) t + \sigma W(t) \quad \text{since} \quad W(0) = 0. \]
Now since $f = \ln S$ we can write

$$\ln S(t) - \ln S_0 = \left(\mu - \frac{\sigma^2}{2}\right) t + \sigma W(t)$$

where $S_0 = S(0)$ is the initial stock price.

This means $\ln S(t)$ has a normal distribution with mean $\ln S_0 + \left(\mu - \frac{\sigma^2}{2}\right) t$ and variance $\sigma^2 t$. 
Example 2. Consider a stock with an initial price of 40, an expected return of 16% and a volatility of 20%. Find the probability distribution of $\ln S$ in six months.

We have

$$\ln S(T) \sim N \left( \ln S_0 + \left( \mu - \frac{\sigma^2}{2} \right) T, \sigma^2 T \right)$$

Answer: $\ln S(0.5) \sim N (3.759, 0.020)$
Recall that if the random variable $X$ has a normal distribution with mean $\mu$ and variance $\sigma^2$, then the probability density function is

$$p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp \left( -\frac{(x - \mu)^2}{2\sigma^2} \right)$$

The probability density function of $X = \ln S(t)$ is

$$\frac{1}{\sqrt{2\pi\sigma^2 t}} \exp \left( -\frac{(x - \ln S_0 - (\mu - \sigma^2/2)t)^2}{2\sigma^2 t} \right)$$
Definition. The model of a stock \(dS = \mu S dt + \sigma S dW\) is known as a geometric Brownian motion.

The random function \(S(t)\) can be found from

\[
\ln(S(t)/S_0) = \left(\mu - \frac{\sigma^2}{2}\right) t + \sigma W(t)
\]

Stock price at time \(t\):

\[
S(t) = S_0 e^{\left(\mu - \frac{\sigma^2}{2}\right) t + \sigma W(t)}
\]

Or

\[
S(t) = S_0 e^{\left(\mu - \frac{\sigma^2}{2}\right) t + \sigma \sqrt{t} X} \quad \text{where} \quad X \sim N(0, 1)
\]
Below we plot the lognormal distribution function for $\mu = 0$, $\sigma = 0.4$ and $t = 1$. 

$$S(t) = S_0 e^{\mu t + \sigma^2 t}.$$ 

$\ln \mathcal{N} \left( \left( \mu - \frac{\sigma^2}{2} \right) t, \sigma^2 t \right)$
1. Financial Derivatives
2. European Call and Put Options
3. Payoff Diagrams, Short Selling and Profit
A Derivative is a financial instrument whose value depends on the values of other underlying variables. Other names are financial derivative, derivative security, derivative product. A stock option, for example, is a derivative whose value is dependent on a stock price.

Examples: forward contracts, futures, options, swaps, CDS, etc.

Options are very attractive to investors, both for speculation and for hedging.
**Derivatives**

**Definition**

**European call option** gives the holder the right (not obligation) to buy underlying asset at a prescribed time $T$ for a specified (strike) price $E$.

**European put option** gives its holder the right (not obligation) to sell underlying asset at a prescribed time $T$ for a specified (strike) price $E$.

The question is:

“What does this actually mean?”
Consider a three-month European call option on a BP share with a strike price $E = 15$ ($T = 0.25$). If you enter into this contract you have the right but not the obligation to buy one share for $E = 15$ in a three months time.

Whether you exercise your right depends on the stock price in the market at time $T$:

- If the stock price is above £15, say £25, you can buy the share for £15, and sell it immediately for £25, making a profit of £10.
- If the stock price is below £15, there is no financial sense to buy it. The option is worthless.
We denote by $C(S, t)$ the value of European call option and $P(S, t)$ the value of European put option.

**Definition**

Payoff Diagram is a graph of the value of the option position at expiration $t = T$ as a function of the underlying stock price $S$.

Call price at $t = T$:

$$C(S, T) = \max (S - E, 0)$$

$$= \begin{cases} 
0, & S \leq E, \\
S - E, & S > E, 
\end{cases}$$
Put price at $t = T$:

$$P(S, T) = \max (E - S, 0) = \begin{cases} E - S, & S \leq E, \\ 0, & S > E, \end{cases}$$

If a trader thinks that the stock price is on the rise, he can make money by purchasing a call option without buying the stock. If a trader believes the stock price is on the decline, he can make money by buying put options.
The profit (gain) of a call option holder (buyer) at time $T$ is
\[ \max (S - E, 0) - C_0 e^{rT}, \]
where $C_0$ is the initial call option price at $t = 0$.

Example:
Find the stock price on the exercise date in three months, for a European call option with strike price £10 to give a gain (profit) of £14 if the option is bought for £2.25, financed by a loan with continuously compounded interest rate of 5%

Solution:
\[ 14 = S(T) - 10 - 2.25 \times e^{0.05 \times \frac{1}{4}}, \]
\[ S(T) = 26.28 \]

For the holder of European put option, the profit at time $T$ is
\[ \max (E - S, 0) - P_0 e^{rT} \]
Portfolios and Short Selling

**Definition**

Short selling is the practice of selling assets that have been borrowed from a broker with the intention of buying the same assets back at a later date to return to the broker.

This technique is used by investors who try to profit from the falling price of a stock.

**Definition**

Portfolio is a combination of assets, options and bonds.

We denote by $\Pi$ the value of a portfolio. Example: $\Pi = 2S + 4C - 5P$.

It means that the portfolio consists of long position in two shares, long position in four call options and a short position in five put options.
**Option positions**

- $C(S, T)$ Long Call
- $P(S, T)$ Long Put
- $-C(S, T)$ Short Call
- $-P(S, T)$ Short Put
Straddle is the purchase of a call and a put on the same underlying security with the same maturity time $T$ and strike price $E$. The value of portfolio is $\Pi = C + P$.

- Straddle is effective when an investor is confident that a stock price will change dramatically, but is uncertain of the direction of price move.
Example of large profits: $S_0 = 40$, $E = 40$, $C_0 = 2$, $P_0 = 2$.
Can you find the expected return if the stock price at $T$ is given by the following tree?

\[
\begin{array}{c}
S_0 = 40 \\
\quad \quad \quad p = \frac{3}{4} 60 \\
\quad \quad p = \frac{1}{4} 20
\end{array}
\]

Ans: 400%
Trading Strategies: Straddle, Bull Spread, etc.

Bond and Risk-Free Interest Rate

No Arbitrage Principle
**Trading Strategies Involving Options**

**Straddle** is the purchase of a call and a put on the same underlying security with the same maturity time $T$ and strike price $E$.

The value of portfolio is $\Pi = C + P$

- **Straddle** is effective when an investor is confident that a stock price will change dramatically, but is uncertain of the direction of price move.

- **Short Straddle**, $\Pi = -C - P$, profits when the underlying security changes little in price before the expiration $t = T$.

**Barings Bank** was the oldest bank in London until its collapse in 1995. It happened when the bank’s trader, Nick Leeson, took short straddle positions and lost 1.3 billion dollars.
Bull Spread

Bull spread is a strategy that is designed to profit from a moderate rise in the price of the underlying security.

