Similarity and Diagonalization

Similar Matrices

Let A and B be $n \times n$ matrices. We say that A ***is similar*** to B if there is an invertible $n \times n$ matrix P such that $P^{-1}AP = B$. If A is similar to B, we write $A \sim B$.

Remarks

- If $A \sim B$, we can write, equivalently, that $A = PBP^{-1}$ or $AP = PB$.
- If $A \sim B$, we can write, equivalently, that $A = PBP^{-1}$ or $AP = PB$.
- The matrix P depends on A and B. It is not unique for a given pair of similar matrices A and B. To see this, simply take $A = B = I$, in which case $I \sim I$, since $P^{-1}IP = I$ for *any* invertible matrix P.

Theorem 4.21. Let A, B and C be $n \times n$ matrices.

a. $A \sim A$.

b. If $A \sim B$, then $B \sim A$.

c. If $A \sim B$ and $B \sim C$, then $A \sim C$.

This means that $\sim$ is an equivalence relation. The main problem is to find a "good" representative in each equivalence class.

The "real" meaning of $P^{-1}AP$ is that this is the matrix of the same linear transformation (given in the standard basis by the matrix A) in a different basis, which consists of the columns of P. This really much better explains why many properties are the same for A and $P^{-1}AP$.

Theorem 4.22. Let A and B be $n \times n$ matrices with $A \sim B$. Then

a. $\det A = \det B$.

b. A is invertible if and only if B is invertible.

c. A and B have the same rank.

d. A and B have the same characteristic polynomial.

e. A and B have the same eigenvalues.

Diagonalization

Definition. An $n \times n$ matrix A is *diagonalizable* if there is a diagonal matrix D such that A is similar to D — that is, if there is an invertible matrix P such that $P^{-1}AP = D$.

Note that the eigenvalues of D are its diagonal elements, and these are the same eigenvalues as for A.

Theorem 4.23. Let A be an $n \times n$ matrix. Then A is diagonalizable if and only if A has n linearly independent eigenvectors.

More precisely, there exists an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$ if and only if the columns of P are n linearly independent eigenvectors of A and the diagonal entries of D are the eigenvalues of A corresponding to the eigenvectors in P in the same order.

Theorem 4.25. If A is an $n \times n$ matrix with n distinct eigenvalues, then A is diagonalizable.

...Since eigenvectors for distinct eigenvalues are lin. indep. by Th. 4.20.

Theorem 4.24. Let A be an $n \times n$ matrix and let

$$\lambda_1, \lambda_2, \ldots, \lambda_k$$

be distinct eigenvalues of A. If $\mathcal{B}_i$ is a basis for the eigenspace E_{λ_i}, then

$$\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \cdots \cup \mathcal{B}_k$$

(i.e., the total collection of basis vectors for all of the eigenspaces) is linearly independent.

Lemma 4.26. If A is an $n \times n$ matrix, then the geometric multiplicity of each eigenvalue is less than or equal to its algebraic multiplicity.

Theorem 4.27. The Diagonalization Theorem Let A be an $n \times n$ matrix whose distinct eigenvalues are $\lambda_1, \lambda_2, \ldots, \lambda_k$. The following statements are equivalent:

a. A is diagonalizable.

b. The union $\mathcal{B}$ of the bases of the eigenspaces of A (as in Theorem 4.24) contains n vectors (which is equivalent to $\sum_{i=1}^{k} \dim E_{\lambda_i} = n$).

c. The algebraic multiplicity of each eigenvalue equals its geometric multiplicity **and all eigenvalues are real numbers — this condition is missing in the textbook!**.

In these theorems the eigenvalues are supposed to be real numbers, although for real matrices there may be some complex roots of the characteristic polynomial (in fact, these theorems remain valid for vector spaces and matrices over $\mathbb{C}$ — then, of course, one does not need the condition that the eigenvalues be all real).

Theorem 4.27 and Th. 4.23 actually give a method to decide whether A is diagonalizable, and if yes, to find P such that $P^{-1}AP$ is diagonal: the columns of P are vectors of bases of the eigenspaces.

Example. For $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$ the characteristic polynomial is

$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 2 & 2 \\ 2 & 1-\lambda & 2 \\ 2 & 2 & 1-\lambda \end{vmatrix} =$

$(1-\lambda)^3 + 8 + 8 - 4(1-\lambda) - 4(1-\lambda) - 4(1-\lambda) = \cdots = -(\lambda-5)(\lambda+1)^2$. Thus, eigenvalues are 5 and -1.

