Euler's Method and Stability

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LAST WEEK...

Topics:
- Errors and precision
- Flow control - if, else, for, do
- Functions - how and where to use them
- Pointers - dynamic allocation, pass by reference

Aims - week 2:
- Understand precision and errors within a program.
- Use functions, loops and if statement to control a program.
- Understand the concept of a pointer.
Topics:
- Initial value problems for ODEs
- Euler’s Method for ODEs
- Stability
- Using the standard libraries

Aims - week 3:
- Be able to solve an initial value ODE problem
- Understand where truncation errors come from
- Be aware of the concept of stability in a numerical method
- Be able to use standard libraries to store your data
Here we will look at the solution of ordinary differential equations of the type, say

\[ \frac{dy}{dx} = f(x, y), \quad a \leq x \leq b \]
The Initial Value Problem

Here we will look at the solution of ordinary differential equations of the type, say

\[ \frac{dy}{dx} = f(x, y), \quad a \leq x \leq b \]

subject to an initial condition

\[ y(a) = \alpha \]
**Examples**

Consider the problem:

\[
\frac{dy}{dx} = y \left(1 - \frac{y}{4}\right), \quad x \geq 0
\]

with the initial condition

\[y(0) = 1\]
**EXAMPLES**

**Consider the problem:**

\[ \frac{dy}{dx} = y \left(1 - \frac{y}{4}\right), \quad x \geq 0 \]

with the initial condition

\[ y(0) = 1 \]

- How can we go about solving it?
The simplest method to solve an ODE is the Euler method.

In order to solve, we must discretise the problem – make a continuous (infinite) problem discrete (finite).

Divide up the interval \([a, b]\) into \(n\) equally spaced intervals.

\[ a \quad \underbrace{\; x_0 \quad x_1 \quad x_2 \quad \ldots \quad x_i \quad \ldots \quad x_n \;}_{h} \quad b \]
Suppose $y(x)$ is the unique solution to the ODE, and is twice differentiable.

Apply a Taylor series approximation around $x_i$ then we have

$$y(x_{i+1}) = y(x_i) + y'(x_i)h + \frac{1}{2}y''(\xi)$$

where $x_i \leq \xi \leq x_{i+1}$.
SOLVING AN INITIAL VALUE PROBLEM

- Suppose $y(x)$ is the unique solution to the ODE, and is twice differentiable
- Apply a Taylor series approximation around $x_i$ then we have

\[ y(x_{i+1}) = y(x_i) + f(x_i, y(x_i))h + \frac{1}{2}y''(\xi) \]

where $x_i \leq \xi \leq x_{i+1}$.

- Since $y$ is a solution to the ODE, we can replace $y'$ by the function $f(x, y)$
The Euler Method

- Assume $w_i$ is our approximation to $y$ at $x_i$, then

  $$W_0 = \alpha$$

- Then to find all subsequent values of $w$, set the remainder to zero in the previous equation to obtain:

  $$w_{i+1} = w_i + hf(x_i, w_i), \text{ for each } i = 0, 1, \ldots, n - 1$$
The Euler Method

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- Try solving the example problem...
We would like to be able to compare the errors for different methods.

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We can use the local truncation error – difference between equation and the approximation.

For the Euler method we have:

\[
\tau_{i+1}(h) = \frac{y_{i+1} - (y_i + hf(x_i, y_i))}{h} = \frac{y_{i+1} - y_i}{h} - f(x_i, y_i)
\]
We can calculate the truncation error as

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and if $y''$ is bounded by the constant $M$ on the interval $[a, b]$ then

$$|\tau_{i+1}(h)| \leq \frac{h}{2}M.$$
We can calculate the truncation error as

\[ \tau_{i+1}(h) = \frac{h}{2} y''(\xi_i), \]

and if \( y'' \) is bounded by the constant \( M \) on the interval \([a, b]\) then

\[ |\tau_{i+1}(h)| \leq \frac{h}{2} M. \]

Hence the truncation error is \( O(h) \).

A method with truncation error \( O(h^p) \) is called an order \( p \) method.
AN EXAMPLE PROBLEM:

- Suppose the initial-value problem
  \[ y' = -50y, \quad 0 \leq x \leq 1, \quad y(0) = 1 \]
  is approximated by the Euler method so that
  \[ y_{i+1} = y_i + hf(x_i, y_i). \]
- The exact solution to the problem is
  \[ y(x) = e^{-50x} \]
- This is a stiff problem.
What is stability?

Defining stability

The exact solution

\[ y(x) = e^{-50x} \]
Now we generate the numerical solution with $h = 0.05$, we have that

- $y_0 = 1$
- $y_1 = 1 + (0.05) \cdot (-50 \cdot 1) = -1.5$
- $y_2 = -1.5 + (0.05) \cdot (-50 \cdot -1.5) = 2.25$

and so on...

Here errors are growing, the method is unstable.
Now we take \( h = 0.04 \), then

- \( y_0 = 1 \)
- \( y_1 = 1 + (0.04) \cdot (-50 \cdot 1) = -1 \)
- \( y_2 = -1 + (0.04) \cdot (-50 \cdot -1) = 1 \)

and so on...

Here the solution oscillates, and errors grow slowly. The method is unstable.
Next take \( h = 1/30 \), then
\[
y_0 = 1
\]
\[
y_1 = 1 + (1/30) \cdot (−50 \cdot 1) = −2/3
\]
\[
y_2 = −2/3 + (1/30) \cdot (−50 \cdot −2/3) = 4/9
\]
and so on...
Here the solution oscillates, but errors reduce slowly. The method is in some sense stable.
What is stability?

Defining stability

Unstable Solutions

\[ y(x) = e^{-50x} \]

-4
-2
0
2
4
0 0.2 0.4 0.6 0.8 1

x

y(x)
A stable solution is one in which errors or perturbations are damped down.

Think of the first case with $h = 0.05$:

- The error from calculating $y_1$ is amplified at $y_2$.
- Given the exact value for $y_1$, the error at $y_2$ would be much smaller.

If we set $h$ small enough the errors are bounded and the method gives satisfactory results.
What is stability?

Defining stability

Some Stable Solutions

\[ y(x) = e^{-50x} \]

- For \( n = 50 \)
- For \( n = 100 \)
**CONSISTENCY**

**Definition:**
A one-step difference method with truncation error $\tau_i(h)$ at the $i$th step is said to be consistent with the difference equation it approximates if

$$\lim_{h \to 0} \max_i |\tau_i(h)| = 0.$$ 

- This is a local definition.
- We compare the exact value to the difference approximation given that we know the exact value at $y_{i-1}$.
A one-step difference method is said to be convergent with respect to the difference equation it approximates if

$$\lim_{h \to 0} \max_i |w_i - y(x_i)| = 0,$$

where $w_i$ is the approximate value at $x_i$ to the exact value $y_i = y(x_i)$.

- This is a global definition.
- Given that only $y_0$ is exact, the errors at the $i$th position tend to zero as we reduce the step size.
**Stability**

**Definition:**
A stable method is one whose results depend continuously on the initial data.

- Consider a one step method in the form

  \[ w_{i+1} = w_i + h\phi(x_i, y_i, h), \]

  such that \( \phi \) is continuous and satisfies a Lipschitz condition. Then the method is stable as defined above.

- If the method is consistent, then it is convergent if and only if it is stable.
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