History Matching of Petroleum Reservoirs Using a Level Set Technique

Oliver Dorn and Rosmary Villegas
Modelling and Numerical Simulation Group, Universidad Carlos III de Madrid, Avenida de la Universidad 30, Leganes 28911, Spain

Abstract.
We describe a novel technique for the characterization of structured reservoirs from production data. A level set technique is used in order to be able to incorporate interfaces between different lithofacies in the reservoir. The new algorithm is able to reconstruct, in addition to the shapes and topologies of the individual lithofacies, also their internal permeability profiles simultaneously from the given production data. Here either a pixel-by-pixel or a parameterized model can be used for describing each of the internal profiles, and any hybrid combination of them. If \( n \) lithofacies are present in the reservoir, \( n - 1 \) level set functions are employed during the characterization. Overlapping regions of these \( n - 1 \) level set functions are treated by a special technique tailored to the reservoir characterization application. Here our approach differs significantly from standard techniques for describing more than two values by a level set representation. We also describe techniques for testing the existence and characteristic features of certain geometrical structures (such as channels or barriers) inside the reservoir by suitable topological perturbations of the describing level set function. Numerical results in 2D are presented which demonstrate the performance of our novel scheme for a variety of simulated but realistic situations which are of importance in reservoir engineering applications.

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1. Introduction

Reservoir characterization by history matching of production data has a long history, see for example the contributions [1, 3, 5, 10, 11, 13, 14, 27, 30] and the many references given there. The reservoir engineer needs to understand the geological properties inside the Earth in order to optimize the production process of active oil fields. Certainly, the Earth has a very complex structure and it is impossible to determine from few production data all the details of the reservoir. On the other hand, it would be desirable to be able to identify at least some important large-scale structural features of the reservoir. For example, many reservoirs are composed of different geological regions (lithofacies) which are separated by clearly defined interfaces or contain certain channel or barrier structures. Some of these interfaces can be determined by analysis of seismic data, but not all of them. Traditional history matching techniques are not able to resolve these interfaces due to the way these tools are constructed. Typically an inverse problem is solved for determining pixel-values (voxel-values in 3D applications) of petrophysical parameters (e.g. porosity and permeability) inside the reservoir which minimize, when plugged into a reservoir simulator, the mismatch between simulated and actual production data in some sense. Due to the need of very strong regularization (caused by the ill-posedness of the inverse problem and the sparsity of available production data) in most cases an oversmoothed reconstruction is delivered from which it is impossible to detect any clearly defined region boundaries without additional image segmentation tools. Image segmentation techniques, on the other hand, typically change these profiles without taking into account production data, such that the segmented images do not honor anymore production data and are therefore -to the least- suboptimal.

In this work we present a novel reservoir characterization technique which aims at directly providing structured profiles of physical parameters (here permeability) from production data without the need of any segmentation postprocessing techniques. In addition, our technique is able to incorporate smoothly varying region-internal profiles in each area of the reservoir.

Another important aspect of reservoir characterization is that the history matching problem typically is vastly underdetermined and that a unique solution does not exist. Therefore, in order to plan future oil production, it is often desirable to have not just one reconstruction available (given for example by a classical Tikhonov-Philips scheme) which honors the available information, but a broad variety of possible reconstructions. Traditional reservoir characterization techniques yield typically those reconstructions which do not incorporate any interfaces. In this sense, our technique is intended to be a generalization of classical history matching techniques which adds the flexibility of incorporating certain interfaces and therefore certain types of structural features into the inversion which honor production data. The classical pixel-/voxel-based schemes can then be considered as a special case of our new scheme where only one 'lithofacie' (i.e. one region) is present in the reservoir.

For describing the fluid flow in the reservoir, we use a simplified two phase
incompressible flow model of oil and water [29]. We mention that the inversion algorithm
developed here can without problems be applied to more complex (e.g. three-phase flow)
models, as long as suitable forward and adjoint simulators are available for those models.
In order to avoid the so-called inverse crime, the data is generated with a streamline
method, while during the reconstruction process an independent IMPES method is used
to solve the forward problem. This guarantees that the ‘exact’ solution, plugged into the
reservoir simulator, still produces simulated data which are sufficiently different from the
reference data which are used as input for the inversion. This makes the inversion process
more realistic, since reservoir simulators typically are not able to ‘exactly’ reproduce real
data due to the various types of model and measurement noise included in real data.

We use a level set technique [2, 4, 9, 18, 19, 21, 22, 25, 26] for modeling interfaces of
regions with different geophysical properties in the Earth. The topology of the regions
separated by these interfaces is a-priori unknown. Using the available production data,
and some prior information from well-logs affecting physical parameters close to the well
locations, we try to recover a binary (or, in our generalized approach, multi-valued) map
of permeability values in the reservoir.

In the literature there exist several approaches focusing on alternative automatic
history matching techniques based on geological shape definitions. One example is
the use of shape triangularization [23, 24] which, however, requires to have prior
information of the topology and approximate facies locations available. Other methods
use geostatistical approaches [12, 20], which generally work well but are computationally
very expensive. Alternative level set based approaches have been proposed very recently
in the literature as well [15, 16, 17, 18]. Our scheme, as presented here, differs in several
aspects from those approaches. We apply a so-called ‘adjoint scheme’ for calculating the
sensitivities during the reconstruction, we initialize the reservoir model using stochastic
and deterministic initializations based on prior information (parameter values measured
at the well locations), we allow for quite arbitrary internal permeability profiles during
the reconstruction task, we consider reservoirs with an arbitrary (but typically small)
number of different lithofacies, and we incorporate the possibility of testing the existence
and properties of certain geometrical objects in the reservoir. Moreover, as regularization
tool we use an adapted filtering operator to be applied in each step of the inversion
problem.

The paper is organized as follows. Following this section 1 where we have given a
short introduction into the history matching problem associated with geological shape
reconstruction in reservoir characterization, we present in section 2 the reservoir models,
specifically a mathematical description of the forward model and the adjoint model
which we use during the inversion process. Section 3 introduces into our inverse problem
and gives a theoretical derivation of the basic algorithms which we use for the inversion
of the geological shapes and internal profiles. As part of this section, subsection 3.1
presents the theoretical derivation of the basic algorithm for the automatic identification
of geological regions and, in addition, smoothly varying permeability distributions inside
these regions. The subsection 3.2 explains the regularization scheme which we apply in
our work for the reconstruction of the level set functions which describe the interfaces between lithofacies. This regularization scheme basically assures that highly oscillating parts of the final interfaces are avoided. In subsection 3.3 we focus on parameterized internal permeability profiles. As a special case of that, in subsection 3.4 constant and linearly varying profiles are briefly addressed. In section 4 we present our technique for treating situations with more than two lithofacies. Here we introduce a new technique which uses \( n - 1 \) level set functions for describing \( n \) lithofacies. Section 5 discusses possibilities for adding objects with certain geometrical shapes to the reconstruction at a given step of the evolution. For adding such an object we apply a perturbation to the level set function which creates such an object without destroying the smoothness properties of the level set function. In section 6 we present then various numerical experiments which demonstrate the performance of the new techniques for simulated but realistic situations, and in section 7 we give some conclusions.

