Lecture 18

1. Measure of Future Values of Interest Rate

2. Term Structure of Interest Rates (Yield Curve)
Recall that the solution of the zero-coupon bond is

\[ V(t) = F \exp \left( - \int_t^T r(s) ds \right). \]

Now let us introduce the notation \( V(t, T) \) for bond prices. Bond prices are usually quoted at time \( t \) for different values of \( T \).

Let us differentiate \( V(t, T) \) with respect to \( T \):

\[
\frac{\partial V(t, T)}{\partial T} = F \exp \left( - \int_t^T r(s) ds \right) (-r(T)) = -V(t, T)r(T),
\]

therefore

\[
r(T) = -\frac{1}{V(t, T)} \frac{\partial V(t, T)}{\partial T},
\]

This is the interest rate at the future date \( T \) (forward rate).
Term Structure of Interest Rates (Yield Curve)

We define

\[ Y(t, T) = -\frac{\ln(V(t, T)) - \ln(V(T, T))}{T - t}, \]

as a measure of the future values of interest rate, where \( V(t, T) \) is taken from financial data.

Then we can write

\[ Y(t, T) = -\frac{\ln \left( F \exp \left( -\int_t^T r(s)ds \right) \right) - \ln F}{T - t} \]

so that

\[ Y(t, T) = \frac{1}{T - t} \int_t^T r(s)ds \]
We can say that \( Y(t, T) \) is the average value of the interest rate \( r(t) \) in the time interval \([t, T]\). Therefore the bond price can be written as

\[
V(t, T) = F e^{-Y(t,T)(T-t)}
\]

We define the term structure of interest rates (yield curve):

\[
Y(0, T) = -\frac{\ln(V(0, T)) - \ln(V(T, T))}{T} = \frac{1}{T} \int_0^T r(s) ds
\]

as the average value of interest rate in the future.
Assume that the instantaneous interest rate $r(t)$ is

$$r(t) = r_0 + at,$$

where $r_0$ and $a$ are positive constants.

Bond Price:

$$V(t, T) = Fe^{-\int_t^T r(s)ds} = Fe^{-\int_t^T (r_0+as)ds}.$$

$$V(t, T) = Fe^{-r_0(T-t)-\frac{a}{2}(T^2-t^2)}.$$

Term structure of interest rate:

$$Y(0, T) = \frac{1}{T} \int_0^T r(s)ds = r_0 + \frac{aT}{2}.$$
Risk of Default

There exists a risk of a default bond, \( V(t, T) \), when the principal is not paid to lender as promised by the borrower.

How can we take this into account?

Consider a 1 year bond, \( V(0, 1) \), that has probability \( p \) of defaulting on repayments.

Bond Tree:

\[ V(0, 1) \]

\[ p \]

\[ 0 \]

\[ 1 - p \]

\[ F \]

Price:

\[ V(0, 1) = e^{-r}(F(1 - p) + 0.p) \]

and therefore the yield is

\[ Y(0, 1) = -\ln(e^{-r} F(1 - p)) + \ln F \]

\[ Y(0, 1) = r - \ln(1 - p) \]
Risk of Default

In this case the bond has a yield of the form

\[ Y(0, 1) = r + s \]

and the positive parameter \( s \) is called the yield spread w.r.t risk-free interest rate \( r \).

Let us find it:

\[
  s = -\ln(1 - p) = p + O(p^2) \approx p
\]

which means that the spread is approximately the probability of default in that year.

In fact, if we model default as a Poisson process with intensity \( \lambda(t) \) we find the yield spread is

\[
  s(T) = \frac{1}{T} \int_0^T \lambda(s) ds
\]