Eigenspace E_{-1}: $(A - (-1)I)\vec{x} = \vec{0}$; $\begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$; $x_1 = -x_2 - x_3$,

where x_2, x_3 are free var.; $E_{-1} = \left\{ \begin{bmatrix} -s-t \\ s \\ t \end{bmatrix} \mid s, t \in \mathbb{R} \right\}$;

a basis of E_{-1}: $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$.

Eigenspace E_5: $(A - 5I)\vec{x} = \vec{0}$; $\begin{bmatrix} -4 & 2 & 2 \\ 2 & -4 & 2 \\ 2 & 2 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$; solve this

system....: $x_1 = x_2 = x_3$, where x_3 is a free var.; $E_5 = \left\{ \begin{bmatrix} t \\ t \\ t \end{bmatrix} \mid t \in \mathbb{R} \right\}$;

a basis of E_5: $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$.

Together the dimensions add up to 3, so $\mathcal{B}_5 \cup \mathcal{B}_{-1}$ is a basis of $\mathbb{R}^3$, so A is diagonalizable.

Let $P = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$; then $P^{-1}AP = \begin{bmatrix} 5 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$.

(Note that is we arrange the eigenvectors in a different order, then the eigenvalues on the diagonal must be arranged accordingly: let $Q = \begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$;

then $Q^{-1}AQ = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 5 \end{bmatrix}$.)

Example. For $A = \begin{bmatrix} 3 & 20 & 29 \\ 0 & 1 & 82 \\ 0 & 0 & 7 \end{bmatrix}$ the eigenvalues are 3, 1, and 7. Since they are distinct, the matrix is diagonalizable.

(To find that P such that $P^{-1}AP = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 7 \end{bmatrix}$, one still needs to solve those linear systems $(A - (\lambda)I)\vec{x} = \vec{0}$......).

Example. For $A = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix}$ the eigenvalue is 3 of alg. multiplicity 3. Eigenspace E_3: $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \vec{x} = \vec{0}$; matrix has rank 2, so $\dim E_3 = 1$. So A is not digonalizable.

Example. Use diagonalization to find A^{100} for $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$. Eigenvalues are.... -1 and 3. Eigenspace E_3: $\begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix} \vec{x} = \vec{0}$; $x_1 = x_2$; basis $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$. Eigenspace E_{-1}: $\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \vec{x} = \vec{0}$; $x_1 = -x_2$; basis $\left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$. Let $P = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$; then $P^{-1}AP = D = \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix}$. Now, $A = PDP^{-1}$, so $A^{100} = (PDP^{-1})^{100} = PDP^{-1}PDP^{-1} \cdots PDP^{-1} = PD^{100}P^{-1} = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix}^{100} \begin{bmatrix} -1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} =$
$\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3^{100} \end{bmatrix} \begin{bmatrix} -1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} = \begin{bmatrix} -1 & 3^{100} \\ 1 & 3^{100} \end{bmatrix} \begin{bmatrix} -1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} =$
$(1/2) \begin{bmatrix} 3^{100}+1 & 3^{100}-1 \\ 3^{100}-1 & 3^{100}+1 \end{bmatrix}$.

Orthogonality in $\mathbb{R}^n$

We introduce the **dot product** of vectors in $\mathbb{R}^n$ by setting

$$\vec{u} \cdot \vec{v} = \vec{u}^T \vec{v};$$

that is, if

$$\vec{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} \text{ and } \vec{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$$

then

$$\vec{u} \cdot \vec{v} = \vec{u}^T \vec{v} = \begin{bmatrix} u_1 & \cdots & u_n \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n.$$

The dot product is frequently called **scalar product** or **inner product**; we shall use the latter term in a slightly more general context. Notice the following properties of the dot product which can be easily checked directly or immediately follow from the properties of matrix multiplication. They hold for arbitrary vectors $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^n$ and arbitrary scalar λ.