2. The reservoir characterization problem

2.1. The reservoir model

Our simplified model for two-phase flow in porous media for reservoir engineering is given as

\[
\phi \frac{\partial S_w}{\partial t} - \nabla \cdot \left[ T_w (\nabla p_w + \rho_w g k) \right] = Q_w \quad \text{in} \quad \Omega \times [0, t_f] \tag{1}
\]

\[
\phi \frac{\partial S_o}{\partial t} - \nabla \cdot \left[ T_o (\nabla p_o + \rho_o g k) \right] = Q_o \quad \text{in} \quad \Omega \times [0, t_f]. \tag{2}
\]

These two conservation laws for water (subscript \( w \)) and oil (subscript \( o \)), considered as incompressible fluids in a porous medium, are typically augmented by the two additional equations

\[
P_{cwo} = p_o - p_w; \tag{3}
\]

\[
S_w + S_o = 1. \tag{4}
\]

This yields four equations \( (1)-(4) \) in the four unknowns \( p_w, p_o, S_w \) and \( S_o \). Hereafter, the subindex ‘\( w \)’ stands for ‘water’, and the subindex ‘\( o \)’ stands for ‘oil’. Equation (3) links the water and oil pressures \( (p_w \) and \( p_o \), resp.) in the medium by the capillary pressure \( P_{cwo} \). Equation (4) links the saturations \( S_w \) of water and \( S_o \) of oil and indicates that the porous medium is fully saturated. Gravity effects are taken into account by the terms \( \rho_w g k \) and \( \rho_o g k \). These two terms, together with the capillary pressure \( P_{cwo} \), are incorporated in our forward modeling code, but are assumed to be small and will be neglected when deriving the algorithm for solving the inverse problem. \( \Omega \subset \mathbb{R}^n \) \( (n = 2, 3) \) is the modeling domain with boundary \( \partial \Omega \), and \( [0, t_f] \) is the time interval for which production data is available. We denote by \( \phi(x) \) the porosity, and by \( T_o, T_w \)
and $T$ the transmissibilities, which are known functions of the permeability $K$ and the water saturation $S_w$:

$$T_w = K(x) \frac{K_{rw}(S_w)}{\mu_w}; \quad T_o = K(x) \frac{K_{ro}(S_w)}{\mu_o}; \quad T = T_w + T_o.$$  (5)

Here, the relative permeabilities $K_{rw}(S_w)$ and $K_{ro}(S_w)$ are typically available as tabulated functions, and $\mu_w$ and $\mu_o$ denote the viscosities of each phase. $Q_o$, $Q_w$ and $Q = Q_o + Q_w$ define the oil flow, the water flow and the total flow, respectively, which are measured at the well positions. Equations (1)–(4) are solved with appropriate initial conditions, and a no-flux boundary condition on $\partial \Omega$.

When neglecting the gravity terms $\rho_o g k$ and $\rho_w g k$, as well as capillary pressure (such that $p_w = p_o = p$), equations (1)–(4) simplify to the two equations

$$-\nabla \cdot [T \nabla p] = Q \quad \text{in} \quad \Omega \times [0, t_f]$$  (6)

$$\phi \frac{\partial S_w}{\partial t} - \nabla \cdot [T_w \nabla p] = Q_w \quad \text{in} \quad \Omega \times [0, t_f]$$  (7)

for the two unknowns $p$ and $S_w$, where we add the following initial and boundary conditions

$$S_w(x, 0) = S_w^0(x) \quad \text{in} \quad \Omega;$$  (8)

$$p(x, 0) = p^0(x) \quad \text{in} \quad \Omega;$$  (9)

$$\nabla p \cdot \nu = 0 \quad \text{on} \quad \partial \Omega.$$  (10)

Here, $\nu$ is the outward unit normal to $\partial \Omega$. The boundary condition (10) implies no flux across the boundary. Equations (6)–(10) will be our basic model for deriving the shape inversion algorithm. $Q(x, t)$ and $Q_w(x, t)$ define the total flow and the water flow at the wells, respectively. They are given by

$$Q = c T \sum_{j=1}^{\hat{N}_i} (p^{(i)}_{wbj} - p) \delta(x - x^{(i)}_j) + c T \sum_{j=1}^{\hat{N}_p} (p^{(p)}_{wbj} - p) \delta(x - x^{(p)}_j)$$  (11)

$$Q_w = c T \sum_{j=1}^{\hat{N}_i} (p^{(i)}_{wbj} - p) \delta(x - x^{(i)}_j) + c T \sum_{j=1}^{\hat{N}_p} (p^{(p)}_{wbj} - p) \delta(x - x^{(p)}_j)$$  (12)

where $x^{(i)}_j, j = 1, \ldots, \hat{N}_i$, denote the locations of the $\hat{N}_i$ injector wells, $x^{(p)}_j, j = 1, \ldots, \hat{N}_p$, denote the locations of the $\hat{N}_p$ production wells, and $p^{(i)}_{wbj}, p^{(p)}_{wbj}$ are the imposed well bore pressures at the $\hat{N}_i$ injector wells and at the $\hat{N}_p$ production wells, respectively. Here, $c$ is a constant that depends on the well model [6]. Since $p^{(i)}_{wbj}$ are larger than the reservoir pressure at the injector wells, $Q$ and $Q_w$ are positive at the injector wells. Similarly, since $p^{(p)}_{wbj}$ are smaller than the reservoir pressure at the production wells, $Q$ and $Q_w$ are negative at the production wells.

In all the reservoir models discussed in this paper, there are two (incompressible) fluids in the reservoir, water and oil. Certainly, all the techniques developed here can
easily be generalized to more complex reservoir models, which we intend to address in our future research. We use tabulated values for the relative permeabilities $K_{rw}$ and $K_{ro}$ as shown in [11], which correspond to a Corey function with coefficients $n_w = 3$ and $n_o = 2$. The viscosity values for oil and water are $\mu_o = 0.79 \times 10^{-3}$ Pa s and $\mu_w = 0.82 \times 10^{-3}$ Pa s, and the porosity is taken to be constant $\phi = 0.213$ in the reservoir. The pressure values in the reservoir are in the range between 2000 psi (imposed pressure at production wells) and 3500 psi (imposed pressure at injection wells). The numerical physical time-step (which is unrelated to the time-step of the artificial shape evolution) used in the simulator is 0.1 days, and the reservoir is monitored over a period of 120 days. For more details regarding our reservoir simulation tools, we refer again to [11].

2.2. The forward problem

We can now introduce the forward operator of our problem. We write equations (6)-(12) in operator form as

$$\Lambda(K) u = q$$

with $u = (p, S_w)$ and where the right hand side $q$ is defined by the right hand sides of (6), (7), i.e., the expressions given in (11) and (12). Notice that the derivation of our algorithm is not restricted to the assumption of wells modeled by point sources, and that more complex descriptions of the wells (e.g., as interior boundary conditions) can easily be incorporated in the algorithm. We can define the forward operator $A$ mapping the parameter $K$ to the corresponding data $g = Mu$ by

$$A(K) = Mu(K) = M \Lambda(K)^{-1}q$$

where $M$ is the measurement operator given by

$$Mu(K) = \{Q^{(p)}_{w,j}(K)\}_{j=1,...,N_p} = Q_w(K),$$

being the water flow obtained at the production wells. Practically, calculating $\Lambda(K)^{-1}q$ means to run our reservoir simulator on the applied input pressure data with the permeability given as $K$. We will denote the physically measured ‘true data’ by

$$\tilde{g} = M\tilde{u},$$

where $\tilde{u}$ denotes the (unknown) physical state given the correct parameter distribution $\tilde{K}$. Finally, we introduce the ‘residual operator’ $R$ by defining

$$R(K) = A(K) - \tilde{g} = Q_w(K) - \tilde{Q}_w.$$
Let $\rho \in D$ be an arbitrary function in the data space. Then $\mathcal{R}'(K)^* \rho$ is given by

$$
\mathcal{R}'(K)^* \rho = \int_0^{t_f} \left( \frac{T_w}{K} \nabla p \nabla z - z \frac{1}{K} Q_w \right) dt
$$

where $z$ is the solution of the adjoint problem

$$
-\phi \frac{\partial z}{\partial t} + \frac{\partial T_w}{\partial S_w} \nabla p \nabla z - \left( z - \sum_{j=1}^{N_p} \rho \delta(x - x_j^{(p)}) \right) \frac{\partial Q_w}{\partial S_w} = 0 \quad \text{in } \Omega
$$

and $S_w$ and $p$ are the solutions of the forward problem (6)-(10), where both the forward and the adjoint problem are solved with permeability distribution $K$.

