1. Bond Pricing with Known Interest Rates and Coupon Payments

2. Zero-Coupon Bond Pricing
**Bond Pricing Equation**

**Definition**

A bond is a contract that yields a known amount (nominal, principal or face value) on the maturity date, \( t = T \). The bond may pay a coupon (interest payment) at fixed times.

If there is no coupon payment, the bond is known as a zero-coupon bond.

Let us introduce the following notation:

- \( V(t) \) is the value of the bond,
- \( r(t) \) is the interest rate,
- \( K(t) \) is the coupon payment rate.

Equation for the bond price

\[
\frac{dV}{dt} = r(t)V - K(t).
\]

The final condition is: \( V(T) = F \).
Let us consider the case when the coupon payment rate is $K(t) = 0$.

The solution of the equation $\frac{dV}{dt} = r(t)V$ with $V(T) = F$ can be written as

$$V(t) = F \exp \left( - \int_t^T r(s) \, ds \right).$$

Let us show this by integrating $\frac{dV}{V} = r(t) \, dt$ from $t$ to $T$.

$$\ln V(T) - \ln V(t) = \int_t^T r(s) \, ds$$

or

$$\ln \left( \frac{V(t)}{F} \right) = - \int_t^T r(s) \, ds$$
Example

A zero-coupon bond, $V$, issued at time $t = 0$, is worth $V(t = 1) = 1$ at maturity $T = 1$. Find the bond price $V(t)$ at time $t < 1$ and $V(0)$, when the continuous interest rate is $r(t) = t^2$. 
Example

Solution: Since maturity is $T = 1$, one can find

$$\int_{t}^{1} r(s) ds = \int_{t}^{1} s^2 ds = \frac{1}{3} - \frac{1}{3} t^3$$

Therefore

$$V(t) = \exp \left( - \int_{t}^{1} r(s) ds \right) = \exp \left( \frac{t^3 - 1}{3} \right),$$

and

$$V(0) = \exp \left( - \frac{1}{3} \right) = 0.7165$$
Let us consider the case when the continuous coupon payment rate $K(t) > 0$. The solution of the equation $\frac{dV}{dt} = r(t)V - K(t)$ can be written as

$$V(t) = F \exp \left( - \int_t^T r(s) ds \right) + V_1(t),$$

where

$$V_1(t) = C(t) \exp \left( - \int_t^T r(s) ds \right)$$

It can be shown that this gives the explicit solution

$$V(t) = \exp \left( - \int_t^T r(s) ds \right) \left[ F + \int_t^T K(y) \exp \left( - \int_y^T r(s) ds \right) dy \right].$$

See See Examples Sheet 9 for details.