2.1. Show that $A_2(3, 2) = 4$ by writing down a suitable code with four elements and then showing that it is impossible to find one with more elements. [The last part might be slightly easier if you use Lemma 6.]

2.2. Generalize the previous question by showing that $A_2(n, 2) = 2^{n-1}$. This time use the Corollary to Theorem 10 and Proposition 4.

2.3. Prove that the ternary code (that is, a code over $\mathbb{F}_3$)

$$C = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 0 \\ 2 & 0 & 1 \end{pmatrix}$$

is equivalent to the ternary repetition code of length 3.

2.4. Show that if $C$ is a perfect $q$-ary code, where $q$ is a power of a prime number $p$, then the number of codewords must be a power of $p$.

2.5. Prove that every perfect code has odd distance. [This is not too hard to see geometrically: consider the “midpoint” between two codewords. It is trickier to write down properly.]

2.6. We know that $E_n$, the code of all even-weight vectors of $\mathbb{F}_2(n)$ is linear. What are the parameters $[n, k, d]$ of $E_n$? Write down a generator matrix for $E_n$ in standard form.

2.7. Let $H$ be an $r \times n$ matrix over $\mathbb{F}_q$. Prove that the set

$$C = \left\{ \bar{x} \in \mathbb{F}_q^{(n)} \mid \bar{x}H^T = \bar{0} \right\}$$

is a linear code. [Remark: we will show later that every linear code may be defined by means of such a matrix $H$, which is called a parity-check matrix of the code.]

2.8. Show that if $C$ is a binary linear code, then the code obtained by adding an overall parity check to $C$ is also linear.

2.10. Prove that in a binary linear code, either all the codewords have even weight or exactly half have even weight and half have odd weight.

2.11. Let $C_1$ and $C_2$ be binary linear codes having the generator matrices

$$G_1 = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

and

$$G_2 = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix}.$$ 

List the codewords of $C_1$ and $C_2$ and hence find the minimum distance of each code.

2.12. Let $C$ be the ternary linear code with generator matrix

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \end{bmatrix}.$$ 

List the codewords of $C$ and find the minimum distance of $C$. Deduce that $C$ is a perfect code.

2.13. Let $C$ be the binary linear code with generator matrix

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}.$$ 

Find a generator matrix for $C$ in standard form.