Lecture 2: Properties of SDEs

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Lecture 2

1. Properties of Wiener Process
2. Approximation for Stock Price Equation
3. Itô’s Lemma
The probability density function for $W(t)$ is

$$p(y, t) = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{y^2}{2t}\right)$$

and $\mathbb{P}(a \leq W(t) \leq b) = \int_a^b p(y, t) dt$

- Simulations of a Wiener process:
The increment $\Delta W = W(t + \Delta t) - W(t)$ can be written as $\Delta W = X (\Delta t)^{\frac{1}{2}}$, where $X$ is a random variable with normal distribution with zero mean and unit variance: $X \sim N(0,1)$

- $\mathbb{E}\Delta W = 0$ and $\mathbb{E}(\Delta W)^2 = \Delta t$.

Recall: equation for the stock price is

$$dS = \mu S dt + \sigma S dW,$$

then

$$\Delta S \approx \mu S \Delta t + \sigma S X (\Delta t)^{\frac{1}{2}}$$

It means $\Delta S \sim N (\mu S \Delta t, \sigma^2 S^2 \Delta t)$
Example 1. Consider a stock that has volatility 30% and provides expected return of 15% p.a. Find the increase in stock price for one week if the initial stock price is 100.

Answer: \( \Delta S = 0.288 + 4.16X \)

Note: 4.16 is the standard deviation per week
Example 2. Show that the return $\frac{\Delta S}{S}$ is normally distributed with mean $\mu \Delta t$ and variance $\sigma^2 \Delta t$.
We assume that $f(S, t)$ is a smooth function of $S$ and $t$.

Find $df$ if $dS = \mu S dt + \sigma S dW$

- Volatility $\sigma = 0$

$$df = \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial S} dS = \left( \frac{\partial f}{\partial t} + \mu S \frac{\partial f}{\partial S} \right) dt$$

- Volatility $\sigma \neq 0$

Itô’s Lemma:

$$df = \left( \frac{\partial f}{\partial t} + \mu S \frac{\partial f}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} \right) dt + \sigma S \frac{\partial f}{\partial S} dW$$
Example 3. Find the SDE satisfied by $f = S^2$. 