Subspaces, Basis, Dimension, and Rank

Definition. A subspace of $\mathbb{R}^n$ is any collection $S$ of vectors in $\mathbb{R}^n$ such that

1. The zero vector $\vec{0}$ is in $S$.
2. If $\vec{u}$ and $\vec{v}$ are in $S$, then $\vec{u} + \vec{v}$ is in $S$ (that is, $S$ is closed under addition).
3. If $\vec{u}$ is in $S$ and $c$ is a scalar, then $c\vec{u}$ is in $S$ (that is, $S$ is closed under multiplication by scalars).

Remark. Property 1 is needed only to ensure that $S$ is non-empty; for non-empty $S$ property 1 follows from property 3, as $0\vec{a} = \vec{0}$.

Theorem 3.19. Let $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ be vectors in $\mathbb{R}^n$. Then $\text{span}(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k)$ is a subspace of $\mathbb{R}^n$.

Subspaces Associated with Matrices

Definition. Let $A$ be an $m \times n$ matrix.

1. The row space of $A$ is the subspace $\text{row}(A)$ of $\mathbb{R}^n$ spanned by the rows of $A$.
2. The column space of $A$ is the subspace $\text{col}(A)$ of $\mathbb{R}^m$ spanned by the columns of $A$.

If we need to determine if $\vec{b}$ belongs to $\text{col}(A)$, this is actually the same problem as whether $\vec{b} \in \text{span}$ of the columns of $A$; see the method on p. 9.

If we need to determine if $\vec{b}$ belongs to $\text{row}(A)$, then we can apply the same method as above to the columns $\vec{b}^T$ and $\text{col}(A^T)$. Another method for the same task is described in Example 3.41 in the Textbook.

Theorem 3.20. Let $B$ be any matrix that is row equivalent to a matrix $A$. Then $\text{row}(B) = \text{row}(A)$.

See the theorem on p. 13.

Theorem 3.21. Let $A$ be an $m \times n$ matrix and let $N$ be the set of solutions of the homogeneous linear system $A\vec{x} = \vec{0}$. Then $N$ is a subspace of $\mathbb{R}^n$.

Definition. Let $A$ be an $m \times n$ matrix. The null space of $A$ is the subspace of $\mathbb{R}^n$ consisting of solutions of the homogeneous linear system $A\vec{x} = \vec{0}$. It is denoted by $\text{null}(A)$. 
Theorem. Let $B$ be any matrix that is row equivalent to a matrix $A$. Then $\text{null}(B) = \text{null}(A)$.

This is the Fund. Th. on e.r.o.s, see p. 4.

E.g., the set $\{[x_1, x_2, x_3] \mid x_1 + x_2 + x_3 = 0\}$ is automatically a subspace of $\mathbb{R}^3$ — no need to verify those closedness properties 1, 2, 3, as this is the null space of the homogeneous system $x_1 + x_2 + x_3 = 0$ (consisting of one equation).

**Basis**

**Definition.** A *basis* for a subspace $S$ of $\mathbb{R}^n$ is a set of vectors in $S$ that
1. spans $S$ and
2. is linearly independent.

**Remark.** It can be shown that this definition is equivalent to each of the following two definitions:

**Definition’.** A *basis* for a subspace $S$ of $\mathbb{R}^n$ is a set of vectors in $S$ that spans $S$ and is minimal with this property (that is, any proper subset does not span $S$).

**Definition”’.** A *basis* for a subspace $S$ of $\mathbb{R}^n$ is a set of vectors in $S$ that is linearly independent and is maximal with this property (that is, adding any other vector in $S$ to this subset makes the resulting set linearly dependent).

**Method for finding a basis of row($A$).** Reduce $A$ to r.r.e.f. $R$ by e.r.o.s. (We know row($A$) = row($R$).) The non-zero rows of $R$, say, $\vec{b}_1, \ldots, \vec{b}_r$, form a basis of row($R$) = row($A$). Indeed, they clearly span row($R$), as zero rows contribute nothing. The fact that the non-zero rows are linearly independent can be seen from columns with leading 1s: in a linear combination $\sum c_i \vec{b}_i$ the coordinate in the column of the 1st leading 1 is $c_1$, since there are only zeros above and below this leading 1; also the coordinate in the column of the 2nd leading 1 is $c_2$, since there are only zeros above and below this leading 1; and so on. If $\sum c_i \vec{b}_i = \vec{0}$, then we must have all $c_i = 0$.

