Supplementary reading 2: The local immersion theorem

Given a map $f: M^m \to N^n$ of smooth manifolds, then for each point $a \in M$ we can find charts $\phi_1: U_1 \to V_1$ about $a$ in $M$ and $\phi_2: U_2 \to V_2$ about $f(a)$ in $N$. By restriction if necessary we can suppose that $f(V_1) \subseteq V_2$. Let $F = \phi_2^{-1} \circ f \circ \phi_1: U_1 \to U_2$. Then we can think of $F$ as providing a coordinate representation of the map $f$ in a neighbourhood of $a$.

Let $\phi_1(u) = a$. Then recall that

- $f$ is smooth at $a$ when the function $F$ is smooth at $u$;
- $f$ is a local immersion at $a$ when the derivative $dF_0: \mathbb{R}^m \to \mathbb{R}^n$ is an injection or, equivalently, if the $n \times m$ Jacobian matrix $J(F)_u$ has rank $m$.

**Question.** Given that $f$ is a local immersion at $a$, how simple can we make the coordinate map $F$ representing $f$ in a neighbourhood of $a$, by appropriate choice of charts $\phi_1$ and $\phi_2$?

**3.30 Theorem.** Given a smooth map $f: M^m \to N^n$ which is a local immersion at $a \in M$. There there are charts $\phi_1: U_1 \to V_1$ for $M$ about $a \in M$ and $\phi_2: U_2 \to V_2$ for $N$ about $f(a)$ such that

1. $0 \in U_1$ and $\phi_1(0) = a$,
2. $0 \in U_2$ and $\phi_2(0) = f(a)$,
3. $U_1 = U_2 \cap (\mathbb{R}^m \times \{0\})$,
4. $f(V_1) = V_2 \cap f(M)$,
5. $\phi_2^{-1} f \phi_1(x_1, x_2, \ldots, x_m) = (x_1, x_2, \ldots, x_m, 0, \ldots, 0)$.

**Proof.** First of all it is not difficult to construct charts $\phi_1$ and $\phi_2$ satisfying conditions (i) and (ii) of the theorem by starting with any charts. This is left as an exercise.

Furthermore, by restriction of $\phi_1$ if necessary, we can suppose that $f(V_1) \subseteq V_2$.

Write $F = \phi_2^{-1} \circ f \circ \phi_1: U_1 \to U_2$. Then $F(0) = 0$.

Since $f$ is a local immersion at $a$, the derivative $dF_0: \mathbb{R}^m \to \mathbb{R}^n$ has rank $m$. Hence, by Gauss elimination, the $n \times m$ Jacobian matrix $J(F)_0$ is row equivalent to the matrix

\[
\begin{pmatrix}
I_m \\
0_{n-m,m}
\end{pmatrix}
\]

where $I_m$ denotes the $m \times m$ identity matrix and $0_{p,q}$ denotes the $p \times q$ zero matrix.
Recall that row operations correspond to change of basis in the codomain $\mathbb{R}^n$ and so we can assume that
\[ J(F)_0 = \begin{pmatrix} I_m \\ 0 \end{pmatrix}. \]

[Alternatively, we could say the following. Since $J(F)_0$ is row equivalent to $\begin{pmatrix} I \\ 0 \end{pmatrix}$, there is a non-singular $n \times n$ matrix $P$ such that $PJ(F)_0 = \begin{pmatrix} I \\ 0 \end{pmatrix}$. Then, if $\phi_2$ is replaced by $\phi_2 \circ P^{-1}$, $F$ will be replaced by $P \circ F$ (where the linear map $\mathbb{R}^n \to \mathbb{R}^n$ represented by $P$ with respect to the standard basis is also denoted $P$) and so the derivative $dF_0$ is replaced by $P \circ dF_0$ or, equivalently, the Jacobian $J(F)_0$ is replaced by $PJ(F)_0 = \begin{pmatrix} I \\ 0 \end{pmatrix}$.]

Now define the function $G: U_1 \times \mathbb{R}^{n-m} \to \mathbb{R}^n$ by
\[ G(x, z) = F(x) + (0, z), \quad \text{for } x \in U_1, \ z \in \mathbb{R}^{n-m}. \]

To evaluate the Jacobian of this map at $0$ observe that
\[ J((x, z) \mapsto F(x))_0 = \begin{pmatrix} I_m & 0_{m,n-m} \\ 0_{n-m,m} & 0_{n-m,n-m} \end{pmatrix}, \text{ from the above, and} \]
\[ J((x, z) \mapsto (0, z))_0 = \begin{pmatrix} 0_{m,m} & 0_{m,n-m} \\ 0_{n-m,m} & I_{n-m} \end{pmatrix}. \]

Hence, adding these matrices, $J(G)_0 = I_n$.

We now apply The Inverse Function Theorem (3.19). From the above calculation $J(G)_0$ is certainly non-singular and so $G$ is a local diffeomorphism at $0$. This means that there is an open neighbourhood $U$ of $0$ in $U_1 \times \mathbb{R}^{n-m}$ such that $G: U \to G(U)$ is a diffeomorphism.

By restriction if necessary we can suppose that $G(U) \subseteq U_2$.

Then $\phi = \phi_2 \circ G: U \to \phi(U) \subseteq N$ gives a chart about $f(a)$ with $\phi(0) = f(a)$. Write $V = \phi(U)$.

By restriction if necessary we can suppose that $U_1 = U \cap (\mathbb{R}^m \times \{0\})$ so that $f(V_1) = V \cap f(M)$.

Now the charts claimed by the theorem are given by $\phi_1: U_1 \to V_1$ and $\phi: U \to V$. The coordinate formula in part (v) of the theorem may be confirmed as follows.
\[ \phi^{-1} f \phi_1(x) = G^{-1} \phi_2^{-1} f \phi_1(x) \\
= G^{-1} F(x) \\
= (x, 0) \]
as required, since, by definition of $G$, $G(x, 0) = F(x)$.

\[ \square \]