Convergence Properties of Iteration Schemes

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OUTLINE

1 REVIEW

2 MATRIX EQUATIONS
   • Rearranging the Matrix
   • Iteration Matrix Equations

3 CONVERGENCE PROPERTIES
   • The Iteration Matrix
   • Properties of Convergence
   • Further Convergence Properties

4 SUMMARY
1 Review

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4 SUMMARY
The model equations can be rewritten

\[
\frac{w_{i+1,j} - 2w_{i,j} + w_{i-1,j}}{\Delta x^2} + \frac{w_{i,j+1} - 2w_{i,j} + w_{i,j-1}}{\Delta y^2} = f_{i,j},
\]

How up-to-date the values are define different schemes

We can over-relax to improve convergence

\[
\omega_{i,j}^{q+1} = (1 - \omega)\omega_{i,j}^q + \omega\omega_{i,j}^*
\]

and also solve a line (or block) using a direct method.
RELAXATION METHODS

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4. **Summary**
Given some PDE problem, it can always be reduced to the solution of a linear system

\[Ax = b\]

where \(A = (a_{i,j})\) is an \((n \times n)\) matrix, and \(x, b\) are \((n \times 1)\) column vectors.
Let write the matrix $A$ in the form $A = D + L + U$ where $D$, $L$, $U$ are the diagonal matrix, lower and upper triangular parts of $A$, ie

$$
D = \begin{bmatrix}
    a_{1,1} & 0 & & & & \\
    0 & a_{2,2} & 0 & & & \\
    0 & 0 & a_{3,3} & 0 & & \\
    & & & \ddots & \ddots & \\
    & & & & 0 & a_{n-1,n-1} & 0 \\
    & & & & 0 & 0 & a_{n,n}
\end{bmatrix}
$$
$$U = \begin{bmatrix} 0 & a_{1,2} & \cdots & \cdots & \cdots \\ 0 & 0 & a_{2,3} & \cdots & \cdots \\ 0 & 0 & 0 & a_{3,4} & \cdots \\ \vdots & \vdots & \vdots & \ddots & \cdots \\ 0 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & a_{n-1,n} & 0 & 0 \end{bmatrix}$$
Matrix Equation

\[ L = \begin{bmatrix}
0 & 0 \\
0 & a_{2,1} & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots \\
0 & \cdots & a_{n-1,n-2} & 0 & 0 \\
0 & \cdots & 0 & a_{n,n-1} & 0
\end{bmatrix} \]
Then we may write

\[ Ax = b \]

as

\[ Lx + Dx + Ux = b. \]

In this way we can express our iteration schemes in terms of matrices.
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4 SUMMARY
Iterative Methods

- Jacobi iteration may be written as:

\[ x^{(k+1)} = D^{-1}(b - (L + U)x^{(k)}) \]

- Gauss-Seidel iteration may be written as:

\[ x^{(k+1)} = (D + L)^{-1}(b - Ux^{(k)}) \]
Iterative methods

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4. **Summary**
In general an iteration scheme may be written as

\[ x^{(k+1)} = Px^{(k)} + Q, \]

where \( P \) is the iteration matrix.

- For the Jacobi scheme we have

\[ P = P_J = D^{-1}(-L-U). \]

- For the Gauss-Seidel scheme

\[ P = P_G = (D+L)^{-1}(-U). \]
General Iteration Scheme

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4. SUMMARY
The exact solution to the linear system satisfies

\[ x = Px + Q. \]

The approximation at the \( k \)th iteration has some error defined by

\[ x^{(k)} = x + e^{(k)} \]

then the error satisfies the equation

\[ e^{(k+1)} = Pe^{(k)} = P^2e^{(k-1)} = P^{k+1}e^{(0)}. \]
In order that the error diminishes as \( k \to \infty \) we must have

\[ \| P^k \| \to 0 \quad \text{as} \quad k \to \infty, \]

Since

\[ \| P^k \| = \| P \|^k \]

we see that we require

\[ \| P \| < 1. \]
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4. **Summary**
Spectral Radius