Notice that $Q_w$ is nonzero only at the well locations. Therefore, when we assume in the mathematical derivation of the theorem that the permeability is known directly at the wells (these values are available from well-log data), the second term in (18) disappears and we only have to evaluate the first term in order to calculate the update in the rest of the domain $\Omega$. This will be the approach we use in our numerical reconstructions. For more details we refer to [11].

3. The shape reconstruction problem (two lithofacies)

3.1. General internal profiles

We start the description of our new technique by the simplest case where two lithofacies are present in the reservoir. Classical history matching approaches typically formulate a least squares cost functional of the type

$$
J(K) = \frac{1}{2} \| \mathcal{R}(K) \|^2,
$$

possibly augmented by some additional regularization terms (e.g. of the Tikhonov-Philips type) and try to find a (typically smoothly varying over the whole domain) profile $K(x)$ which minimizes this cost.

In our strategy, the level set function $\psi$ will determine the interfaces between the two regions present in the reservoir, whereas the functions $K_i$ and $K_e$ represent the internal
profiles in each of these regions. Notice that all three functions $K_i$, $K_e$ and $\psi$ are defined on the entire reservoir, and that the function $\psi$ determines at each location which of the profiles $K_i$ or $K_e$ applies at this location. Therefore, in our extended inverse problem there are three functions to be determined instead of one in the classical formulation. Even though it seems a harder problem to find three functions now instead of just a single profile in the classical approach, we will see that we gain much flexibility with this extended approach. The increased dimensionality of the inverse problem can be dealt with by applying suitable regularization tools when determining the three unknown functions. We emphasize also that the classical case can be considered as a special case of our model where the level set function is positive (or negative) over the entire reservoir.

To solve the shape reconstruction problem, we will adopt a time evolution approach [25]. As a consequence, $\psi$, $K_i$ and $K_e$ will be functions of an artificial evolution time $t$,

$$
\frac{d\psi}{dt} = f(x, t, \psi, R) \\
\frac{dK_i}{dt} = h_i(x, t, \psi, R), \quad \frac{dK_e}{dt} = h_e(x, t, \psi, R).
$$

The goal is to determine the unknown terms $f$, $h_i$ and $h_e$ such that the cost defined in (23) decreases during the evolution. Using the one-dimensional Heaviside function, we have

$$
K = K(\psi, K_i, K_e) = K_eH(\psi) + K_i(1 - H(\psi)).
$$

Formal differentiation of the cost functional (23) with respect to the artificial time variable $t$ yields

$$
\frac{d\mathcal{J}}{dt} = \frac{d\mathcal{J}}{dK} \frac{\partial K}{\partial \psi} \frac{d\psi}{dt} + \frac{d\mathcal{J}}{dK_i} \frac{dK_i}{dt} + \frac{d\mathcal{J}}{dK_e} \frac{dK_e}{dt}
$$

by the chain rule. Here, the expression $\mathcal{R}'(K)^*\mathcal{R}(K)$ is the adjoint of the linearized residual operator (containing the sensitivities of the data with respect to the parameters) applied to the most recent mismatch in the data. It can be calculated efficiently by using the adjoint scheme outlined in section 2.3. The individual components of (27) can be identified as $\frac{dK}{d\psi} = (K_e - K_i)\delta(\psi)$, $\frac{dK_i}{dK_e} = 1 - H(\psi)$, and $\frac{dK}{dK_e} = H(\psi)$, where we have used that the derivative of the one-dimensional Heaviside function $H(\psi)$ is the one-dimensional Dirac delta function $\delta(\psi)$. We can select descent directions for the cost functional by defining

$$
f_{\text{SD}} = -C_1\chi_{NB}(\psi)(K_e - K_i)\mathcal{R}'(K)^*\mathcal{R}(K) \\
h_{i\text{SD}} = -C_2(1 - H(\psi))\mathcal{R}'(K)^*\mathcal{R}(K) \\
h_{e\text{SD}} = -C_3H(\psi)\mathcal{R}'(K)^*\mathcal{R}(K).
$$
This can be verified by plugging these expressions into equation (27) for \( \frac{dJ}{dt} \). Here, \( C_1, C_2, C_3 \) are some small (possibly time-dependent) positive constants (which also can be zero) modifying the evolution speed of these quantities individually. The function \( \chi_{NB}(\psi) \) has values 1 in a small neighborhood centered around the zero level set (the so-called 'narrowband') and zeros elsewhere. We call this indicator function the 'narrowband function' associated with the zero level set.

We will call the above expressions Steepest-Descent (SD) directions for our cost functional \( J \). Numerically, discretizing (24), (25) by a straightforward finite difference time discretization with time-step \( \tau > 0 \) yields at time \( t \) the update rules

\[
\psi(t + \tau) - \psi(t) = f_{\text{SD}},
\]

\[
K_i(t + \tau) - K_i(t) = h_{\text{isD}},
\]

\[
K_e(t + \tau) - K_e(t) = h_{\text{esD}}.
\]

Similar time discretization schemes will be used for the evolution equations derived further below in this paper without explicitly mentioning it each time we formulate such a scheme.

### 3.2. Regularization and smoothing

So far we have not really insisted in the fact that our level set function should be sufficiently smooth inside the domain of interest. In fact, the adjoint operator \( R'(K)^* \) (and with it the descent directions) have been calculated with respect to general \( L_2 \)-spaces. In our regularization approach developed in our work we inforce that the level set function describing the shapes are smoothly varying to a certain degree. In more details, let us assume that the level set function \( \psi \) is in the Sobolev space \( W_1(\Omega) \) defined by

\[
W_1(\Omega) = \{ \psi : \psi \in L_2(\Omega), \nabla \psi \in L_2(\Omega), \frac{\partial \psi}{\partial \nu} = 0 \text{ at } \partial \Omega \},
\]

\[
\langle v, w \rangle_{W_1(\Omega)} = \alpha \langle v, w \rangle_{L_2(\Omega)} + \beta \langle \nabla v, \nabla w \rangle_{L_2(\Omega)}
\]

with certain positive weighths \( \alpha \) and \( \beta \). In order to formally incorporate our regularization scheme in the algorithm, let us write (26) for fixed \( K_i \) and \( K_e \) as

\[
K = \Pi(\psi) = \begin{cases} 
K_i & \text{in } D \text{ where } \psi \leq 0, \\
K_e & \text{in } \Omega \setminus D \text{ where } \psi > 0.
\end{cases}
\]

Then, the residual operators \( R(K) \) and \( T(\psi) \) are given by

\[
R(K) = g(K) - \tilde{g}, \quad T(\psi) = R(\Pi(\psi)).
\]

Defining the least squares cost functional \( \tilde{J} \) as in (23), namely \( \tilde{J}(K) = \frac{1}{2} \| R(K) \|_2^2 \), and \( \tilde{J}(\psi) = \frac{1}{2} \| T(\psi) \|_2^2 \), we denote

\[
\text{grad}_{\tilde{J},L_2}(K) = R'(K)^* R(K), \quad \text{grad}_{\tilde{J},L_2}(\psi) = T'(\psi)^* T(\psi)
\]

the gradient directions of \( \tilde{J} \) with respect to \( K \) and \( \psi \), respectively. Notice that these gradient directions will depend on the choice of function spaces for the parameter.
functions $K$ and level set functions $\psi$, as indicated in the notation. This will be used when selecting our regularization scheme for the inversion. Formal differentiation by the chain rule yields $T'(\psi) = R'(\Pi(\psi))\Pi'(\psi)$. We have $\Pi'(\psi) = (K_e - K_i)\delta(\psi)$. However, the Dirac delta distribution $\delta(\psi)$ will be approximated by a suitable $L^2$-function. In our numerical implementations, we will use the narrowband function of thickness $d$ for that purpose. In other words, $\delta(\psi) \approx \chi_{\mathcal{B}_d(\Gamma)}$ with $\mathcal{B}_d(\Gamma) = \{x : \text{dist}(x, \Gamma) \leq d/2\}$ and $\chi_D$ denoting the characteristic function of the set $D$. We have