Let us set up a portfolio consisting of a long position in call with strike price $E_1$ and short position in call with $E_2$ such that $E_1 < E_2$.

The value of this portfolio is $\Pi_t = C_t(E_1) - C_t(E_2)$. At maturity $t = T$

$$\Pi_T = \begin{cases} 
0, & S \leq E_1, \\
S - E_1, & E_1 \leq S < E_2, \\
E_2 - E_1, & S \geq E_2
\end{cases}$$

- The holder of this portfolio benefits when the stock price will be above $E_1$. 
We assume the existence of a \textit{risk-free} investment. Examples are US government bonds or deposits in a sound bank. We denote by $B(t)$ the value of this investment.

\begin{definition}
A \textbf{Bond} is a contract that yields a known amount $F$, called the \textbf{face value}, on a known time $T$, called the \textbf{maturity date}. The authorised issuer (for example, government) owes the holder a debt and is obliged to repay the face value at maturity and may also pay interest (the coupon).
\end{definition}
A Zero-coupon bond does not pay any coupons and involves only a single payment at $T$.

The return on a risk free bond can be defined as

$$\frac{dB}{B} = r dt,$$

where $r$ is the risk-free interest rate.

If $B(T) = F$ and $r$ is constant, then $B(t) = F e^{-r(T-t)}$, where $e^{-r(T-t)}$ is often called the discount factor.
No Arbitrage Principle

One of the key principles of financial mathematics is the No Arbitrage Principle.

- There are never opportunities to make risk-free profit.
- Arbitrage opportunity arises when a zero initial investment $\Pi_0 = 0$ is identified that guarantees non-negative payoff in the future such that $\Pi_T > 0$ with non-zero probability.

Arbitrage opportunities may exist in a real market. But, they cannot last for a long time.
1. No-Arbitrage Principle
2. Put-Call Parity
3. Upper and Lower Bounds on Call Options
The key principle of financial mathematics is No Arbitrage Principle.

- All risk-free portfolios must have the same rate of return. Let $\Pi$ be the value of a risk-free portfolio, and $d\Pi$ is its increment during a small period of time $dt$.

- Then
  \[
  \frac{d\Pi}{\Pi} = rdt,
  \]
  where $r$ is the risk-free interest rate.

- Let $\Pi_t$ be the value of the portfolio at time $t$. If $\Pi_T \geq 0$, then $\Pi_t \geq 0$ for $t < T$. 

Let us set up portfolio consisting of long one stock, long one put and short one call with the same $T$ and $E$. The value of this portfolio is $\Pi = S + P - C$.

The payoff for this portfolio is

$$\Pi_T = S + \max (E - S, 0) - \max (S - E, 0) = E$$

The payoff is always the same whatever the stock price is at $t = T$.

Using No Arbitrage Principle, we obtain

$$S_t + P_t - C_t = E e^{-r(T-t)},$$

where $C_t = C(S_t, t)$ and $P_t = P(S_t, t)$. 
This relationship between $S_t$, $P_t$ and $C_t$ is called Put-Call Parity which represents an example of complete risk elimination.

- The Put-Call Parity ($t = 0$): $S_0 + P_0 - C_0 = Ee^{-rT}$.
- It shows that the value of European call option can be found from the value of European put option with the same strike price and maturity:

$$C_0 = P_0 + S_0 - Ee^{-rT}.$$  

Therefore $C_0 \geq S_0 - Ee^{-rT}$ since $P_0 \geq 0$

- $S_0 - Ee^{-rT}$ is the lower bound for call option and

$$S_0 - Ee^{-rT} \leq C_0 \leq S_0$$
Let us illustrate these bounds geometrically.
Example 1. Find a lower bound for a six month European call option with the strike price £35 when the initial stock price is £40 and the risk-free interest rate is 5% p.a.

In this case \( S_0 = 40, \ E = 35, \ T = 0.5, \) and \( r = 0.05. \)

The lower bound for the call option price is \( S_0 - E \exp(-rT), \) or

\[
40 - 35 \exp(-0.05 \times 0.5) = 5.864
\]
Example 2. Consider the situation where the European call option is £4. Show that there exists an arbitrage opportunity.

We establish a zero initial investment $\Pi_0 = 0$ by purchasing one call for £4 and the bond for £36 and selling one share for £40. The portfolio is $\Pi = C + B - S$.

At maturity $t = T$, the portfolio $\Pi = C + B - S$ has the value:

$$\Pi_T = \max(S - E, 0) + 36 \exp(0.05 \times 0.5) - S = \begin{cases} 36.911 - S, & S \leq 35 \\ 1.911, & S > 35 \end{cases}$$

It is clear that $\Pi_T > 0$, therefore there exists an arbitrage opportunity.
1. Upper and Lower Bounds on Put Options
2. Proof of Put-Call Parity by No-Arbitrage Principle
3. Example on Arbitrage Opportunity
Upper and Lower Bounds on Put Option
(Examples Sheet 3):

\[ E e^{-rT} - S_0 \leq P_0 \leq E e^{-rT} \]

Let us illustrate these bounds geometrically.
The value of European put option can be found as

\[ P_0 = C_0 - S_0 + E e^{-rT}. \]

Let us prove this relation by using No-Arbitrage Principle.
Proof of Put-Call Parity

Assume that $P_0 > C_0 - S_0 + Ee^{-rT}$. Then we can show how one can make a risk-free profit (arbitrage opportunity).

We set up the portfolio $\Pi = -P - S + C + B$ such that $\Pi_0 = 0$. At time $t = 0$ we

- sell one put option for $P_0$ (short/write the put option)
- sell one share for $S_0$ (short position)
- buy one call option for $C_0$ (long/hold the call option)
- buy one bond for $B_0 = P_0 + S_0 - C_0 > Ee^{-rT}$
Proof of Put-Call Parity

At maturity $t = T$ the portfolio $\Pi = -P - S + C + B$ has the value

$$\Pi_T = \begin{cases} 
- (E - S) - S + B_0 e^{rT}, & S \leq E, \\
- S + (S - E) + B_0 e^{rT}, & S > E, 
\end{cases} \quad = -E + B_0 e^{rT}$$

Now

$$B_0 > E e^{-rT} \quad \implies \quad \Pi_T > 0.$$

So since $\Pi_0 = 0$ and $\Pi_T > 0$ there exists an arbitrage opportunity.
Now we assume that $P_0 < C_0 - S_0 + Ee^{-rT}$.

We set up the portfolio $\Pi = P + S - C - B$.

At time $t = 0$ we

- buy one put option for $P_0$
- buy one share for $S_0$ (long position)
- sell one call option for $C_0$ (write the call option)
- borrow $B_0 = P_0 + S_0 - C_0 < Ee^{-rT}$
The balance of all these transactions is zero, that is, \( \Pi_0 = 0 \)

At maturity \( t = T \) we have \( \Pi_T = E - B_0e^{rT} \). Since \( B_0 < Ee^{-rT} \), we conclude \( \Pi_T > 0 \).