Moreover, the same is true for any r.e.f. $Q$ (not necessarily reduced r.e.f.): The non-zero rows of $Q$ form a basis of row($Q$) = row($A$).

**1st Method for finding a basis of col($A$).** Use the previous method applied to $A^T$. 
2nd Method for finding a basis of \( \text{col}(A) \). We know that e.r.o.s do not alter the solution set of the homogeneous system \( A\vec{x} = \vec{0} \). Every solution of it can be regarded as a dependence of the columns of \( A \). Thus, after reducing by e.r.o.s to r.r.e.f. \( R \), we shall have exactly the same dependences among the columns as for \( A \). For \( R \), the columns of leading 1s clearly form a basis of \( \text{col}(R) \). Then the corresponding columns of \( A \) will form a basis of \( \text{col}(A) \). (WARNING: \( \text{col}(R) \neq \text{col}(A) \) in general!)

Again, the same is true for any r.e.f. \( Q \) (not necessarily reduced r.e.f.): the columns of leading entrie form a basis of \( \text{col}(Q) \) and therefore the corresponding columns of \( A \) form a basis of \( \text{col}(A) \).

Method for finding a basis of null\( (A) \). Express leading variables via free variables (independent parameters). Give these parameters values as in the columns of the identity matrix \( I_f \), where \( f \) is the number of free variables, and compute the values of leading variables. The resulting \( f \) vectors form a basis of null\( (A) \).

E.g., suppose the general solution (=null\( (A) \)) is
\[
\begin{bmatrix}
  s + 2t + 3u \\
  s \\
  t \\
  2u \\
  u
\end{bmatrix}
\]

(that is, the free variables are \( x_2, x_3, x_5 \), while leading are \( x_1, x_4 \)). We give \( s, t, u \) the values 1, 0, 0, then 0, 1, 0, then 0, 0, 1. The resulting vectors are

\[
\begin{bmatrix}
1 \\
0 \\
0 \\
0
\end{bmatrix} ; \begin{bmatrix}
2 \\
0 \\
0 \\
0
\end{bmatrix} ; \begin{bmatrix}
3 \\
0 \\
2 \\
1
\end{bmatrix}
\]

which form a basis of null\( (A) \).

Dimension and Rank

Theorem 3.23. The Basis Theorem

Let \( S \) be a subspace of \( \mathbb{R}^n \). Then any two bases for \( S \) have the same number of vectors.

Warning: there is blunder in the textbook – the existence of a basis is not proven. A correct statement should be

Theorem 3.23+. The Basis Theorem

Let \( S \) be a non-zero subspace of \( \mathbb{R}^n \). Then

(a) \( S \) has a finite basis;
(b) any two bases for \( S \) have the same number of vectors.
For the examination, no need to have proof. But, for the completeness of exposition, I give a proof of existence of basis, Theorem 3.23+(a), here.

**The existence of basis.** Let $S$ be a non-zero subspace (that is, $S$ does not consist of zero vector only) of $\mathbb{R}^n$. Then $S$ has a basis.

**Proof.** Consider all linearly independent systems of vectors in $S$. Since $S$ contains a non-zero vector $\vec{v} \neq \vec{0}$, there is at least one such system: $\vec{v}$. Now, if $\vec{v}_1, \ldots, \vec{v}_k$ is a system of linearly independent vectors in $S$, we have $k \leq n$ by Theorem 2.8.

We come to a crucial step of the proof: choose a system of linearly independent vectors $\vec{v}_1, \ldots, \vec{v}_k$ in such way that $k$ is maximal possible and consider

$$U = \text{span}(v_1, \ldots, v_k).$$

Observe that $U \subseteq S$. If $U = S$, then $\vec{v}_1, \ldots, \vec{v}_k$ is a basis of $S$ by definition of the basis, and our theorem is proven. Therefore we can assume that $U \neq S$ and chose a vector $\vec{v} \in S \setminus U$ (in $S$ but not in $U$).

**The rest of proof of Theorem 3.23** can be taken from the textbook.

**Definition.** If $S$ is a subspace of $\mathbb{R}^n$, then the number of vectors in a basis for $S$ is called the **dimension** of $S$, denoted $\dim S$.

