1. American Put Option Pricing on Binomial Tree
2. Replicating Portfolio
An American Option is one that may be exercised at any time prior to expire \((t = T)\).

We should determine when it is best to exercise the option.

It is not subjective! It can be determined in a systematic way!

The American put option value must be greater than or equal to the payoff function.

If \(P < \max(E - S, 0)\), then there is obvious arbitrage opportunity.

We can buy stock for \(S\) and option for \(P\) and immediately exercise the option by selling stock for \(E\).

\[E - (P + S) > 0\]
We denote by $P^m_n$ the $n$-th possible value of put option at time-step $m\Delta t$.

- **European Put Option:**
  \[ P^m_n = e^{-r\Delta t} \left( pP_{n+1}^{m+1} + (1 - p)P_n^{m+1} \right). \]
  Here $0 \leq n \leq m$ and the risk-neutral probability $p = \frac{e^{r\Delta t} - d}{u - d}$.

- **American Put Option:**
  \[ P^m_n = \max \left\{ \max (E - S^m_n, 0), e^{-r\Delta t} \left( pP_{n+1}^{m+1} + (1 - p)P_n^{m+1} \right) \right\}, \]
  where $S^m_n$ is the $n$-th possible value of stock price at time-step $m\Delta t$.

- **Final condition:** $P^N_n = \max (E - S^N_n, 0)$, where $n = 0, 1, 2, ..., N$, $E$ is the strike price.
Example

Evaluation of American Put Option on Two-Step Tree:

- We assume that over each of the next two years the stock price either moves up by 20% or moves down by 20%. The risk-free interest rate is 5%.

- Find the value of a 2-year American put with a strike price of $52 on a stock whose current price is $50.

In this case $u = 1.2$, $d = 0.8$, $r = 0.05$, $E = 52$.

Risk-neutral probability: 

$$p = \frac{e^{0.05} - 0.8}{1.2 - 0.8} = 0.6282$$
Replicating Portfolio

The aim here is to calculate the value of call option $C_0$.

Let us establish a portfolio of stocks and bonds in such a way that the payoff of a call option is completely replicated.

Final value: $\Pi_T = C_T = \max (S - E, 0)$

To prevent risk-free arbitrage opportunity, the current values should be identical. We say that the portfolio replicates the option.

The Law of One Price: $\Pi_t = C_t$. 
Consider replicating portfolio of $\Delta$ shares held long and $N$ bonds held short. 
The value of portfolio: $\Pi = \Delta S - NB$. A pair $(\Delta, N)$ is called a trading strategy.

**Task**

How to find $(\Delta, N)$ such that $\Pi_T = C_T$ and $\Pi_0 = C_0$?
Example: One-Step Binomial Model.

Initial stock price is $S_0$. The stock price can either move up from $S_0$ to $S_0u$ or down from $S_0$ to $S_0d$. At time $T$, let the option price be $C_u$ if the stock price moves up, and $C_d$ if the stock price moves down.

- The value of portfolio: $\Pi = \Delta S - NB$.
- When stock moves up: $\Delta S_0u - NB_0e^{rT} = C_u$.
- When stock moves down: $\Delta S_0d - NB_0e^{rT} = C_d$.
- We have two equations for two unknown variables $\Delta$ and $N$.
- Current value: $C_0 = \Delta S_0 - NB_0$.
- Prove: $C_0 = e^{-rT}(pC_u + (1 - p)C_d)$, where $p = \frac{e^{rT} - d}{u - d}$.

(Examples Sheet 5)