Properties of Invertible Matrices

Theorem 3.9.

a. If A is an invertible matrix, then A^{-1} is invertible and
$$(A^{-1})^{-1} = A$$

b. If A is an invertible matrix and c is a nonzero scalar, then cA is an invertible matrix and
$$(cA)^{-1} = \frac{1}{c}A^{-1}$$

c. If A and B are invertible matrices of the same size, then AB is invertible and
$$(AB)^{-1} = B^{-1}A^{-1}$$

d. If A is an invertible matrix, then A^T is invertible and
$$(A^T)^{-1} = (A^{-1})^T$$

e. If A is an invertible matrix, then A^n is invertible for all nonnegative integers n and
$$(A^n)^{-1} = (A^{-1})^n$$

Corollary of part (c). If $A_1, A_2, \ldots, A_k$ are invertible square matrices of the same size, then the product $A_1A_2\cdots A_k$ is also invertible and
$$(A_1A_2\cdots A_k)^{-1} = A_k^{-1}A_{k-1}^{-1}\cdots A_1^{-1}.$$

Elementary Matrices

Definition An *elementary matrix* is any matrix that can be obtained by performing an elementary row operation on an identity matrix.

Theorem 3.10. Let E be the elementary matrix obtained by performing an elementary row operation on I_n. If the same elementary row operation is performed on an $n \times r$ matrix A, the result is the same as the matrix EA.

Theorem 3.11. Each elementary matrix E is invertible, and its inverse is an elementary matrix of the same type (corresponding to the inverse e.r.o. of the e.r.o. which produced E from I_n).

Proof. Let E' be the result of the inverse e.r.o. applied to I_n. We have $I \xrightarrow{\text{e.r.o.}} E \xrightarrow{\text{inverse e.r.o.}} E'E$ by Th.3.10, but this is also $= I_n$, since we reversed e.r.o., so $E'E = I$. Similarly, $I \xrightarrow{\text{inverse e.r.o.}} E' \xrightarrow{\text{e.r.o.}} EE' = I_n$. Thus, $E' = E^{-1}$.

Theorem 3.12. The Fundamental Theorem of Invertible Matrices

Let A be a square $n \times n$ matrix. The following statements are equivalent:

a. A is invertible.

b. $A\vec{x} = \vec{b}$ has a unique solution for every $\vec{b}$ in $\mathbb{R}^n$.

c. $A\vec{x} = \vec{0}$ has only the trivial solution.

d. The reduced row echelon form of A is I_n.

e. A is a product of elementary matrices.

Theorem 3.13. Let A be a square matrix. If B is a square matrix such that $AB = I$ ("right inverse of A"), then A is invertible and $B = A^{-1}$. Also, if C is a square matrix such that $CA = I$, then A is invertible and $C = A^{-1}$.

Method for decomposing a square matrix into a product of elementary matrices and finding the inverse matrix. Let A be $n \times n$ matrix. Reduce A to r.r.e.f., at each step record e.r.o. as multiplication by the corresponding elementary matrix on the left: $A \to E_1 A \to E_2 E_1 A \to \cdots$.

If the r.r.e.f. of A is I_n, then $E_k \cdots E_1 A = I$, whence $A = E_1^{-1} \cdots E_k^{-1}$. Also, then A is invertible and $A^{-1} = (E_1^{-1} \cdots E_k^{-1})^{-1} = (E_k^{-1})^{-1} \cdots (E_1^{-1})^{-1} = E_k \cdots E_1$. (If only A^{-1} is required, then $E_k \cdots E_1 A = I$ already gives $A^{-1} = E_k \cdots E_1$.)

If the r.r.e.f. of A is not I_n, then A is not invertible and cannot be represented as a product of elementary matrices.

Example. $A = \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix} \xrightarrow{(1/2)R_1} \begin{bmatrix} 1 & 3/2 \\ 0 & 1 \end{bmatrix} \xrightarrow{R_1 - (3/2)R_2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$; this is r.r.e.f. $= I$. This is equivalent to $\begin{bmatrix} 1 & -3/2 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \cdot A = I$. Then $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 3/2 \\ 0 & 1 \end{bmatrix}$. Also $A^{-1} = \begin{bmatrix} 1 & -3/2 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ (taking product of inverses in reverse order). We can multiply through to find $A^{-1} = \begin{bmatrix} 1/2 & -3/2 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -3/2 & 1/2 \\ 1 & 0 \end{bmatrix}$.

Theorem 3.14. Let A be a square matrix. If a sequence of elementary row operations reduces A to I, then the same sequence of elementary row operations transforms I into A^{-1}.

Gauss–Jordan Method for finding inverse matrix. Let A be $n \times n$ matrix. Form $n \times 2n$ matrix with left half A and right half I_n. Reduce left half to r.r.e.f. applying the same e.r.o.s to the whole matrix (to rows of length $2n$). If the r.r.e.f. of A (in the left half) becomes I_n, then A is invertible and the right half becomes A^{-1}. If the r.r.e.f. of A is not I_n, then A is not invertible (and the right half is of no use).

Example. $\begin{bmatrix} 0 & 1 & 1 & 0 \\ 2 & 3 & 0 & 1 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 2 & 3 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \xrightarrow{(1/2)R_1} \begin{bmatrix} 1 & 3/2 & 0 & 1/2 \\ 0 & 1 & 1 & 0 \end{bmatrix} \xrightarrow{R_1-(3/2)R_2}$ $\begin{bmatrix} 1 & 0 & -3/2 & 1/2 \\ 0 & 1 & 1 & 0 \end{bmatrix}$; left half is r.r.e.f. $= I$; hence right half is A^{-1}.