This is an arbitrage opportunity!!!
Example on Arbitrage Opportunity

Three months European call and put options with the exercise price £12 are trading at £3 and £6 respectively. The stock price is £8 and interest rate is 5%. Show that there exists arbitrage opportunity.
Example on Arbitrage Opportunity

Solution:

The Put-Call Parity $P_0 = C_0 - S_0 + Ee^{-rT}$ is violated, because 
$6 < 3 - 8 + 12e^{-0.05\times\frac{1}{4}} = 6.851$

To get arbitrage profit we

- buy a put option for £6
- sell a call option for £3
- buy a share for £8
- borrow £11 at the interest rate 5%.

The balance is zero!!
Example: Arbitrage Opportunity

The value of the portfolio $\Pi = P + S - C - B$ at maturity $T = \frac{1}{4}$ is

$$\Pi_T = E - B_0 e^{rT} = 12 - 11e^{0.05 \times \frac{1}{4}} \approx 0.862.$$  

Combination $P + S - C$ gives us £12. We repay the loan £$11e^{0.05 \times \frac{1}{4}}$.

The balance $12 - 11e^{0.05 \times \frac{1}{4}}$ is an arbitrage profit £0.862.
1. One-Step Binomial Model for Option Price
2. Risk-Neutral Valuation
3. Examples
Initial stock price is $S_0$. The stock price can either move up from $S_0$ to $S_0u$ or down from $S_0$ to $S_0d$ ($u > 1; d < 1$).

At time $T$, let the option price be $C_u$ if the stock price moves up, and $C_d$ if the stock price moves down.

The purpose is to find the current price $C_0$ of a European call option.
Now, we set up a portfolio consisting of a long position in $\Delta$ shares and short position in one call

$$\Pi = \Delta S - C$$

**Task**
Let us find the number of shares $\Delta$ that makes the portfolio $\Pi$ risk free.

The value of portfolio when stock moves up is

$$\Delta S_0u - C_u$$

The value of portfolio when stock moves down is

$$\Delta S_0d - C_d$$
If the portfolio $\Pi = \Delta S - C$ is to be risk free, then

$$\Delta S_0 u - C_u = \Delta S_0 d - C_d$$

and therefore the number of shares $\Delta = \frac{C_u - C_d}{S_0 (u - d)}$.

**Task**

Can we use this to find the value of the European Option?

Because the portfolio is risk free for this $\Delta$, the current value $\Pi_0$ can be found by discounting: $\Pi_0 = (\Delta S_0 u - C_u) e^{-rT}$, where $r$ is the risk free interest rate.

On the other hand, the cost of setting up the portfolio is $\Pi_0 = \Delta S_0 - C_0$. Therefore

$$\Delta S_0 - C_0 = (\Delta S_0 u - C_u) e^{-rT}.$$
Finally, the current call option price is

$$C_0 = \Delta S_0 - (\Delta S_0 u - C_u) e^{-rT},$$

where $\Delta = \frac{C_u - C_d}{S_0(u - d)}$.

(No-Arbitrage Argument)

Alternatively

$$C_0 = e^{-rT} (pC_u + (1 - p)C_d),$$

where

$$p = \frac{e^{rT} - d}{u - d}.$$

(Risk-Neutral Valuation)
It is natural to interpret the variable $0 \leq p \leq 1$ as the probability of an up movement in the stock price, and the variable $1 - p$ as the probability of a down movement.

Fair price of a call option $C_0$ is equal to the expected value of its future payoff discounted at the risk-free interest rate. For a put option $P_0$ we have the same result

$$P_0 = e^{-rT} (pP_u + (1 - p)P_d).$$
A stock price is currently $40. At the end of three months it will be either $44 or $36. The risk-free interest rate is 12%. What is the value of three-month European call option with a strike price of $42? Use no-arbitrage arguments and risk-neutral valuation.

In this case $S_0 = 40$, $u = 1.1$, $d = 0.9$, $r = 0.12$, $T = 0.25$, $C_u = 2$, $C_d = 0$. 
**Example**

_No-arbitrage_ arguments: the number of shares

\[
\Delta = \frac{C_u - C_d}{S_0 u - S_0 d} = \frac{2 - 0}{40 \times (1.1 - 0.9)} = 0.25
\]

and the value of call option

\[
C_0 = S_0 \Delta - (S_0 u \Delta - C_u) e^{-rT} =
\]

\[
40 \times 0.25 - (40 \times 1.1 \times 0.25 - 2) \times e^{-0.12 \times 0.25} = 1.266
\]
**Example**

Risk-neutral valuation: one can find the probability $p$

$$p = \frac{e^{rT} - d}{u - d} = \frac{e^{0.12 \times 0.25} - 0.9}{1.1 - 0.9} = 0.6523$$

and the value of call option

$$C_0 = e^{-rT} [pC_u + (1 - p)C_d] = e^{-0.12 \times 0.25} [0.6523 \times 2 + 0] = 1.266$$
1. Risk-Neutral Valuation
2. Risk-Neutral World
3. Two-Steps Binomial Tree
Reminder from Lecture 8...

- Call option price:

\[ C_0 = e^{-rT} \left( pC_u + (1 - p)C_d \right), \]

where \( p = \frac{e^{rT} - d}{u - d} \). No-Arbitrage Principle: \( d < e^{rT} < u \).

- In particular, if \( d > e^{rT} \) then there exists an arbitrage opportunity. We could make money by taking out a bank loan \( B_0 = S_0 \) at time \( t = 0 \) and buying the stock for \( S_0 \).

- We interpret the variable \( 0 \leq p \leq 1 \) as the probability of an up movement in the stock price.

This formula is known as a risk-neutral valuation.

- The probability of up \( q \) or down movement \( 1 - q \) in the stock price plays no role whatsoever! Why???
Let us find the expected stock price at $t = T$:

$$\mathbb{E}_p [S_T] = p S_0 u + (1-p) S_0 d = \frac{e^{rT} - d}{u - d} S_0 u + (1 - \frac{e^{rT} - d}{u - d}) S_0 d = S_0 e^{rT}.$$  

This shows that stock price grows on average at the risk-free interest rate $r$. Since the expected return is $r$, this is a risk-neutral world.

<table>
<thead>
<tr>
<th>In the Real World:</th>
<th>In a Risk-Neutral World:</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathbb{E} [S_T] = S_0 e^{\mu T}$</td>
<td>$\mathbb{E}_p [S_T] = S_0 e^{rT}$</td>
</tr>
</tbody>
</table>

- Risk-Neutral Valuation: $C_0 = e^{-rT} \mathbb{E}_p [C_T]$  
- The option price is the expected payoff in a risk-neutral world, discounted at risk-free rate $r$.  

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Now the stock price changes twice, each time by either a factor of \( u > 1 \) or \( d < 1 \). We assume that the length of the time step is \( \Delta t \) such that \( T = 2\Delta t \). After two time steps the stock price will be \( S_0u^2 \), \( S_0ud \) or \( S_0d^2 \).

