Example: The Trapezium Rule for Integration

Use the trapezium rule to evaluate the integral:

\[ I = \int_{0}^{\infty} \frac{1}{\sqrt{2\pi(1+x)}} e^{-\frac{x^2}{2\pi}} \, dx \]

with \( \kappa = 1, 2, \ldots, 10 \).

Solution:

Some points to consider when solving a problem are:

- **What are the inputs?**
  - We need to specify \( \infty \). How do we do this with a computer?
  - The value of \( \kappa \) needs to be changed so it **must** be an argument to the functions.
  - We shall also need to specify a step size.

- **What are the outputs?**
  - We just require the 10 values of the integral against different values of \( \kappa \).
  - Do we need the values to output to a table or a graph?

- **What functions do we need?**
  - The main function will need to call on an integrate function;
  - The integrate function will need to call on an integrand function.

The way to efficiently solve a problem (i.e. to minimize the risk of failing!) is to work backwards from the lowest function. So in this problem:

I Build and test the integrand function in the main code, then move out to a function.

II Build and test the integrate function in the main code, then move out to a function.

III Implement any input/output and user interface in the main code.

See website for codes.
More on Functions

3.1 Rewrite the example using Simpson rule.

3.2 Now find the integral

\[ I = \int_a^b \frac{1}{\alpha(1 + x)} e^{-\frac{x^2}{s^2}} dx \]

with \( a = 0, \ b = 1, \ \alpha = 0.02, \) and \( s = 0.1. \) What do you have to change in your code? What if \( a, \ b, \ \alpha, \ s \) are to be supplied by the user?

3.3 Overload your square function from earlier tasks to accept as an argument and return each of the standard data types.

3.4 Try to overload a function by just changing the return type. What happens? Try to think about why this won’t work...

3.5 Rewrite your code from question 2.2 so that your sin function definition is contained in a namespace Mymath:

```cpp
namespace Mymath {
    double sin(double x);
} // end namespace
```

Now plot the value from your sin function against the value from sin function in the cmath library differentiating between the two using their second names.

3.6 Amounts \( A_1, A_2, \ldots, A_n \) are placed in an investment account at the beginning of month 1, 2, \ldots, \( n, \) respectively. Let \( B_n \) be the balance in the account just after amount \( A_n \) has been invested at the end of the month \( n. \) The average annual return rate \( r \) (assuming monthly interest compounding) of this investment account is a root to the equation:

\[ B_n = \sum_{i=1}^{n} A_i \left(1 + \frac{r}{12}\right)^{n-i}. \]

(i) Write a function to calculate \( r \) the annual return rate if you are given \( B_n, A_1, \ldots, A_n. \) If a man invest 100 dollars each month for the first year, then 200, 300, 400 and 500 dollars each month for the second, third, fourth, and fifth years respectively, and has a balance of 28000 dollars after the 60th investment, what is the average annual return rate. Hint: use nested multiplication to calculate \( B_n \) for a given \( r, \) and Newton Secant root finding algorithm to find \( r. \)

Standard Libraries

3.7 Create a new vector with 10 elements, assign values to them and print them to screen.

3.8 Write a function add with the following definition:
Implement the function so that it return a new vector that has all the elements in $a$ followed by all the elements of $b$.

3.9 Write a function $createGrid$ passing a grid vector $x$, the number of divisions $n$, and range $x_{\min}$ and $x_{\max}$. On return, vector $x$ should satisfy:

$$x_i = x_{\min} + idx, \quad \text{for} \quad i = 0, 1, \ldots, n \quad dx = \frac{x_{\max} - x_{\min}}{n}$$

Remember that number of divisions in a grid is not the same as number of points.

3.10 Write a function $integrate$ to find the area underneath the function $y(x)$ where the values of $x_i$ and $y_i = y(x_i)$ are stored in vectors. Use the trapezium rule so that grid spacing may be unequal. Test it on the following data:

$$x = (-0.1, 0, 0.25, 0.5, 1) \quad y = (1.2, 1.6, 2.5, 2.1, 1.8)$$

**Numerical Solutions to ODEs**

3.11 The Euler method for integrating an ODE

$$\frac{dy}{dx} = f(x, y), \quad y(a) = \alpha$$

on the interval $x \in [a, b]$ is given by the algorithm

$$w_{i+1} = w_i + hf(x_i, w_i), \quad i = 0, 1, \ldots, n - 1$$

where $w_i$ is the approximation to $y(x_i)$, $x_i = a + ih$ and $h = (b - a)/n$ is the step size. Write a program to integrate the following differential equation

$$\frac{dy}{dx} = xe^{3x} - 2y, \quad 0 \leq x \leq 1, \quad y(0) = 0.$$

Using an appropriate step size, output a table of values with $x_i$ and $w_i$ in the range $a \leq x \leq b$.

3.12 Now using a loop (or otherwise) run your program with increasing values of $n$. Show that your approximation $w_n$ converges to the real solution $y(b)$ by putting some appropriate results in a table.