Boundary Conditions for Call and Put Options

Exact Solution to the Black-Scholes Equation
We use $C(S, t)$ and $P(S, t)$ for call and put options. Boundary conditions are applied for zero stock price $S = 0$ and $S \to \infty$.

- Boundary conditions for a call option:

  $$C(0, t) = 0 \quad \text{and} \quad C(S, t) \to S \quad \text{as} \quad S \to \infty$$

  The call option will never be exercised if $S = 0$. The call option is certain to be exercised as $S \to \infty$.

- Boundary conditions for a put option:

  $$P(0, t) = E e^{-r(T-t)} \quad \text{and} \quad P(S, t) \to 0 \quad \text{as} \quad S \to \infty$$

  If $S_t = 0$ then the put is just the present value of $E$. The put option is never exercised as $S \to \infty$. 
The Black-Scholes equation

\[
\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = 0
\]

with appropriate final and boundary conditions has the explicit solution:

\[
C(S, t) = SN(d_1) - Ee^{-r(T-t)}N(d_2),
\]

where

\[
N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{y^2}{2}} dy \quad \text{(cumulative normal distribution)}
\]

and

\[
d_1 = \frac{\ln(S/E) + (r + \sigma^2/2)(T - t)}{\sigma \sqrt{T - t}}, \quad d_2 = d_1 - \sigma \sqrt{T - t}.
\]
Call Price Premium

Maximum Value
Actual Value
Speculative Value
Intrinsic Value

$E$

$S_T$
Example

Calculate the price of a three-month European call option on a stock with a strike price of £25 when the current stock price is £21.6. The volatility is 35% and the risk-free interest rate is 1% p.a.
In this case $S_0 = 21.6$, $E = 25$, $T = 0.25$, $\sigma = 0.35$ and $r = 0.01$. The value of a call option is $C_0 = S_0 N(d_1) - E e^{-rT} N(d_2)$.

First we compute the values of $d_1$ and $d_2$:

\[
d_1 = \frac{\ln\left(\frac{S_0}{E}\right) + (r + \sigma^2/2)T}{\sigma \sqrt{T}} = \frac{\ln(21.6/25) + (0.01 + (0.35)^2/2) \times 0.25}{0.35 \times \sqrt{0.25}} \approx -0.7335
\]

\[
d_2 = d_1 - \sigma \sqrt{T} = -0.7335 - 0.35 \times \sqrt{0.25} \approx -0.9085
\]

Since

\[
N(-0.7335) \approx 0.2316, \quad N(-0.9085) \approx 0.1818,
\]

we obtain

\[
C_0 \approx 21.6 \times 0.2316 - 25 \times e^{-0.01 \times 0.25} \times 0.1818 \approx 0.4689
\]
The Limit of Higher Volatility

Let us find the limit

$$\lim_{\sigma \to \infty} C(S, t).$$

We know that

$$C(S, t) = SN(d_1) - Ee^{-r(T-t)}N(d_2),$$

and

$$d_1 = \frac{\ln(S/E) + (r + \sigma^2/2)(T - t)}{\sigma\sqrt{T - t}}$$

can be written as

$$d_1 = \frac{\ln(S/E)}{\sigma\sqrt{T - t}} + \frac{r\sqrt{T - t}}{\sigma} + \frac{\sigma\sqrt{T - t}}{2}$$

Then in the limit $\sigma \to \infty$, $d_1 \to \infty$. 

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Now since \( d_2 = d_1 - \sigma \sqrt{T-t} \), in the limit \( \sigma \to \infty \), \( d_2 \to -\infty \).

Thus \( \lim_{\sigma \to \infty} N(d_1) = 1 \) and \( \lim_{\sigma \to \infty} N(d_2) = 0 \).

Therefore

\[
\lim_{\sigma \to \infty} C(S, t) = S.
\]

and this is the upper bound (or maximum value) for the call option!!!