$$T'(\psi) = \Pi'(\psi)^R(\Pi(\psi))^\ast.$$  

Notice that $T'(\psi)^\ast$ maps into $L^2(\Omega)$ but not necessarily into $W^1_1(\Omega)$. In order to make sure that our updates and therefore the evolving level set function are in the smaller space of smooth functions $W^1_1(\Omega)$, we replace the adjoint operator $T'(\psi)^\ast$ by a new adjoint operator $T'(\psi)^{\circ}$ which maps back from the data space into this space $W^1_1(\Omega)$ with its associated weighted inner product $\langle v, w \rangle_{W^1_1(\Omega)}$. Here, $\alpha \geq 1$ and $\beta > 0$ are carefully chosen regularization parameters. Typical values chosen in our numerical experiments are for example $\alpha = 1$, $\beta = 0.08$. Following these lines we arrive at the new gradient directions

$$T'(\psi)^{\circ} = (\alpha I - \beta \Delta)^{-1} T'(\psi)^\ast, \quad \text{grad}_{\mathcal{D}, W^1_1} (\psi) = T'(\psi)^{\circ} T_j(\psi). \quad (35)$$

See, e.g., [8] for details. The positive definite operator $(\alpha I - \beta \Delta)^{-1}$ has the effect of 'projecting' the gradient $T'(\psi)^\ast T(\psi)$ from $L^2(\Omega)$ towards the smaller space $W^1_1(\Omega)$. In fact, different choices of the weighting parameters $\alpha$ and $\beta$ visually have the effect of 'smearing out' the unregularized updates to a different degree. In particular, high-frequency oscillations or discontinuities of the updates for the level set function are removed, which yields smoother level set functions and therefore shapes with more regular boundaries. We use this gradient direction in our reconstruction schemes discussed in this paper. See also the general discussions on regularization schemes led in [8]. A similar regularization scheme is also applied for the internal permeability profiles $K_i$ and $K_e$ in each region. For more details regarding this regularization technique for permeability profiles we refer to [11].

### 3.3. Parameterized internal profiles

So far the internal profiles $K_i$ and $K_e$ have been considered to be arbitrary functions. Often a-priori knowledge is available which restricts the choice to a smaller subset of parameterized functions according to a small set of basis functions. This can happen, for example, when the reservoir engineer has some prior information available describing the general trend of the parameters inside each region. Moreover, restricting the selection of these internal profiles to a smaller subset of parameterized functions has the effect of stabilizing the inversion, which makes such a choice an attractive alternative to smoothly varying internal profiles. In the following we show that parameterized internal profiles can be incorporated easily in our shape-based reconstruction technique.
For the theoretical development let us assume that the two internal profiles can be written in the parameterized form

\[ K_i(x, y) = \sum_{j=1}^{N_i} \alpha_j a_j(x, y), \quad K_e(x, y) = \sum_{k=1}^{N_e} \beta_k b_k(x, y), \]  

where \( a_j \) and \( b_k \) are our selected basis functions for each of the two domains \( D \) and \( \Omega - D \), respectively. In the inverse problem, we need to estimate now the level set function \( \psi \) and the weights \( \alpha_j \) and \( \beta_k \) which can reproduce the measured data in some sense. In order to obtain an (artificial) evolution of the unknown quantities \( \psi, \alpha_j \) and \( \beta_k \), we consider the following three general evolution equations for the level set function and for the weight parameters:

\[ \frac{d\psi}{dt} = f(x, t, \psi, \mathcal{R}), \]  
\[ \frac{d\alpha_j}{dt} = g_j(t, \psi, \mathcal{R}), \quad \frac{d\beta_k}{dt} = h_k(t, \psi, \mathcal{R}). \]

In the same way as before the goal is to define the unknown terms \( f, g_j \) and \( h_k \) such that the mismatch in the production data decreases during the evolution. For this purpose, we reformulate the cost functional now as

\[ J(K(\psi, \alpha_j, \beta_k)) = \frac{1}{2} \| \mathcal{R}(K(\psi, \alpha_j, \beta_k)) \|^2, \]

where \( \alpha_j \) denotes the weight parameters for region \( D \) and \( \beta_k \) denotes the weight parameters for region \( \Omega - D \). Formal differentiation of this cost functional with respect to the artificial time variable \( t \) yields, in a similar way as before, the descent directions [33]:

\[ f_{SD}(x) = -C_1 \chi_{NB}(\psi)(K_e - K_i)\mathcal{R}'(K)^*\mathcal{R}(K), \]  
\[ g_{jSD}(t) = -C(\alpha_j) \int_{\Omega} a_j(1 - H(\psi))\mathcal{R}'(K)^*\mathcal{R}(K)dx, \]  
\[ h_{kSD}(t) = -C(\beta_k) \int_{\Omega} b_k H(\psi)\mathcal{R}'(K)^*\mathcal{R}(K)dx, \]

where \( C_1, C(\alpha_j) \) and \( C(\beta_k) \) are constants which are used for steering the speed of evolution for each of the unknowns \( \psi, \alpha_j \) and \( \beta_k \) individually.

### 3.4. Constant or linearly varying internal profiles

Let us consider two special cases of parameterized profiles. To start with, we describe a linear permeability profile which can be applied in cases where a certain trend is expected for the physical parameters in one or both of the regions. Let us pick the profile inside region \( D \) as an example. Here we have, denoting \( x = (x, y) \),

\[ N_i = 3, a_1 = 1, a_2 = x, a_3 = y \Rightarrow K_i(x, y) = \alpha_1 + \alpha_2 x + \alpha_3 y. \]
Applying the theory presented above yields the descent directions:

\[
g_{1SD}(t) = - C_1 \int_D R'(K)^* R(K) \, dx \, dy
\]

\[
g_{2SD}(t) = - C_2 \int_D xR'(K)^* R(K) \, dx \, dy
\]

\[
g_{3SD}(t) = - C_3 \int_D yR'(K)^* R(K) \, dx \, dy,
\]

where the constants \(C_1, C_2\) and \(C_3\) steer the speed of evolution of each of the parameters.

The second example which we want to mention is in fact a special case of the just described linear profile. In many situations it is convenient to represent the permeability function in a given subregion by some average value, such that it does not depend on the location inside this region. This amounts to the choice of \(\alpha_2 = \alpha_3 = 0\) and \(C_2 = C_3 = 0\) in the linear profile. We will use this representation often in certain stages of our reconstruction schemes presented in this paper.

4. The shape reconstruction problem (more than two lithofacies)

In this part of the paper we present a new methodology for characterizing reservoirs with more than two lithofacies during an automatic history matching process. In the previous section 3.1 we have used one level set function in order to describe situations with two different lithofacies. The idea is now to use more than one level set function for describing more than two lithofacies.

Several techniques have been discussed in the literature for image segmentation problems and for some medical imaging problems where more than one level set function are used for describing regions with more than two values. We refer to [8] for an overview of these different techniques. In particular, in [19] a technique has been presented where \(n\) \((n > 1)\) level set functions are used for describing \(2^n\) different values in an inverse scattering experiment. The idea of that work is to encode the different regions by a combination of signs of the \(n\) describing level set functions. Each level set function can be either positive or negative, such that we arrive at \(2^n\) different combinations. Allowing for the possibility that the physical quantity (such as permeability) assumes identical values in two or more of these regions, situations can be modeled where at most \(2^n\) different values occur. The main drawback of this methodology in our application of reservoir characterization is caused by the way how the regions in the domain change value during the evolution of the \(n\) level set functions. For example, when two regions of different value merge together in this formulation such that the combination of signs of the various describing level set functions changes in the merging area, a third region is created whose value is completely unrelated to the values of the two merging regions. Its new value depends solely on the way how the different regions have been distributed at the beginning of the algorithm over the different sign combinations of the level set functions. Consequently, in our numerical experiments this technique did not converge well for the reservoir characterization problem, which motivated us to develop
an alternative technique as follows.