- $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$ (commutativity).
- $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$
- $\vec{u} \cdot (\lambda \vec{v}) = \lambda(\vec{v} \cdot \vec{u})$ (The last two properties are referred to as **linearity** of the dot product.)
- $\vec{u} \cdot \vec{u} = u_1^2 + \cdots + u_n^2$ and therefore $\vec{u} \cdot \vec{u} \geq 0$. Moreover, if $\vec{u} \cdot \vec{u} = 0$ then $\vec{u} = \vec{0}$.

We define the **length** (or **norm**) $\|\vec{v}\|$ of vector

$$\vec{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$$

by

$$\|\vec{v}\| = \sqrt{\vec{v} \cdot \vec{v}} = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$$

Orthogonal and Orthonormal Sets of Vectors

A set of vectors

$$\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$$

in $\mathbb{R}^n$ is called an **orthogonal set** if all pairs of distinct vectors in the set are orthogonal – that is, if

$$\vec{v}_i \cdot \vec{v}_j = 0 \text{ whenever } i \neq j \text{ for } i, j = 1, 2, \ldots, k$$

The standard basis

$$\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n$$

in $\mathbb{R}^n$ is an orthogonal set, as is any subset of it. As the first example illustrates, there are many other possibilities.

Example 5.1 Show that $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is an orthogonal set in $\mathbb{R}^3$ if

$$\vec{v}_1 = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

Solution We must show that every pair of vectors from this set is orthogonal. This is true, since

$$\vec{v}_1 \cdot \vec{v}_2 = 2(0) + 1(1) + (-1)(1) = 0$$

$$\vec{v}_2 \cdot \vec{v}_3 = 0(1) + 1(-1) + (1)(1) = 0$$

$$\vec{v}_1 \cdot \vec{v}_3 = 2(1) + 1(-1) + (-1)(1) = 0$$

Theorem 5.1. If

$$\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$$

is an orthogonal set of nonzero vectors in $\mathbb{R}^n$, then these vectors are linearly independent.

Proof If $c_1, c_2, \ldots, c_k$ are scalars such that

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k = \vec{0},$$

then

$$(c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k) \cdot \vec{v}_i = \vec{0} \cdot \vec{v}_i = 0$$

or, equivalently,

$$c_1(\vec{v}_1 \cdot \vec{v}_i) + \cdots + c_i(\vec{v}_i \cdot \vec{v}_i) + \cdots + c_k(\vec{v}_k \cdot \vec{v}_i) = 0 \qquad (1)$$

Since

$$\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$$

is an orthogonal set, all of the dot products in equation (1) are zero, except $\vec{v}_i \cdot \vec{v}_i$. Thus, equation (1) reduces to

$$c_i(\vec{v}_i \cdot \vec{v}_i) = 0$$

Now, $\vec{v}_i \cdot \vec{v}_i \neq 0$ because $\vec{v}_i \neq \vec{0}$ by hypothesis. So we must have $c_i = 0$. The fact that this is true for all $i = 1, \ldots, k$ implies that

$$\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$$

is a linearly independent set. □

Remark. Thanks to the Theorem 5.1, we know that if a set of vectors is orthogonal, it is automatically linearly independent. For example, we can immediately deduce that the three vectors in Example 5.1 are linearly independent. Contrast this approach with the work needed to establish their linear independence directly!

An **orthogonal basis** for a subspace W of $\mathbb{R}^n$ is a basis of W that is an orthogonal set.

Example 5.2. The vectors

$$\vec{v}_1 = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

from Example 5.1 are orthogonal and, hence, linearly independent. Since any three linearly independent vectors in $\mathbb{R}^3$ form a basis in $\mathbb{R}^3$, by the Fundamental Theorem of Invertible Matrices, it follows that $\vec{v}_1, \vec{v}_2, \vec{v}_3$ is an orthogonal basis for $\mathbb{R}^3$.