**Remark.** The zero vector $\vec{0}$ by itself is always a subspace of $\mathbb{R}^n$. (Why?) Yet any set containing the zero vector (and, in particular, \{\vec{0}\}) is linearly dependent, so \{\vec{0}\} cannot have a basis. We define $\dim \{\vec{0}\}$ to be 0.

**Examples.** 1) As we know, the $n$ standard unit vectors form a basis of $\mathbb{R}^n$; thus, $\dim \mathbb{R}^n = n$.

2) If $\vec{v}_1, \ldots, \vec{v}_k$ are linearly independent vectors, then they form a basis of $\text{span}(\vec{v}_1, \ldots, \vec{v}_k)$, so then $\dim \text{span}(\vec{v}_1, \ldots, \vec{v}_k) = k$.

We shall need a slightly more general result:

**Theorem 3.23++.** (a) If $v_1, \ldots, v_k$ are linearly independent vectors in a subspace $S$, then they can be included in (complemented to) a basis of $S$; in particular, $k \leq \dim S$.

(b) If one subspace is contained in another, $S \subseteq T$, then $\dim S \leq \dim T$. If both $S \subseteq T$ and $\dim S = \dim T$, then $S = T$.

**Example.** If we have some $n$ linearly independent vectors $\vec{v}_1, \ldots, \vec{v}_n$ in $\mathbb{R}^n$, they must also form a basis of $\mathbb{R}^n$, as the dimension of their span is $n$ and we can apply Theorem 3.23++(b).
Theorem 3.24.  The row and column spaces of a matrix $A$ have the same dimension.

Definition  The rank of a matrix $A$ is the dimension of its row and column spaces and is denoted by rank$(A)$.

Theorem 3.25.  For any matrix $A$,

$$\text{rank } (A^T) = \text{rank } (A)$$

Definition  The nullity of a matrix $A$ is the dimension of its null space and is denoted by nullity$(A)$.

Theorem 3.26.  The Rank–Nullity Theorem

If $A$ is an $m \times n$ matrix, then

$$\text{rank } (A) + \text{nullity } (A) = n$$

Theorem 3.27.  The Fundamental Theorem of Invertible Matrices

Let $A$ be an $n \times n$ matrix. The following statements are equivalent:

a.  $A$ is invertible.

b.  $A\vec{x} = \vec{b}$ has a unique solution for every $\vec{b}$ in $\mathbb{R}^n$.

c.  $A\vec{x} = \vec{0}$ has only the trivial solution.

d.  The reduced row echelon form of $A$ is $I_n$.

e.  $A$ is a product of elementary matrices.

f.  rank$(A) = n$.

g.  nullity$(A) = 0$.

h.  The column vectors of $A$ are linearly independent.

i.  The column vectors of $A$ span $\mathbb{R}^n$.

j.  The column vectors of $A$ form a basis for $\mathbb{R}^n$.

k.  The row vectors of $A$ are linearly independent.

l.  The row vectors of $A$ span $\mathbb{R}^n$.

m.  The row vectors of $A$ form a basis for $\mathbb{R}^n$. 
Coordinates

Theorem 3.29. Let $S$ be a subspace of $\mathbb{R}^n$ and let

$$\mathcal{B} = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$$

be a basis for $S$. For every vector $\vec{v}$ in $S$, there is exactly one way to write $\vec{v}$ as a linear combination of the basis vectors in $\mathcal{B}$:

$$\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k$$

Definition. Let $S$ be a subspace of $\mathbb{R}^n$ and let

$$\mathcal{B} = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$$

be a basis for $S$. Let $\vec{v}$ be a vector in $S$, and write

$$\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k$$

Then $c_1, c_2, \ldots, c_k$ are called the coordinates of $\vec{v}$ with respect to $\mathcal{B}$, and column vector

$$[\vec{v}]_\mathcal{B} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{bmatrix}$$

is called the coordinate vector of $\vec{v}$ with respect to $\mathcal{B}$. (Note, although $\vec{v}$ has $n$ “original” coordinates as a vector in $\mathbb{R}^n$, the same vector in the basis $\mathcal{B}$ of the subspace $S$ of dimension $k$ has $k$ coordinates.)