The call option expires after two time steps producing payoffs \( C_{uu} \), \( C_{ud} \) and \( C_{dd} \) respectively.
The purpose is to calculate the option price \( C_0 \) at the initial node of the tree. We apply the risk-neutral valuation backward in time: \( C_u = e^{-r\Delta t} (pC_{uu} + (1 - p)C_{ud}) \), \( C_d = e^{-r\Delta t} (pC_{ud} + (1 - p)C_{dd}) \).

Current option price: \( C_0 = e^{-r\Delta t} (pC_u + (1 - p)C_d) \).
Substitution gives

\[ C_0 = e^{-2r\Delta t} \left( p^2 C_{uu} + 2p(1 - p)C_{ud} + (1 - p)^2 C_{dd} \right), \]

where \( p^2, 2p(1 - p) \) and \( (1 - p)^2 \) are the probabilities in a risk-neutral world that the upper, middle, and lower final nodes are reached.

Finally, the current call option price is

\[ C_0 = e^{-rT} \mathbb{E}_p [C_T], \quad T = 2\Delta t. \]

The current put option price can be found in the same way:

\[ P_0 = e^{-2r\Delta t} \left( p^2 P_{uu} + 2p(1 - p)P_{ud} + (1 - p)^2 P_{dd} \right) \]

or

\[ P_0 = e^{-rT} \mathbb{E}_p [P_T]. \]
Consider six months European put with a strike price of £32 on a stock with current price £40. There are two time steps and in each time step the stock price either moves up by 20% or moves down by 20%. Risk-free interest rate is 10%. Find the current option price.
Two-Step Binomial Tree Example

We have $u = 1.2$, $d = 0.8$, $\Delta t = 0.25$, and $r = 0.1$. Risk-neutral probability $p = \frac{e^{r\Delta t} - d}{u - d} = \frac{e^{0.1 \times 0.25} - 0.8}{1.2 - 0.8} = 0.5633$.

The possible stock prices at final nodes are $40 \times (1.2)^2 = 57.6$, $40 \times 1.2 \times 0.8 = 38.4$, and $40 \times (0.8)^2 = 25.6$.

We obtain $P_{uu} = 0$, $P_{ud} = 0$ and $P_{dd} = 32 - 25.6 = 6.4$.

Thus the value of put option is

$$P_0 = e^{-2 \times 0.1 \times 0.25} \times \left(0 + 0 + (1 - 0.5633)^2 \times 6.4\right) = 1.1610.$$
1 Binomial Model for Stock Price
2 Option Pricing on Binomial Tree
3 Matching Volatility $\sigma$ with $u$ and $d$
Continuous random model for the stock price:
\[ dS = \mu S dt + \sigma S dW \]

The binomial model for the stock price is a discrete time model:

- The stock price \( S \) changes only at discrete times \( \Delta t, 2\Delta t, 3\Delta t, \ldots \)
- The price either moves up \( S \rightarrow Su \) or down \( S \rightarrow Sd \) with \( d < e^{r\Delta t} < u \).
- The probability of up movement is \( q \).
Let us build up a tree of possible stock prices. The tree is called a binomial tree, because the stock price will either move up or down at the end of each time period. Each node represents a possible future stock price.

We divide the time to expiration $T$ into several time steps of duration $\Delta t = T/N$, where $N$ is the number of time steps in the tree.
Example: Let us sketch the binomial tree for $N = 4$. 
We introduce the following notations:

- \( S_{m}^{n} \) is the \( n \)-th possible value of stock price at time-step \( m\Delta t \).

Then \( S_{m}^{n} = u^{n}d^{m-n}S_{0}^{0} \), where \( n = 0, 1, 2, \ldots, m \).

\( S_{0}^{0} \) is the stock price at the time \( t = 0 \). Note that \( u \) and \( d \) are the same at every node in the tree.

For example, at the third time-step \( 3\Delta t \), there are four possible stock prices: \( S_{0}^{3} = d^{3}S_{0}^{0} \), \( S_{1}^{3} = ud^{2}S_{0}^{0} \), \( S_{2}^{3} = u^{2}dS_{0}^{0} \) and \( S_{3}^{3} = u^{3}S_{0}^{0} \).

At the final time-step \( N\Delta t \), there are \( N + 1 \) possible values of stock price.
We denote by $C^m_n$ the $n$-th possible value of call option at time-step $m \Delta t$.

- **Risk Neutral Valuation (backward in time):**
  
  $C^m_n = e^{-r \Delta t} \left( pC^m_{n+1} + (1 - p)C^m_{n+1} \right)$.

  Here $0 \leq n \leq m$ and $p = \frac{e^{r \Delta t u} - d}{u - d}$.

- **Final condition:** $C^N_n = \max \left( S^N_n - E, 0 \right)$, where

  $n = 0, 1, 2, ..., N$, $E$ is the strike price.

The current option price $C^0_0$ is the expected payoff in a risk-neutral world, discounted at risk-free rate $r$:

$C^0_0 = e^{-rT} \mathbb{E}_p [C_T]$. 
Example: $N = 4$. 
We assume that the stock price starts at the value $S_0$ and the time step is $\Delta t$. Let us find the expected stock price, $\mathbb{E}[S]$, and the variance of the return, $\text{var} \left[ \frac{\Delta S}{S} \right]$, for continuous and discrete models.

- Expected stock price: Continuous model: $\mathbb{E}[S] = S_0 e^{\mu \Delta t}$. 
  On the binomial tree: $\mathbb{E}[S] = q S_0 u + (1 - q) S_0 d$.

First equation: $q u + (1 - q) d = e^{\mu \Delta t}$. 
Matching volatility $\sigma$ with $u$ and $d$

- Variance of the return: Continuous model:
  \[
  \text{var} \left[ \frac{\Delta S}{S} \right] = \sigma^2 \Delta t \text{ (Lecture 2)}
  \]
  On the binomial tree: \[
  \text{var} \left[ \frac{\Delta S}{S} \right] = q(u - 1)^2 + (1 - q)(d - 1)^2 - [q(u - 1) + (1 - q)(d - 1)]^2 = qu^2 + (1 - q)d^2 - [qu + (1 - q)d]^2.
  \]
  Recall: \[
  \text{var} [X] = \mathbb{E} [X^2] - [\mathbb{E} (X)]^2.
  \]

  Second equation: \[
  qu^2 + (1 - q)d^2 - [qu + (1 - q)d]^2 = \sigma^2 \Delta t.
  \]

- Third equation: \[
  u = d^{-1}.
  \]
Matching volatility $\sigma$ with $u$ and $d$

From the first equation we find $q = \frac{e^{\mu \Delta t} - d}{u - d}$.

This is the probability of an up movement in the real world. Substituting this probability into the second equation, we obtain

$$e^{\mu \Delta t}(u + d) - ud - e^{2\mu \Delta t} = \sigma^2 \Delta t.$$  

Using $u = d^{-1}$, we get

$$e^{\mu \Delta t}(u + \frac{1}{u}) - 1 - e^{2\mu \Delta t} = \sigma^2 \Delta t.$$  

This equation can be reduced to the quadratic equation. (Examples Sheet 4, part 5).
Matching volatility $\sigma$ with $u$ and $d$

From

$$e^{\mu \Delta t} (u + \frac{1}{u}) - 1 - e^{2\mu \Delta t} = \sigma^2 \Delta t.$$ 

one can obtain $u \approx e^{\sigma \sqrt{\Delta t}} \approx 1 + \sigma \sqrt{\Delta t}$ and $d \approx e^{-\sigma \sqrt{\Delta t}}$.

