Lecture 3

1. Distribution for $\ln S(t)$

2. Solution to Stochastic Differential Equation for Stock Price

3. Examples
Example 1. Find the stochastic differential equation (SDE) for

\[ f = \ln S \]

by using Itô’s Lemma:

\[
df = \left( \frac{\partial f}{\partial t} + \mu S \frac{\partial f}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} \right) dt + \sigma S \frac{\partial f}{\partial S} dW.\]
Differential of $\ln S$

We obtain

$$df = \left( \mu - \frac{\sigma^2}{2} \right) dt + \sigma dW$$

and this is a constant coefficient SDE.

Therefore integration from 0 to $t$ gives

$$f - f_0 = \left( \mu - \frac{\sigma^2}{2} \right) t + \sigma W(t) \quad \text{since} \quad W(0) = 0.$$
Now since $f = \ln S$ we can write

$$\ln S(t) - \ln S_0 = \left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W(t)$$

where $S_0 = S(0)$ is the initial stock price.

This means $\ln S(t)$ has a normal distribution with mean $\ln S_0 + \left(\mu - \frac{\sigma^2}{2}\right)t$ and variance $\sigma^2 t$. 
Example 2. Consider a stock with an initial price of 40, an expected return of 16% and a volatility of 20%. Find the probability distribution of \( \ln S \) in six months.

We have

\[
\ln S(T) \sim N \left( \ln S_0 + \left( \mu - \frac{\sigma^2}{2} \right) T, \sigma^2 T \right)
\]

Answer: \( \ln S(0.5) \sim N(3.759, 0.020) \)
Recall that if the random variable $X$ has a normal distribution with mean $\mu$ and variance $\sigma^2$, then the probability density function is

$$p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x - \mu)^2}{2\sigma^2}\right)$$

The probability density function of $X = \ln S(t)$ is

$$\frac{1}{\sqrt{2\pi\sigma^2t}} \exp\left(-\frac{(x - \ln S_0 - (\mu - \sigma^2/2)t)^2}{2\sigma^2t}\right)$$
Definition. The model of a stock \( dS = \mu S dt + \sigma S dW \) is known as a geometric Brownian motion.

The random function \( S(t) \) can be found from

\[
\ln\left(\frac{S(t)}{S_0}\right) = \left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W(t)
\]

Stock price at time \( t \):

\[
S(t) = S_0 e^{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W(t)}
\]

Or

\[
S(t) = S_0 e^{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma \sqrt{t}X} \quad \text{where} \quad X \sim N(0, 1)
\]
Below we plot the lognormal distribution function for $\mu = 0$, $\sigma = 0.4$ and $t = 1$. 

$$S(t) = \frac{S_t}{S_0} e^{\mu t} 2e^{\mu t} 3e^{\mu t}$$

$$\ln \mathcal{N} \left( \left( \mu - \frac{\sigma^2}{2} \right) t, \sigma^2 t \right)$$