- From linear algebra

\[ ||P|| < 1 \]

- is equivalent to the requirement that

\[ \rho(P) = \max_i |\lambda_j| < 1 \]

where \( \lambda_i \) are the eigenvalues of the matrix \( P \), and \( \rho(P) \) is called the spectral radius of \( P \).
Note also that for large $k$

$$||e^{(k+1)}|| = \rho||e^{(k)}||$$

- How many iterations does it take to reduce the initial error by a factor $\epsilon$?
- We need $q$ iterations where $q$ is the smallest value for which $\rho^q < \epsilon$ giving

$$q \leq q_d = \frac{\log \epsilon}{\log \rho}.$$
CONVERGENCE RATE

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$$q \leq q_d = \frac{\log \epsilon}{\log \rho}.$$
Thus iteration matrices where the spectral radius is close to 1 will converge slowly.

For the model problem it can be shown that for Jacobi iteration

\[ \rho = \rho(P_J) = \frac{\cos \left( \frac{\pi}{n} \right) + \beta^2 \cos \left( \frac{\pi}{m} \right)}{1 + \beta^2}, \]

and for Gauss-Seidel

\[ \rho = \rho(P_G) = [\rho(P_J)]^2. \]
If we take \( n = m, dx = dy \) and \( n \gg 1 \) then for Jacobi iteration we have

\[
q_d = \frac{\log \epsilon}{\log \left(1 - \frac{\pi^2}{2n^2}\right)} = -\frac{2n^2}{\pi^2} \log \epsilon
\]

For Gauss-Seidel we have that \( \rho(P_G) = [\rho(P_J)]^2 \) so that

\[
q_d = -\frac{n^2}{\pi^2} \log \epsilon
\]

Gauss-Seidel converges twice as fast as Jacobi.
Convergence Properties

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- Gauss-Seidel converges twice as fast as Jacobi.
Optimal SOR

- We can relate the eigenvalues $\lambda$ of the point SOR iteration matrix to the eigenvalues $\mu$ of the point Jacobi iteration matrix by the equation

\[(\lambda + \omega - 1)^2 = \lambda \omega^2 \mu^2\]

- From this equation it can be proved that the optimum value of omega is

$$\omega_b = \frac{2}{1 + \sqrt{1 - [\rho(\mathbf{P}_J)]^2}}$$

- There exist methods to estimate $\rho(\mathbf{P}_J)$, but for the model problem we have a formula.
We can relate the eigenvalues $\lambda$ of the point SOR iteration matrix to the eigenvalues $\mu$ of the point Jacobi iteration matrix by the equation

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There exist methods to estimate $\rho(P_J)$, but for the model problem we have a formula.
For point SOR on the model problem with optimum values and $n = m$ it can be shown that

$$
\rho = \frac{1 - \sin \frac{\pi}{n}}{1 + \sin \frac{\pi}{n}}
$$

giving

$$
q_d = -\frac{n}{2\pi} \log \epsilon.
$$
Convergence Rates for Line SOR

For line SOR on the model problem with optimum values and \( n = m \) it can be shown that

\[
\rho = \left( \frac{1 - \sin \frac{\pi}{n}}{1 + \sin \frac{\pi}{n}} \right)^2
\]

giving

\[
q_d = -\frac{n}{2\sqrt{2\pi}} \log \epsilon.
\]

- Line SOR converges \( \sqrt{2} \) times faster than point SOR for the model problem.
- When choosing a method we must decide whether faster convergence compensates for extra computations required.
For line SOR on the model problem with optimum values and $n = m$ it can be shown that

$$\rho = \left( \frac{1 - \sin \frac{\pi}{n}}{1 + \sin \frac{\pi}{n}} \right)^2$$

giving

$$q_d = -\frac{n}{2\sqrt{2\pi}} \log \epsilon.$$
A linear system may be expressed as

\[ Lx + Dx + Ux = b. \]

so that in a general iterative scheme may be written

\[ x^{(k+1)} = Px^{(k)} + Q, \]

The condition for convergence on the iteration matrix \( P \) is

\[ ||P|| < 1. \]