These are the values of $u$ and $d$ obtained by Cox, Ross, and Rubinstein in 1979.

Recall: $e^x \approx 1 + x$ for small $x$. 

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American Put Option Pricing on Binomial Tree

Replicating Portfolio
An American Option is one that may be exercised at any time prior to expire \( t = T \).

We should determine when it is best to exercise the option.

It is not subjective! It can be determined in a systematic way!

The American put option value must be greater than or equal to the payoff function.

If \( P < \max(E - S, 0) \), then there is obvious arbitrage opportunity.

We can buy stock for \( S \) and option for \( P \) and immediately exercise the option by selling stock for \( E \).

\[ E - (P + S) > 0 \]
American Put Option on a Binomial Tree

We denote by $P^m_n$ the $n$-th possible value of put option at time-step $m\Delta t$.

- **European Put Option:**
  \[
  P^m_n = e^{-r\Delta t} \left( pP^m_{n+1} + (1 - p)P^m_{n+1} \right) .
  \]
  Here $0 \leq n \leq m$ and the risk-neutral probability
  \[
  p = \frac{e^{r\Delta t} - d}{u - d}.
  \]

- **American Put Option:**
  \[
  P^m_n = \max \left\{ \max(E - S^m_n, 0), e^{-r\Delta t} \left( pP^m_{n+1} + (1 - p)P^m_{n+1} \right) \right\} ,
  \]
  where $S^m_n$ is the $n$-th possible value of stock price at time-step $m\Delta t$.

- **Final condition:**
  \[
  P^N_n = \max \left( E - S^N_n, 0 \right) ,
  \]
  where $n = 0, 1, 2, \ldots, N$, $E$ is the strike price.
Example

Evaluation of American Put Option on Two-Step Tree:

- We assume that over each of the next two years the stock price either moves up by 20% or moves down by 20%. The risk-free interest rate is 5%.
- Find the value of a 2-year American put with a strike price of $52 on a stock whose current price is $50.

In this case \( u = 1.2, \ d = 0.8, \ r = 0.05, \ E = 52. \)

Risk-neutral probability: \[ p = \frac{e^{0.05} - 0.8}{1.2 - 0.8} = 0.6282 \]
The aim here is to calculate the value of call option $C_0$.

Let us establish a portfolio of stocks and bonds in such a way that the payoff of a call option is completely replicated.

Final value: $\Pi_T = C_T = \max (S - E, 0)$

To prevent risk-free arbitrage opportunity, the current values should be identical. We say that the portfolio replicates the option.

The Law of One Price: $\Pi_t = C_t$. 
Consider replicating portfolio of $\Delta$ shares held long and $N$ bonds held short.

The value of portfolio: $\Pi = \Delta S - NB$. A pair $(\Delta, N)$ is called a trading strategy.

**Task**

How to find $(\Delta, N)$ such that $\Pi_T = C_T$ and $\Pi_0 = C_0$?
Example: One-Step Binomial Model.

Initial stock price is $S_0$. The stock price can either move up from $S_0$ to $S_0u$ or down from $S_0$ to $S_0d$. At time $T$, let the option price be $C_u$ if the stock price moves up, and $C_d$ if the stock price moves down.

- The value of portfolio: $\Pi = \Delta S - NB$.
- When stock moves up: $\Delta S_0u - NB_0e^{rT} = C_u$.
- When stock moves down: $\Delta S_0d - NB_0e^{rT} = C_d$.
- We have two equations for two unknown variables $\Delta$ and $N$.
- Current value: $C_0 = \Delta S_0 - NB_0$.
- Prove: $C_0 = e^{-rT}(pC_u + (1 - p)C_d)$, where $p = \frac{e^{rT} - d}{u - d}$.

(Examples Sheet 5 )
The Black-Scholes model for option pricing was developed by Fischer Black, Myron Scholes in the early 1970’s. This model is the most important result in financial mathematics.
The Black-Scholes model is used to calculate a call price using: stock price, strike price $E$, volatility, time to expiration, and risk free interest rate.

The Black-Scholes model involves several explicit assumptions.

Over the years since the first derivation they have all been relaxed to try and make the model more realistic.
Black - Scholes Assumptions

Assumptions

- One can borrow and lend cash at a constant risk-free interest rate.
- The stock price follows a Geometric Brownian motion with constant expected return and volatility.
- No transaction costs.
- The stock does not pay dividends.
- Securities are perfectly divisible (i.e. one can buy any fraction of a share of stock).
- No restrictions on short selling.
Basic Notation

We denote by $V(S, t)$ the value of an option. We use the notations $C(S, t)$ and $P(S, t)$ for the value of a call and a put when the distinction is important.

Task

To derive the famous Black-Scholes Equation:

$$\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = 0$$

First, we set up a portfolio consisting of a long position in one option and a short position in $\Delta$ shares. The value of the portfolio is $\Pi = V - \Delta S$.

Task

To find the number of shares that makes this portfolio risk free.
The change in the value of this portfolio in the time interval $dt$ is: $d\Pi = dV - \Delta dS$, where $dS = \mu S dt + \sigma S dW$.

Using Itô’s lemma:

$$dV = \left( \frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \mu S \frac{\partial V}{\partial S} \right) dt + \sigma S \frac{\partial V}{\partial S} dW$$

we get

$$d\Pi = \left( \frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \mu S \frac{\partial V}{\partial S} - \Delta \mu S \right) dt + \left( \sigma S \frac{\partial V}{\partial S} - \Delta \sigma S \right) dW$$

**How to eliminate risk?**

Choose $\Delta = \frac{\partial V}{\partial S}$. 
This choice results in a risk-free portfolio $\Pi = V - \frac{\partial V}{\partial S} S$ whose increment is

$$d\Pi = \left( \frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt$$

**No-Arbitrage Principle**

The return on a risk-free portfolio **must** be $r dt$.

so we get

$$\frac{d\Pi}{\Pi} = r dt$$

and

$$\frac{\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}}{V - \frac{\partial V}{\partial S} S} = r$$
Black-Scholes Equation

Which we can write as

$$\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} = r \left( V - \frac{\partial V}{\partial S} S \right)$$

and finally we obtain the Black-Scholes equation

$$\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + r S \frac{\partial V}{\partial S} - rV = 0$$

Scholes received the 1997 Nobel Prize in Economics. It was not awarded to Black in 1997, because he died in 1995. Black received a Ph.D. in applied mathematics from Harvard University.
If a PDE is of backward type, we must impose a final condition at \( t = T \). For a call option, we have 
\[
C(S, t = T) = \max(S - E, 0).
\]

**Figure:** Plot of a European Call Option value against stock price.
1. Boundary Conditions for Call and Put Options

2. Exact Solution to the Black-Scholes Equation
We use $C(S, t)$ and $P(S, t)$ for call and put options. Boundary conditions are applied for zero stock price $S = 0$ and $S \to \infty$.