Instead of describing $2^n$ regions by $n$ level set functions, we use $n - 1$ level set functions for describing $n$ different regions. Since the number of different regions is typically small in reservoir characterization problems, and since the same forward and adjoint calculation is used in order to extract descent directions for all the level set functions simultaneously, there is practically no additional numerical cost involved in evolving these $n - 1$ level set functions compared to the evolution of just one level set function. We will describe our technique in the following using an example of $n = 4$ and assuming that the profile inside each region is constant. Extensions to more general cases are straightforward and can be done by employing the techniques described in the previous sections of this paper to each of these $n$ regions.

Given 4 level set functions $\psi_1, \ldots, \psi_4$, we define the parameter (permeability) distribution inside the reservoir by

$$K = K_1(1 - H(\psi_1))H(\psi_2)H(\psi_3) + K_2H(\psi_1)(1 - H(\psi_2))H(\psi_3) + K_3H(\psi_1)H(\psi_2)(1 - H(\psi_3)) + K_4H(\psi_1)H(\psi_2)H(\psi_3)$$

where, as already mentioned, the permeability values $K_\nu$, $\nu = 1, \ldots, 4$ are assumed constant inside each region. The four lithofacies are represented as

$$D_1 = \{ x, \psi_1 \leq 0 \text{ y } \psi_2 > 0 \text{ y } \psi_3 > 0 \}$$

$$D_2 = \{ x, \psi_2 \leq 0 \text{ y } \psi_3 > 0 \text{ y } \psi_1 > 0 \}$$

$$D_3 = \{ x, \psi_3 \leq 0 \text{ y } \psi_1 > 0 \text{ y } \psi_2 > 0 \}$$

$$D_4 = \{ x, \psi_1 > 0 \text{ y } \psi_2 > 0 \text{ y } \psi_3 > 0 \}$$.

In other words, a point in the reservoir corresponds to the lithofacies $D_l$, $(l = 1, \ldots, n-1)$ if $\psi_l$ has negative sign and all the other level set functions have positive sign. In addition, one lithofacie (which here is referred to as the 'background' lithofacie with index $l = n$) corresponds to those points where none of the level set functions has a negative sign. Notice that typically this definition does not yield a complete covering of the whole domain $\Omega$ by the four ($n$) lithofacies. Those regions inside the domain where more than one level set function are negative are recognized as so-called 'critical regions' and are introduced here for providing a smooth evolution from the initial guess to the final reconstruction. Inside these critical regions we apply an average permeability value of all those lithofacies (identified by those level set functions which have a negative sign inside the corresponding critical region) which 'contribute' to the critical region. In our numerical experiments these critical regions are allowed to be created and modified.
freely during the evolution for reducing the mismatch in the data, but ideally should be reduced or completely removed upon convergence of our shape evolution in order to arrive at final reconstructions which only show the four given lithofacies. For this purpose, we have tested two different strategies for handling these critical regions in our reconstructions as described further below.

In our example of \( n - 1 = 3 \) different level set functions, we calculate descent directions as follows. The cost functional assumes the form

\[
J(K(\psi_1, \psi_2, \psi_3)) = \frac{1}{2} \| R(K(\psi_1, \psi_2, \psi_3)) \|^2.
\]  

(49)

We look for a family of evolution laws which reduce this least squares cost simultaneously. These evolution laws are

\[
\begin{align*}
\frac{d\psi_1}{dt} &= f_1(x, t, \ldots); \\
\frac{d\psi_2}{dt} &= f_2(x, t, \ldots); \\
\frac{d\psi_3}{dt} &= f_3(x, t, \ldots),
\end{align*}
\]  

(50)

where \( f_1, f_2 \) y \( f_3 \) are the individual forcing terms. Formal differentiation of (49) with respect to the artificial time variable yields by the chain rule

\[
\frac{dJ}{dt} = \sum_{\nu=1}^{4} \frac{dJ}{dK} \frac{\partial K}{\partial \psi_\nu} \frac{d\psi_\nu}{dt}.
\]  

(51)

Using the standard inner products we get

\[
\frac{dJ}{dt} = \left\langle R'(K)^* R(K), \frac{\partial K}{\partial \psi_1} f_1 + \frac{\partial K}{\partial \psi_2} f_2 + \frac{\partial K}{\partial \psi_3} f_3 \right\rangle_{L^2}.
\]  

(52)

The individual components \( \frac{\partial K}{\partial \psi_\nu}, \nu = 1, 2, 3 \), can be calculated easily from (47) by formal differentiation, for example

\[
\begin{align*}
\frac{\partial K}{\partial \psi_1} &= K_1(-\delta(\psi_1))H(\psi_2)H(\psi_3) + K_2\delta(\psi_1)(1 - H(\psi_2))H(\psi_3) \\
&\quad + K_3\delta(\psi_1)H(\psi_2)(1 - H(\psi_3)) + K_4\delta(\psi_1)H(\psi_2)H(\psi_3) \\
&\quad + \frac{K_1 + K_3}{2} \delta(\psi_1)(1 - H(\psi_2))(1 - H(\psi_3)) \\
&\quad + \frac{K_1 + K_2}{2} (-\delta(\psi_1))H(\psi_2)(1 - H(\psi_3)) \\
&\quad + \frac{K_1 + K_2}{3} (-\delta(\psi_1))(1 - H(\psi_2))H(\psi_3) \\
&\quad + \frac{K_1 + K_2 + K_3}{3} (-\delta(\psi_1))(1 - H(\psi_2))(1 - H(\psi_3)).
\end{align*}
\]  

(53)

Therefore descent directions are given by

\[
f_l = -(R'(K)^* R(K)) \frac{\partial K}{\partial \psi_l}, \quad l = 1, 2, 3.
\]  

(54)

Replacing in (53), (54) the Dirac delta distribution at the interfaces by the corresponding narrowband functions, and smoothing the obtained expressions by the regularization operator discussed in subsection 3.2, we arrive at the descent directions which we use in our numerical experiments.

Following the above recipe, we will typically find a permeability distribution which minimizes the cost functional allowing during the evolution the creation and modification
of the critical regions. This strategy will help avoiding some local minima. However, in
the final steps of the reconstruction the idea is to reduce these critical regions as much
as possible in order to identify for each point in the domain one of the four lithofacies
to which this point corresponds. As already mentioned, we have tested two different
strategies in order to reduce or eliminate these critical regions during the final stage of
the evolution, as described in the following.

The first strategy consist of applying an additional penalty to the cost functional
for reducing the size of these critical regions. So, the expression for \( f_l \) is modified as
\[
\begin{equation}
\begin{aligned}
\nonumber f_l &\rightarrow f_l + P_l, \\
\end{aligned}
\end{equation}
\]
where \( P_l \) is given by
\[
\begin{equation}
\begin{aligned}
P_l &= \hat{C}_l \{(1 - H(\psi_l))(1 - H(\psi_m))H(\psi_n) \\
\nonumber &+ (1 - H(\psi_l))H(\psi_m)(1 - H(\psi_n)) \\
\nonumber &+ (1 - H(\psi_l))(1 - H(\psi_m))(1 - H(\psi_n)) \} \text{smoothed}
\end{aligned}
\end{equation}
\]
with \((l, m, n) \in \{(1, 2, 3), (2, 3, 1), (3, 1, 2)\}\) and \(0 < \hat{C}_l \ll 1\) are some small positive
constants. The subscript \textit{smoothed} indicates that we are smoothing these additional
terms in order to arrive at an evolution of the level set functions as \( H_1 \)-functions as
described in subsection 3.2. Since these additional contributions to the forcing terms
are positive valued inside the critical regions and fall rapidly towards zero outside of
them they tend to gradually reduce the size of these critical regions.