Theorem 5.2 Let

$$\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$$

be an orthogonal basis for a subspace W of $\mathbb{R}^n$ and let $\vec{w}$ be any vector in W. Then the unique scalars $c_1, c_2, \ldots, c_k$ such that

$$\vec{w} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k$$

are given by

$$c_i = \frac{\vec{w} \cdot \vec{v}_i}{\vec{v}_i \cdot \vec{v}_i} \text{ for } i = 1, \ldots, k$$

Proof Since

$$\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$$

is a basis for W, we know that there are unique scalars $c_1, c_2, \ldots, c_k$ such that

$$\vec{w} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k$$

(from Theorem 3.29). To establish the formula for c_i, we take the dot product of this linear combination with $\vec{v}_i$ to obtain

$$\vec{w} \cdot \vec{v}_i = (c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k) \cdot \vec{v}_i$$

$$= c_1(\vec{v}_1 \cdot \vec{v}_i) + \cdots + c_i(\vec{v}_i \cdot \vec{v}_i) + \cdots + c_k(\vec{v}_k \cdot \vec{v}_i)$$

$$= c_i(\vec{v}_i \cdot \vec{v}_i)$$

since $\vec{v}_j \cdot \vec{v}_i = 0$ for $j \neq i$. Since $\vec{v}_i \neq \vec{0}$, $\vec{v}_i \cdot \vec{v}_i \neq 0$. Dividing by $\vec{v}_i \cdot \vec{v}_i$, we obtain the desired result. □

A **unit** vector is a vector of unit length. Notice that if $\vec{v} \neq \vec{0}$ then

$$\vec{u} = \frac{\vec{v}}{\|v\|}$$

is a unit vector collinear (directed along the same line) as $\vec{v}$:

$$\vec{v} = \|\vec{v}\|\vec{u}.$$

A set of vectors in $\mathbb{R}^n$ is an **orthonormal set** if it is an orthogonal set of unit vectors. An **orthonormal basis** for a subspace W of $\mathbb{R}^n$ is a basis of W that is an orthonormal set.

Theorem 5.3 Let

$$\{\vec{q}_1, \vec{q}_2, \ldots, \vec{q}_k\}$$

be an orthonormal basis for a subspace W of $\mathbb{R}^n$ and let $\vec{w}$ be any vector in W. Then

$$\vec{w} = (\vec{w} \cdot \vec{q}_1)\vec{q}_1 + (\vec{w} \cdot \vec{q}_2)\vec{q}_2 + \cdots + (\vec{w} \cdot \vec{q}_k)\vec{q}_k$$

and this representation is unique.

Theorem 5.4. The columns of an $m \times n$ matrix Q form an orthonormal set if and only if $Q^T Q = I_n$.

Proof. We need to show that

$$(Q^T Q)_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

Let $\vec{q}_i$ denote the ith column of Q (and, hence, the ith row of Q^T). Since the (i, j) entry of $Q^T Q$ is the dot product of the ith row of Q^T and the jth column of Q, it follows that

$$(Q^T Q)_{ij} = \vec{q}_i \cdot \vec{q}_j \qquad (2)$$

by the definition of matrix multiplication.

Now the columns of Q form an orthonormal set if and only if

$$\vec{q}_i \cdot \vec{q}_j = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

which, by equation (2) holds if and only if

$$(Q^T Q)_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

This completes the proof. □

If the matrix Q in Theorem 5.4 is a **square** matrix, is has a special name.

An $n \times n$ matrix Q whose columns form an orthonormal set is called an **orthogonal matrix**.

The most important fact about orthogonal matrices is given by the next theorem.

Theorem 5.5. A square matrix Q is orthogonal if and only if $Q^{-1} = Q^T$.

Proof. By Theorem 5.4, Q is orthogonal if and only if $Q^T Q = I$. This is true if and only if Q is invertible and $Q^{-1} = Q^T$, by Theorem 3.13. □

Example

Each of the following matrices is orthogonal:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}, \qquad \begin{bmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & -\cos\alpha \end{bmatrix}$$

Theorem 5.6. Let Q be an $n \times n$ matrix. The following statements are equivalent:

a. Q is orthogonal.

b. $||Q\vec{x}|| = ||\vec{x}||$ for every $\vec{x}$ in $\mathbb{R}^n$.

c. $Q\vec{x} \cdot Q\vec{y} = \vec{x} \cdot \vec{y}$ for every $\vec{x}$ and $\vec{y}$ in $\mathbb{R}^n$.

Theorem 5.7. If Q is an orthogonal matrix, then its rows form an orthonormal set.

Theorem 5.8. Let Q be an orthogonal matrix.

a. Q^{-1} is orthogonal.

b. $\det Q = \pm 1$.

c. If λ is an eigenvalue of Q, then $|\lambda| = 1$.

d. If Q_1 and Q_2 are orthogonal $n \times n$ matrices, then so is $Q_1 Q_2$.