- Boundary conditions for a call option:

\[
C(0, t) = 0 \quad \text{and} \quad C(S, t) \to S \quad \text{as} \quad S \to \infty
\]

The call option will \textit{never} be exercised if $S = 0$. The call option is \textit{certain} to be exercised as $S \to \infty$.

- Boundary conditions for a put option:

\[
P(0, t) = E e^{-r(T-t)} \quad \text{and} \quad P(S, t) \to 0 \quad \text{as} \quad S \to \infty
\]

If $S_t = 0$ then the put is just the present value of $E$. The put option is \textit{never} exercised as $S \to \infty$. 
The Black-Scholes equation

\[
\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = 0
\]

with appropriate final and boundary conditions has the explicit solution:

\[
C(S, t) = SN(d_1) - Ee^{-r(T-t)}N(d_2),
\]

where

\[
N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{y^2}{2}} dy \quad \text{(cumulative normal distribution)}
\]

and

\[
d_1 = \frac{\ln(S/E) + (r + \sigma^2/2)(T - t)}{\sigma \sqrt{T - t}}, \quad d_2 = d_1 - \sigma \sqrt{T - t}.
\]
Calculate the price of a three-month European call option on a stock with a strike price of £25 when the current stock price is £21.6. The volatility is 35% and the risk-free interest rate is 1% p.a.
Example

In this case $S_0 = 21.6$, $E = 25$, $T = 0.25$, $\sigma = 0.35$ and $r = 0.01$. The value of a call option is $C_0 = S_0 N(d_1) - E e^{-rT} N(d_2)$.

First we compute the values of $d_1$ and $d_2$:

\[
\begin{align*}
    d_1 &= \frac{\ln(S_0/E) + (r + \sigma^2/2)T}{\sigma \sqrt{T}} \\
    &= \frac{\ln(21.6/25) + (0.01 + (0.35)^2/2) \times 0.25}{0.35 \times \sqrt{0.25}} \\
    &\approx -0.7335
\end{align*}
\]

\[
\begin{align*}
    d_2 &= d_1 - \sigma \sqrt{T} = -0.7335 - 0.35 \times \sqrt{0.25} \approx -0.9085
\end{align*}
\]

Since

\[
N(-0.7335) \approx 0.2316, \quad N(-0.9085) \approx 0.1818,
\]

we obtain

\[
C_0 \approx 21.6 \times 0.2316 - 25 \times e^{-0.01 \times 0.25} \times 0.1818
\]

\[
C_0 \approx 0.4689
\]
Let us find the limit
\[ \lim_{\sigma \to \infty} C(S, t). \]

We know that
\[ C(S, t) = SN(d_1) - Ee^{-r(T-t)}N(d_2), \]
and
\[ d_1 = \ln\left(\frac{S}{E}\right) + \frac{(r + \sigma^2/2)(T - t)}{\sigma\sqrt{T - t}} \]
can be written as
\[ d_1 = \frac{\ln(S/E)}{\sigma\sqrt{T - t}} + \frac{r\sqrt{T - t}}{\sigma} + \frac{\sigma\sqrt{T - t}}{2} \]

Then in the limit \( \sigma \to \infty, d_1 \to \infty. \)
Now since $d_2 = d_1 - \sigma \sqrt{T-t}$, in the limit $\sigma \to \infty$, $d_2 \to -\infty$.

Thus $\lim_{\sigma \to \infty} N(d_1) = 1$ and $\lim_{\sigma \to \infty} N(d_2) = 0$.

Therefore

$$\lim_{\sigma \to \infty} C(S, t) = S.$$ 

and this is the upper bound (or maximum value) for the call option!!!
1. Δ-Hedging

2. The Greeks
Let us show that $\Delta = \frac{\partial C}{\partial S} = N(d_1)$.

First, find the derivative $\Delta = \frac{\partial C}{\partial S}$ by using the explicit solution for the European call $C(S, t) = SN(d_1) - Ee^{-r(T-t)}N(d_2)$.

$$\Delta = \frac{\partial C}{\partial S} = N(d_1) + SN'(d_1) \frac{\partial d_1}{\partial S} - Ee^{-r(T-t)}N'(d_2) \frac{\partial d_2}{\partial S}$$

$$= N(d_1) + \left( SN'(d_1) - Ee^{-r(T-t)}N'(d_2) \right) \frac{\partial d_1}{\partial S}$$

We need to prove

$$\left( SN'(d_1) - Ee^{-r(T-t)}N'(d_2) \right) = 0.$$ 

See Examples Sheet 7.
Find the value of $\Delta$ for a 6-month European call option on a stock with a strike price equal to the current stock price ($t = 0$). The interest rate is 6% p.a. and the volatility $\sigma = 0.16$. 


Now for a delta hedge on a call option we have $\Delta = N(d_1)$, where

$$d_1 = \frac{\ln(S/E) + (r + \sigma^2/2)(T - t)}{\sigma \sqrt{T - t}}.$$

We find

$$d_1 = \frac{\ln(1) + (0.06 + (0.16)^2 \times 0.5) \times 0.5}{0.16 \times \sqrt{0.5}} \approx 0.3217$$

and therefore

$$\Delta = N(0.3217) \approx 0.6262$$
Let us find the Delta for a European put option by using the put-call parity:

\[ S + P - C = Ee^{-r(T-t)}. \]

Differentiate it with respect to \( S \) to get

\[ 1 + \frac{\partial P}{\partial S} - \frac{\partial C}{\partial S} = 0. \]

Then rearrange to get

\[ \frac{\partial P}{\partial S} = \frac{\partial C}{\partial S} - 1 = N(d_1) - 1. \]
The option value: $V = V(S, t; \sigma, r, T)$.

Greeks represent the sensitivities of options to a change in an underlying variable or parameter on which the value of an option is dependent.

- **Delta:**
  \[
  \Delta = \frac{\partial V}{\partial S}
  \]
  measures the rate of change of option value with respect to changes in the underlying stock price.

- **Gamma:**
  \[
  \Gamma = \frac{\partial^2 V}{\partial S^2} = \frac{\partial \Delta}{\partial S}
  \]
  measures the rate of change in $\Delta$ with respect to changes in the underlying stock price.

See Examples Sheet 7. \[
\Gamma = \frac{N'(d_1)}{S\sigma\sqrt{T-t}}.
\]
Greeks

- Vega:
  \[ \frac{\partial V}{\partial \sigma} \] measures the sensitivity to volatility \( \sigma \).
  One can show that \[ \frac{\partial V}{\partial \sigma} = S \sqrt{T-t} N'(d_1) \]

- Rho:
  \[ \rho = \frac{\partial V}{\partial r} \] measures the sensitivity to interest rate \( r \).
  One can show that \[ \rho = E(T-t)e^{-r(T-t)} N(d_2) \].