In the second strategy, the constant permeability values inside the critical regions
are not fixed as in the regions \( D_1-D_4 \), but, in the later stages of our evolution scheme,
evolve as constants following the descent directions (44) of section 3.4. This means that,
instead of eliminating the critical regions from the reservoir, we are aiming at modifying
the values inside these critical regions and the region boundaries until they approach
the correct lithofacie values and lithofacie boundary locations.

5. Topological perturbations

Often certain structures are present in a reservoir (as for example channels or barriers)
and the reservoir engineer has a very good idea about the existence and the form of these
structures. Therefore, it might be desirable to take the existence of such a structure
into account during the history matching process. In other cases, there might be the
possibility of such a structure but the existence and correct shape and orientation of
the structure need to be tested against production data. The level set technique which
we present in this paper offers the possibility of incorporating such prior knowledge into
the characterization process. In the following we present one possible way of testing the
presence and orientation of structures against production data. We will focus in our
numerical experiments on channel and barrier structures. The technique which we use
is based on topological perturbations of the reservoir. A similar technique has already
been used by our group in earlier work for treating low-sensitivity regions in an efficient
way [34], such that we will keep the formal derivation short and focus on the current application of structure detection and specification.

For the creation of artificial channels or barriers we define three disjoint regions $\Omega_1$, $\Omega_2$, and $\Omega_3$ such that $\Omega_3$ is just the region where we want to introduce the new channel or barrier, $\Omega_2$ is a small neighbourhood of this region (excluding $\Omega_3$ itself), and $\Omega_1$ is the rest of the domain $\Omega$. Therefore, we have $\Omega = \Omega_3 \cup \Omega_2 \cup \Omega_1$ with $\Omega_i \cap \Omega_j = \emptyset$ for $i \neq j$. See figure 1. We will furthermore introduce the following three characteristic functions

$$
\chi_{12}(x) = \begin{cases} 
1 & \text{in } \Omega_1 \cup \Omega_2 \\
0 & \text{outside } \Omega_1 \cup \Omega_2 
\end{cases}, \quad \chi_3(x) = \begin{cases} 
1 & \text{in } \Omega_3 \\
0 & \text{outside } \Omega_3 
\end{cases},
$$

$$
\chi_{23}(x) = \begin{cases} 
1 & \text{in } \Omega_2 \cup \Omega_3 \\
0 & \text{outside } \Omega_2 \cup \Omega_3 
\end{cases}.
$$

With these characteristic functions defined, we pose the task (when we want to add a certain structure into the domain) to minimize the following cost functional:

$$
\min \Xi(\Phi) = \frac{a}{2} \| \chi_{23} \nabla \Phi \|^2 + \frac{b}{2} \| \chi_{12}(\Phi - \Psi) \|^2 + \frac{c}{2} \| \chi_3(\Phi - \mu) \|^2,
$$

(57)

where the positive constants $a$, $b$ and $c$ are weighting parameters which can be chosen freely. In our numerical experiments, for example, we have made good experience with the values $a = 0.44$, $b = 1$, $c = 0.6$, $\mu = -200$ and $\gamma = 0.4$ (see equation (59)), but these values might change from case to case. The minimizer $\Phi$ will replace $\Psi$ as our level set function. Notice that the third term in (57) penalizes distance of $\Phi$ from the (negative) value $\mu$ inside the zone $\Omega_3$ where we want to introduce the channel or barrier. The second term penalizes the distance of the new function $\Phi$ to the old one $\Psi$ outside of $\Omega_3$, and the first term has the effect of smoothing the new level set function $\Phi$ inside of $\Omega_3$ and its neighborhood $\Omega_2$. The resulting level set function $\Phi$ is therefore expected to be a smoothly varying function (as it was $\Psi$) in the whole domain $\Omega$, to coincide approximately with $\Psi$ in the largest part of $\Omega$ (namely in $\Omega_1$) and to decrease in a smooth way towards the target value $\mu < 0$ inside of $\Omega_2$ such that inside of $\Omega_3$ a channel or barrier is created with value $\mu < 0$ for the level set function. See again figure 1 for an illustration of this change in the level set function. Notice that it is not necessary that the boundaries of the created channel or barrier are very exact since the following evolution of the level set function will modify it according to the production data.

In our numerical experiments the above described minimization problem is solved by employing a straightforward gradient method. As shown in [34] a descent direction for this cost functional is given by

$$
grad_{\Phi} \Xi(\Phi) = a \chi_{23} \Delta \Phi - b \chi_{12}(\Phi - \Psi) - c \chi_3(\Phi - \mu)
$$

(58)

such that the gradient scheme can be implemented as follows:

$$
\Phi^{(0)} = \Psi
$$

$$
\Phi^{(n+1)} = \Phi^{(n)} + \gamma \left[ a \chi_{23} \Delta \Phi^{(n)} - b \chi_{12}(\Phi^{(n)} - \Psi) - c \chi_3(\Phi^{(n)} - \mu) \right],
$$

(59)

with some step-size $\gamma$ (which also could have been absorbed in the weights $a$, $b$, $c$). Here we will assume that $\frac{\partial \Phi}{\partial n} = 0$ at the computational domain boundary $\partial \Omega$. 
6. Numerical experiments

In order to verify the performance of our shape based inversion algorithm using level sets for realistic examples we have investigated various test cases in two spatial dimensions. These will be discussed in this section. In all cases presented in this paper, the dimensions of the reservoir are 600 m by 600 m discretized into $25 \times 25$ grid cells. There are 9 production wells and 4 injection wells arranged as an array of 4 so-called five-spot patterns (one five-spot pattern consists here of one injection well surrounded by 4 production wells, see for example the upper row images of Fig. 2, where '+' corresponds to production wells and 'o' indicates injector wells). All physical parameters of the reservoir are assumed to be (approximately) known except of the topology of the individual regions and the internal permeability profiles. The remaining physical parameters are chosen as it has been described at the end of section 2.1. We mention here again that the data are created using a numerical streamline simulator, whereas an IMPES simulator is used throughout the reconstruction algorithm in order to avoid the inverse crime.

6.1. Reservoirs with two lithofacies: a hybrid case.

In this subsection we present numerical results of a situation where the reservoir consists of two lithofacies, one having a smoothly varying internal profile and the other one being better described by a parameterized profile. We will describe two different strategies, one being sequential and the other one simultaneous, for reconstructing this profile from production data.

In the top left image of figure 2 we present the reference model of our hybrid reservoir. The permeability in the yellow/red region (assumed to be sandstone in the range of 900 - 1300 mili-Darcy (mD)) does not follow any specifically defined parameterized form, whereas the permeability in the blue region (assumed to be shaly
sandstone in the range of 250 - 500 mD) follows a general linear trend. In more details, we assume that inside the blue region the correct reference permeability \( K_e(x, y) \) (in mili-Darcy) is given by the synthetic profile

\[
K_e(x, y) = 200 + 5 \frac{x}{\Delta x} + 5 \frac{y}{\Delta y} + 20 \sin \left( \frac{x}{\Delta x \pi} \right).
\]

(60)

Notice that this profile follows roughly a linear trend, but that it also contains a noise term \( 20 \sin \left( \frac{x}{\Delta x \pi} \right) \) which intends to model the imperfections of real world applications. In the reconstruction we expect from our algorithm that it is able to recover, in addition to the correct shapes and the smoothly varying profile inside the sandstone region, as well the parameters \( \beta_1 \approx 200, \beta_2 \approx 5 \) and \( \beta_3 \approx 5 \) modeling the profile inside the shaly sandstone region. These latter parameters, however, might be adjusted by the algorithm in order to account for the unknown term \( 20 \sin \left( \frac{x}{\Delta x \pi} \right) \) which does not exactly fit the linear profile. Moreover, since the complete inverse problem is underdetermined, it is also possible that inaccuracies in the recovered shape boundaries are compensated by slightly inexact values of these reconstructed linear profile parameters.

