1 Options on Dividend-Paying Stock

2 American Put Option
We assume that in a time \( dt \) the underlying stock pays out a dividend \( D_0 S dt \) where \( D_0 \) is a constant dividend yield.

Now, we set up a portfolio consisting of a long position in one call option and a short position in \( \Delta \) shares.

The value is \( \Pi = C - \Delta S \).

The change in value of this portfolio in the time interval \( dt \):

\[
d\Pi = dC - \Delta dS - \Delta D_0 S dt.
\]
Using Itô’s Lemma:

\[ dC = \left( \frac{\partial C}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} \right) dt + \frac{\partial C}{\partial S} dS \]

we find

\[ d\Pi = \left( \frac{\partial C}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} - \Delta D_0 S \right) dt + \frac{\partial C}{\partial S} dS - \Delta dS \]

We can eliminate the random component in \( d\Pi \) by choosing \( \Delta = \frac{\partial C}{\partial S} \).
This choice results in a risk-free portfolio \( \Pi = C - S \frac{\partial C}{\partial S} \)
whose increment is

\[
d\Pi = \left( \frac{\partial C}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} - D_0 S \frac{\partial C}{\partial S} \right) dt.
\]

No-Arbitrage Principle: the return from this portfolio must be \( r dt \).

\[
\frac{d\Pi}{\Pi} = r dt \quad \text{or} \quad \frac{\partial C}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} - D_0 S \frac{\partial C}{\partial S} = r \left( C - S \frac{\partial C}{\partial S} \right).
\]

Thus we obtain the modified Black-Scholes PDE:

\[
\frac{\partial C}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + (r - D_0) S \frac{\partial C}{\partial S} - rC.
\]
Let us find the solution to the modified Black-Scholes equation in the form

\[ C(S, t) = e^{-D_0(T-t)}C_1(S, t). \]

**Task**

Prove that \( C_1 \) satisfies the Black-Scholes equation with \( r \) replaced by \( r - D_0 \).

If we substitute \( C(S, t) = e^{-D_0(T-t)}C_1(S, t) \) into the modified Black-Scholes equation, we find the equation for \( C_1 \) in the form

\[ \frac{\partial C_1}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C_1}{\partial S^2} + (r - D_0)S \frac{\partial C_1}{\partial S} - (r - D_0)C_1. \]

The auxiliary function \( C_1(S, t) \) is the value of a European Call option with interest rate \( r - D_0 \).
Examples Sheet 8: show that the modified Black-Scholes equation has the explicit solution for the European call

\[ C(S, t) = S e^{-D_0(T-t)} N(d_{10}) - E e^{-r(T-t)} N(d_{20}), \]

where

\[ d_{10} = \frac{\ln(S/E) + (r - D_0 + \sigma^2/2)(T-t)}{\sigma \sqrt{T-t}} \]

and

\[ d_{20} = d_{10} - \sigma \sqrt{T-t} \]
Recall that an American Option is one that may be exercised at any time prior to expiry ($t = T$).

- The American put option value must be greater than or equal to the payoff function.

- If $P < \max(E - S, 0)$, then there is an obvious arbitrage opportunity.

- Valuing a contract such as this can be very difficult and there are no explicit analytic solutions for the American put option.
The American put problem can be written as a free boundary problem (commonly found in fluid mechanics).

We divide the price axis $S$ into two distinct regions:

$$0 \leq S \leq S_f(t) \quad \text{and} \quad S_f(t) < S < \infty,$$

where $S_f(t)$ is the exercise boundary. Note that we do not know \textit{a priori} the value of $S_f(t)$.

When $S > S_f(t)$, early exercise is not optimal and $P(S, t)$ satisfies the Black-Scholes equation.

The boundary conditions at $S = S_f(t)$ are

$$P(S_f(t), t) = \max(E - S_f(t), 0), \quad \frac{\partial P}{\partial S}(S_f(t), t) = -1.$$