The Greeks are important tools in financial risk management. Each Greek measures the sensitivity of the value of derivatives or a portfolio to a small change in a given underlying parameter.
Lecture 15

1. Black-Scholes Equation and Replicating Portfolio
2. Static and Dynamic Risk-Free Portfolio
The aim is to show that the option price $V(S, t)$ satisfies the Black-Scholes equation

$$\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = 0$$

Consider replicating portfolio of $\Delta$ shares held long and $N$ bonds held short. The value of the portfolio is: $\Pi = \Delta S - N B$. Recall that a pair ($\Delta, N$) is called a trading strategy.

How to find ($\Delta, N$) such that $\Pi_t = V_t$?

- SDE for a stock price $S(t)$: $dS = \mu S dt + \sigma S dW$.

- Equation for a bond price $B(t)$: $dB = rB dt$. 
Replicating Portfolio

By using Itô’s lemma, we find the change in option value

\[ dV = \left( \frac{\partial V}{\partial t} + \mu S \frac{\partial V}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) + \sigma S \frac{\partial V}{\partial S} dW \]

By using self-financing requirement that \( d\Pi = \Delta dS - N dB \), we find the change in portfolio value as

\[ d\Pi = \Delta (\mu S dt + \sigma S dW) - N r B dt = (\Delta \mu S - r NB) dt + \Delta \sigma S dW \]

Equating the last two equations \( d\Pi = dV \), we obtain

\[ \Delta = \frac{\partial V}{\partial S}, \quad - r NB = \frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \]

Since \( NB = \Delta S - \Pi = \left( \frac{\partial V}{\partial S} S - V \right) \), we get the classical Black-Scholes equation

\[ \frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + r S \frac{\partial V}{\partial S} - r V = 0. \]
Let us remember the Put-Call Parity. We set up the portfolio consisting of a long position in one stock, long position in one put and a short position in one call both with the same maturity and strike price. Then the value of the portfolio is

$$\Pi = S + P - C.$$ 

The payoff for this portfolio is

$$\Pi_T = S + \max(E - S, 0) - \max(S - E, 0) = E$$

The payoff is always the same whatever the stock price is at $t = T$. Using the No-Arbitrage Principle, we obtain

$$S_t + P_t - C_t = E e^{-r(T-t)}.$$ 

This is an example of complete risk elimination.

**Definition**

The risk of a portfolio is the variance of the return.
Dynamic Risk-free Portfolio

Put-Call Parity is an example of complete risk elimination when we carry out only one transaction in call/put options and underlying security.

Let us consider the dynamics risk elimination procedure.

We could set up a portfolio consisting of a long position in one call option and a short position in $\Delta$ shares.

The value is $\Pi = C - \Delta S$.

We can eliminate the random component in $\Pi$ by choosing

$$\Delta = \frac{\partial C}{\partial S}.$$ 

This is a $\Delta$-hedging strategy! It required a continuous rebalancing of the number of shares in the portfolio $\Pi$. 
Options on Dividend-Paying Stock

American Put Option
We assume that in a time $dt$ the underlying stock pays out a dividend $D_0 S dt$ where $D_0$ is a constant dividend yield.

Now, we set up a portfolio consisting of a long position in one call option and a short position in $\Delta$ shares.

The value is $\Pi = C - \Delta S$.

The change in value of this portfolio in the time interval $dt$:

$$d\Pi = dC - \Delta dS - \Delta D_0 S dt.$$
Using Itô’s Lemma:

\[
dC = \left( \frac{\partial C}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} \right) dt + \frac{\partial C}{\partial S} dS
\]

we find

\[
d\Pi = \left( \frac{\partial C}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} - \Delta D_0 S \right) dt + \frac{\partial C}{\partial S} dS - \Delta dS
\]

We can eliminate the random component in \(d\Pi\) by choosing \(\Delta = \frac{\partial C}{\partial S}\).
This choice results in a risk-free portfolio \( \Pi = C - S \frac{\partial C}{\partial S} \) whose increment is

\[
d\Pi = \left( \frac{\partial C}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} - D_0 S \frac{\partial C}{\partial S} \right) dt.
\]

No-Arbitrage Principle: the return from this portfolio must be \( rdt \).

\[
\frac{d\Pi}{\Pi} = rdt \quad \text{or} \quad \frac{\partial C}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} - D_0 S \frac{\partial C}{\partial S} = r \left( C - S \frac{\partial C}{\partial S} \right).
\]

Thus we obtain the modified Black-Scholes PDE:

\[
\frac{\partial C}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + (r - D_0) S \frac{\partial C}{\partial S} - rC.
\]
Let us find the solution to the modified Black-Scholes equation in the form

\[ C(S, t) = e^{-D_0(T-t)} C_1(S, t). \]

**Task**

Prove that \( C_1 \) satisfies the Black-Scholes equation with \( r \) replaced by \( r - D_0 \).

If we substitute \( C(S, t) = e^{-D_0(T-t)} C_1(S, t) \) into the modified Black-Scholes equation, we find the equation for \( C_1 \) in the form

\[
\frac{\partial C_1}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C_1}{\partial S^2} + (r - D_0) S \frac{\partial C_1}{\partial S} - (r - D_0) C_1.
\]

The auxiliary function \( C_1(S, t) \) is the value of a European Call option with interest rate \( r - D_0 \).
Examples Sheet 8: show that the modified Black-Scholes equation has the explicit solution for the European call

\[ C(S, t) = S e^{-D_0(T-t)} N(d_{10}) - E e^{-r(T-t)} N(d_{20}), \]

where

\[ d_{10} = \frac{\ln(S/E) + (r - D_0 + \sigma^2/2)(T-t)}{\sigma \sqrt{T-t}} \]

and

\[ d_{20} = d_{10} - \sigma \sqrt{T-t} \]
Recall that an **American Option** is one that may be exercised at any time prior to expiry \( (t = T) \).

- The American put option value must be greater than or equal to the payoff function.

- If \( P < \max(E - S, 0) \), then there is an obvious arbitrage opportunity.

- Valuing a contract such as this can be very difficult and there are **no** explicit analytic solutions for the American put option.
American Put Option

The American put problem can be written as a free boundary problem (commonly found in fluid mechanics).

We divide the price axis $S$ into two distinct regions:

$$0 \leq S \leq S_f(t) \quad \text{and} \quad S_f(t) < S < \infty,$$

where $S_f(t)$ is the exercise boundary. Note that we do not know a priori the value of $S_f(t)$.

When $S > S_f(t)$, early exercise is not optimal and $P(S, t)$ satisfies the Black-Scholes equation.

The boundary conditions at $S = S_f(t)$ are

$$P(S_f(t), t) = \max(E - S_f(t), 0), \quad \frac{\partial P}{\partial S}(S_f(t), t) = -1.$$
1. Bond Pricing with Known Interest Rates and Coupon Payments

2. Zero-Coupon Bond Pricing
**Bond Pricing Equation**

**Definition**
A bond is a contract that yields a known amount (nominal, principal or face value) on the maturity date, $t = T$. The bond may pay a coupon (interest payment) at fixed times.

If there is no coupon payment, the bond is known as a **zero-coupon bond**.

Let us introduce the following notation:

$V(t)$ is the value of the bond, $r(t)$ is the interest rate, and $K(t)$ is the coupon payment rate.