The initial guess for our algorithm is shown in the top right image of figure 2. This initial guess is constructed using well-log data, putting an initial average permeability value of shaly sandstone inside a small neighborhood of those wells which are located in a shaly sandstone region, and putting an initial average permeability value of sandstone in the rest of the reservoir. The corresponding initial level set function is a signed distance function for this initial profile.

6.1.1. First strategy

The first strategy is divided into four stages.

In the first stage we are looking for an average permeability value inside both regions (sandstone and shaly sandstone) as well as for a corresponding interface between these regions, see figure 2. As line search criterion for the change of the interface between the regions we aim at changing not more than five pixels in each step. This gives us a controlled evolution of the interfaces without additional computational cost. The first row of figure 2 shows on the left the reference permeability profile and on the right the initialization derived from well-log data. The second row shows on the left the final reconstruction of this first stage after 700 iterations and on the right the cost evolution. It can be observed that the cost roughly stabilizes after about 200-300 iterations, which could be used as stopping criterion. However, the final row shows the evolution of the mean permeability value inside the sandstone region (left) and the shaly sandstone region (right). Those values become practically stationary after iteration number 600. Therefore, we decided to choose as final reconstruction of this stage the estimate at iteration 700.

Once obtained a rough first estimate for the mean parameter values and the shape in the first stage, the second stage uses the prior information about the existing trend in the shaly sandstone region by applying a linearly parameterized profile in that region and determining the expansion parameters \( \beta_1, \beta_2 \) and \( \beta_3 \) from the data. As initialization of this linear profile inside the shaly sandstone region in this second stage we apply a least
squares parameter fitting of $\beta_1$, $\beta_2$ and $\beta_3$ to the well data which correspond to those wells which are located in the estimated shaly sandstone area of stage I. The remaining unknowns (the sandstone area and the interface) do not change during this stage. The evolution of the three parameters and of the cost when minimizing the mismatch to production data, as well as the final reconstruction of this stage II, can be seen in figure 3.

During the third stage of the algorithm we then focus on the determination of a smoothly varying profile (pixel-by-pixel updates following the Fréchet derivative) inside the sandstone region which minimizes the mismatch to the production data. Here we keep the reconstructed linear profile inside the shaly sandstone region and the interface constant. The evolution of the cost and the final reconstruction (after 700 iterations) of this third stage can be seen in figure 4. For more details on the pixel-based reconstruction scheme which is applied here inside the sandstone region we refer to the previous work in [11] where it has been used as a stand-alone tool for characterizing petroleum reservoirs.
from production data.

In the fourth stage of the inversion algorithm, all parameters of the inversion problem (i.e. the level set function, the three parameters inside the shaly sandstone region and the smoothly varying profile inside the sandstone region) are updated simultaneously in order to find a joint optimal solution which minimizes the given cost functional. Figure 5 shows on the top left the reference profile, on the top right the final reconstruction of stage III which is at the same time the initialization of this stage IV, on the bottom left the final reconstruction of stage IV, and on the bottom right the evolution of the cost during this last stage of the algorithm.
6.1.2. Second strategy  In this second strategy, all unknowns of the inversion problem are evolved simultaneously from the first iteration to the end of the evolution until the cost is minimized. In other words, compared to the previously explained first strategy, we skip here the initial three stages and start immediately with stage IV of that scheme using as initial guess the profile obtained from well-log data (see the top right image of figure 6, where in addition the least squares fit of the linear profile parameters to the permeability data measured at the well locations will be applied in the 'blue region' before starting the evolution). Certainly, this strategy is the more elegant approach for solving the inversion problem at hand. However, we mention that the application of simultaneous updates for all unknowns of this complex inversion problem right from the beginning requires a very careful calibration of the individual step-sizes with respect to each other. Otherwise it might result in an unstable or suboptimal evolution of the different unknowns. This risk is reduced in the previously demonstrated strategy of applying different stages of increasing complexity for the inversion.

In more details, in this second stage the reconstruction algorithm applies in each step the following updates simultaneously from just one forward and adjoint calculation: (i) the update for the level set function using formulas (24), (31), (35); (ii) the update for the internal permeability profile in the sandstone region which is done pixel by pixel using formulas (25), (29); (iii) the update for the three parameters \(\beta_1, \beta_2\) and \(\beta_3\) of the parameterized profile inside the shaly sandstone area using expressions (38), (42).

Figure 6 shows the results obtained using the simultaneous strategy for reconstructing this hybrid case. As mentioned, the initial permeability distribution
Figure 6. Hybrid case. Second strategy. Left column from top to bottom: reference model, final reconstruction and evolution of parameter values $\beta_1$, $\beta_2$, $\beta_3$; right column from top to bottom: initial guess, evolution of the least squares data misfit and the initial (red solid), final (black dashed) and reference (black solid) total water production rate in $m^3/s$. The complete evolution can be seen in the animated file DVmovie2.gif.

is displayed in the top right, and the reference profile can be seen on the top left. The second row displays the final reconstruction (left) and the evolution of the cost (right). The third row displays on the left the evolution of the parameters in the shaly sandstone region and on the right the total water production rate in $m^3/s$. On the right, the upper curve (in red) represents the water production rate for the initial guess, whereas the dashed line is the final water production rate (after 200 iterations) which coincides almost completely with the black solid line representing the reference total water production rate. Notice that the algorithm recovers good estimates not only for the shapes, but also for the medium profile parameters where the red solid curve corresponds to the parameter $\beta_1$ (being related to the scale displayed on the right of the image) and the black curves correspond to $\beta_2$ and $\beta_3$ (being related to the scale displayed on the left of the graph).

We want to mention at this point that in both strategies mentioned above the least squares fit of the linear profile parameters to those parameters measured at the well positions already yields good estimates for them as it could have been expected. These
estimated parameter values are only used in our algorithm so far as initial guesses, whereas the succeeding evolution of these parameters is driven solely by the production data without further consideration of the measured permeability values at the well locations. It is certainly possible to incorporate these measured permeability values at well locations as additional information (e.g. as ‘priors’) also during the parameter evolution in form of penalty terms to the cost functional which drives the evolution. Possible strategies for doing so will be addressed in our future research.

6.2. Reconstruction of more than two lithofacies

Next we present a numerical example to test the proposed new methodology for reconstructing more than two lithofacies. The reference model shown in figure 7 consists of four lithofacies with the four different permeability values 200 milli-Darcy (mD), 600 mD, 1000 mD and 2000 mD, respectively. The unknowns to be reconstructed in this example are the topologies and shapes of these four lithofacies. As explained in section

**Figure 7.** More than two lithofacies. Critical regions evolve freely with fixed average values. First row: Real permeability (left); initial guess (right). Central row (step size 7 pixels) and bottom row (step size 4 pixels): final reconstruction (left) and evolution of cost (right). The complete evolutions can be seen in the animated files DVmovie3a.gif and DVmovie3b.gif
the idea is now to evolve three level set functions simultaneously.

**Shape evolution without extra treatment of critical regions.**

First we investigate the performance of the algorithm as explained in section 4 by letting the critical regions evolve freely and allowing for final reconstructions which include these critical regions as ‘regions of uncertainty’.