Equation for the bond price

$$\frac{dV}{dt} = r(t)V - K(t).$$

The final condition is: $V(T) = F$. 

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Let us consider the case when the coupon payment rate is $K(t) = 0$.

The solution of the equation $\frac{dV}{dt} = r(t)V$ with $V(T) = F$ can be written as

$$V(t) = F \exp \left( - \int_{t}^{T} r(s) ds \right).$$

Let us show this by integrating $\frac{dV}{V} = r(t)dt$ from $t$ to $T$.

$$\ln V(T) - \ln V(t) = \int_{t}^{T} r(s) ds$$

or

$$\ln \left( \frac{V(t)}{F} \right) = - \int_{t}^{T} r(s) ds$$
A zero-coupon bond, \( V \), issued at time \( t = 0 \), is worth \( V(t = 1) = 1 \) at maturity \( T = 1 \). Find the bond price \( V(t) \) at time \( t < 1 \) and \( V(0) \), when the continuous interest rate is

\[ r(t) = t^2. \]
Solution: Since maturity is $T = 1$, one can find

$$\int_t^1 r(s)ds = \int_t^1 s^2 ds = \frac{1}{3} - \frac{1}{3}t^3$$

Therefore

$$V(t) = \exp \left( - \int_t^1 r(s)ds \right) = \exp \left( \frac{t^3 - 1}{3} \right),$$

and

$$V(0) = \exp \left( - \frac{1}{3} \right) = 0.7165$$
Let us consider the case when the continuous coupon payment rate $K(t) > 0$. The solution of the equation $\frac{dV}{dt} = r(t)V - K(t)$ can be written as

$$V(t) = F \exp \left( - \int_t^T r(s)ds \right) + V_1(t),$$

where

$$V_1(t) = C(t) \exp \left( - \int_t^T r(s)ds \right)$$

It can be shown that this gives the explicit solution

$$V(t) = \exp \left( - \int_t^T r(s)ds \right) \left[ F + \int_t^T K(y) \exp \left( \int_y^T r(s)ds \right) dy \right],$$

See See Examples Sheet 9 for details.
Lecture 18

1. Measure of Future Values of Interest Rate
2. Term Structure of Interest Rates (Yield Curve)
Recall that the solution of the zero-coupon bond is

\[ V(t) = F \exp \left( - \int_t^T r(s) \, ds \right). \]

Now let us introduce the notation \( V(t, T) \) for bond prices. Bond prices are usually quoted at time \( t \) for different value of \( T \).

Let us differentiate \( V(t, T) \) with respect to \( T \):

\[
\frac{\partial V(t, T)}{\partial T} = F \exp \left( - \int_t^T r(s) \, ds \right) (-r(T)) = -V(t, T)r(T),
\]

therefore

\[
r(T) = -\frac{1}{V(t, T)} \frac{\partial V(t, T)}{\partial T},
\]

This is the interest rate at the future date \( T \) (forward rate).
We define

\[ Y(t, T) = - \frac{\ln(V(t, T)) - \ln(V(T, T))}{T - t}, \]

as a measure of the future values of interest rate, where \( V(t, T) \) is taken from financial data.

Then we can write

\[ Y(t, T) = - \ln \left( F \exp \left( - \int_{t}^{T} r(s) ds \right) \right) - \ln F \]

so that

\[ Y(t, T) = \frac{1}{T - t} \int_{t}^{T} r(s) ds \]
We can say that $Y(t, T)$ is the average value of the interest rate $r(t)$ in the time interval $[t, T]$. Therefore the bond price can be written as

$$V(t, T) = Fe^{-Y(t,T)(T-t)}$$

We define the term structure of interest rates (yield curve):

$$Y(0, T) = -\frac{\ln(V(0, T)) - \ln(V(T, T))}{T} = \frac{1}{T} \int_0^T r(s)ds$$

as the average value of interest rate in the future.
Assume that the instantaneous interest rate $r(t)$ is

$$r(t) = r_0 + at,$$

where $r_0$ and $a$ are positive constants.

Bond Price:

$$V(t, T) = Fe^{-\int_t^T r(s)ds} = Fe^{-\int_t^T (r_0+as)ds}.$$

$$V(t, T) = Fe^{-r_0(T-t) - \frac{a}{2}(T^2-t^2)}.$$

Term structure of interest rate:

$$Y(0, T) = \frac{1}{T} \int_0^T r(s)ds = r_0 + \frac{aT}{2}.$$
There exists a risk of a default bond, $V(t, T)$, when the principal is not paid to lender as promised by the borrower. How can we take this into account?

Consider a 1 year bond, $V(0, 1)$, that has probability $p$ of defaulting on repayments.

\[
\begin{align*}
\text{Bond Tree:} \\
V(0, 1) & \quad 0 \\
& \quad \downarrow \\
& \quad p \\
& \quad 1 - p \\
& \quad F
\end{align*}
\]

\[
\begin{align*}
\text{Price:} \\
V(0, 1) &= e^{-r}(F(1 - p) + 0.p) \\
\text{and therefore the yield is} \\
Y(0, 1) &= -\ln(e^{-r} F(1 - p)) + \ln F \\
Y(0, 1) &= r - \ln(1 - p)
\end{align*}
\]
Risk of Default

In this case the bond has a yield of the form

\[ Y(0, 1) = r + s \]

and the positive parameter \( s \) is called the yield spread w.r.t risk-free interest rate \( r \).

Let us find it:

\[ s = -\ln(1 - p) = p + O(p^2) \approx p \]

which means that the spread is approximately the probability of default in that year.

In fact, if we model default as a Poisson process with intensity \( \lambda(t) \) we find the yield spread is

\[ s(T) = \frac{1}{T} \int_0^T \lambda(s)ds \]
1. Asian Options

2. Derivation of a PDE for Asian Options
Asian Options

Definition

An *Asian Option* is a contract giving the holder the right to buy/sell an underlying asset for its average price over some prescribed period.

The floating strike Asian put option has the final condition:

\[ V(S, T) = \max \left( S - \frac{1}{T} \int_0^T S(t) \, dt, 0 \right). \]

We introduce a new variable:

\[ I(t) = \int_0^t S(t) \, dt \quad \text{or} \quad \frac{dI}{dt} = S(t). \]

The final condition can now be written as

\[ V(S, I, T) = \max \left( S - \frac{I}{T}, 0 \right). \]
Using Itô’s lemma:

\[ dV = \left( \frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt + \frac{\partial V}{\partial S} dS + \frac{\partial V}{\partial I} dI \]

and setting up a hedging portfolio we find

\[ d\Pi = \left( \frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \right) dt + \frac{\partial V}{\partial S} dS - \Delta dS + \frac{\partial V}{\partial I} dI \]

We can eliminate the random component in \( d\Pi \) by choosing \( \Delta = \frac{\partial V}{\partial S} \).
By using No-Arbitrage Principle, and the equation

\[ dI = Sdt \]

we can obtain the modified Black-Scholes PDE for the Asian option price:

\[
\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV + \frac{S \partial V}{\partial I} = 0
\]

The value of an Asian option must be calculated numerically (no analytic solution!).