We initialize the evolution using the permeability values at the well locations and applying them in a small neighbourhood of these well locations. The background value is the value which appears more than the others at the well locations (in this case 600 mD). See the top right image of the figure 7. The top left image of this figure shows the corresponding reference permeability profile. The first strategy consist of evolving the initial level set function from the initial guess without eliminating the critical regions using different step sizes. In the second and third rows of figure 7 are displayed the results obtained applying two different step size criteria for the level set function during the evolution process. The first step size criterion (second row) changes maximally 7 pixel values in each step of the evolution, whereas the second step size criterion (bottom row) allows for not more than 4 pixels to change value in each step. On the right hand side of the rows two and three, the evolution of the cost for these two step size criteria is shown. We observe that the change of 7 pixel values represents already quite a large step-size such that the cost is reduced significantly during the first iterations until it reaches a low value, but then it starts increasing slightly since large steps are enforced also at later iterations. Reducing the step size to the change of only 4 pixels restores the descent property of the evolution but decreases the cost at a slower rate. We mention that, additionally, during the evolution in the latter case (4 pixels changing) the level set functions are rescaled in each step such that their minimum values remain constant inside the domain which might have a small effect as well on the evolution. Which of the final reconstructions actually is the ‘better’ or ‘more useful’ one is difficult to say in this case.

**Shape evolution with simultaneous evolution of permeability values inside the critical regions.**

One possible strategy for handling the critical regions is to assign to them the average value as given in expression (47) only at the beginning of the evolution and then let these values evolve according to the theory presented in section 3.4 during the rest of the reconstruction together with the level set functions. The values inside the lithofacies $D_1, \ldots, D_4$ do not change during this evolution.

The results of the numerical experiments which we have performed in order to test this strategy are displayed in figure 8. We have tested four different cases.

In the top left image of the figure the real permeability distribution is displayed and in the top right the initial guess. In the remaining rows of the figure we show on the left four final reconstructions and on the right the corresponding evolutions of the cost. In more details, in the second row we apply a time-step criterion of 2 pixels changing in each step and in the third row of 4 pixels changing per step, both without
Figure 8. More than two lithofacies. Values inside critical regions are corrected in each step. First row: real permeability (left); initial guess (right). Rows two to five: final reconstruction on the left and evolution of cost on the right for several different cases. Row two: step size 2 without rescaling of level set function. Row three: step size 4 without rescaling. Row four: step size 2 with rescaling. Row five: step size 4 with rescaling. The complete evolutions can be seen in the animated files DVmovie4a.gif-DVmovie4d.gif.
Figure 9. More than two lithofacies. Values inside critical regions are corrected in each step and penalization of critical region is applied. First row: Real permeability (left); initial guess (right). Second row: final reconstruction (left) and evolution of cost (right). The complete evolution can be seen in the animated file DVmovie5.gif

re-scaling of the level set function. In rows four and five we show the corresponding results (in the same order as before) with re-scaling (the minimum is fixed to a value of $-200$) of the level set function in each step. It can be seen that the critical regions are reduced in all cases, but that the corresponding evolutions lead to slightly different reconstructions (which we might interpret as different 'local minima'). This is probably due to the severe ill-posedness of the problem and the sparsity of the data.

Shape evolution with additional penalization of area of critical regions.

This final example is chosen for showing how the penalty technique described in the expressions (55) and (56) performs for the elimination of critical regions during the evolution of more than one level set function. Notice that also the 'true reservoir' has been chosen here slightly different from the previous one. Compared to the previous example three wells are now situated in a different region such that they yield different well-logs and therefore the initial guess has changed.

In figure 9 we display the results obtained by applying this additional regularization technique. As before, on the top left the real permeability distribution is displayed and on the top right we show the initial guess. On the bottom the final reconstruction obtained by this technique (left) and the evolution of the cost (right) are displayed. Also here the critical region is practically removed in the final reconstruction and the overall reconstruction is satisfactory, even though some parts of the reservoir (with smaller sensitivity to the data) could not be characterized correctly. Again, this is probably due to the sparsity of the data and the ill-posedness of the inverse problem.
6.3. Reconstruction of channel or barrier structures

Here we present numerical examples for our strategy for testing the existence of certain geometrical shapes in the reservoir based on topological perturbations (see section 5). For simplicity we concentrate here on a reservoir which is assumed to have a piecewise constant permeability profile with unknown values. The goal of the algorithm is to recover the topology and shapes of the regions together with their correct permeability values. For this purpose, we apply the shape evolution algorithm based on level sets and the evolution of constant permeability values (as described in subsection 3.4) simultaneously until the algorithm converges. The result of this strategy is displayed in figure 10. In the first row of the figure the real or reference model (left) and the initial guess (right) based on well information are shown. The second row shows the final reconstruction (left) and the corresponding evolution of the cost (right). In the third row we show the evolution of the estimated permeability values in the sandstone (left) and shaly sandstone (right) region.

As it can be seen in the final reconstruction, the barrier which is present in the reservoir has not been recovered completely. The reservoir engineer might have some idea about the existence of a barrier in the reservoir and might therefore want to test if in fact the barrier can be confirmed by the production data and, if it exists, determine some of the geometrical properties (orientation, width, length) of the barrier. For this purpose, our algorithm provides the opportunity to add a barrier of arbitrary length, width and orientation to the reservoir at the location where the reservoir engineer assumes that it should be located. Practically the technique explained in section 5 is applied at some step of the algorithm for perturbing the current level set function such that the barrier is introduced. Then, the level set function continues its evolution starting out from the perturbed guess. The idea is that the added barrier will evolve and deform in the intent to satisfy the data. It might either change orientation, or disappear or just adjust its boundaries depending on the correctness of the assumptions which are incorporated in the added barrier. We mention at this point that a similar strategy has been applied in our previous work in order to handle objects hidden in areas of low sensitivity (for details see [34]).

The first numerical experiment tests the ideal situation where the guess of the reservoir engineer is correct and the seed barrier is put at the correct location with the correct orientation. Figure 11 shows the results in this case. On the top left the reference profile is shown, on the top right the initial guess (which is the final result of figure 10), on the bottom left the final reconstruction of this additional step, and on the bottom right the evolution of the cost during these additional iterations after adding the test barrier. Obviously, adding a test barrier with the correct location and orientation opens the way for the algorithm to arrive at an improved reconstruction. Notice that the final cost after this additional evolution is not significantly lower than the final cost of the previous step shown in figure 10. We interpret this fact by saying that adding the guess for the barrier helped us to get out of one (local) minimum and arriving at
Figure 10. Barriers, first stage. Shape and mean value reconstruction for each lithofacie. Reference permeability (top left); initial guess (top right); final reconstruction (central left); evolution of the cost functional (central right); evolution of the mean sandstone permeability value (bottom left) and of the mean shaly sandstone permeability value (bottom right). The complete evolution can be seen in the animated file DVmovie6.gif

In order to further test the algorithm (and the non-uniqueness of the underlying inverse problem), we have done a second numerical experiment where we add a seed barrier which has, in contrary to the reference barrier with a 45° inclination, a horizontal orientation. The results are shown in figure 12, with the same arrangement of images as before. Surprisingly, it can be observed that the horizontal barrier persists and, even worse, the cost value of the final reconstruction (showing a horizontal barrier) is lower than the final cost values of the previous two reconstructions. Also here we explain the result by the sparsity of the available production data and the ill-posedness of the inverse problem. The noise in the data can easily change the solution in regions of low sensitivity since the effect of these changes to the data is at the noise level.
Figure 11. Barriers, second stage: first numerical experiment. Top row: Real permeability (left); initial guess (right). Bottom row: Final reconstruction after adding an inclined barrier and evolving the shape (left); evolution of the cost functional in this second stage (right). The complete evolution can be seen in the animated file DVmovie7.gif

Figure 12. Barriers, second stage: second numerical experiment. As in figure 11, but after adding a horizontal seed barrier. The complete evolution can be seen in the animated file DVmovie8.gif
7. Conclusions

We have introduced in this paper a new reconstruction scheme for the history matching problem of reservoirs with an arbitrary number of lithofacies using a level set technique. In each of the lithofacies a smoothly varying permeability profile (either pixel-by-pixel or parameterized) can be assumed and is reconstructed as well from the data. In this sense, our new algorithm is a generalization of classical reservoir characterization algorithms which do not incorporate sharp interfaces between lithofacies in the reservoir (which in our algorithm amounts to the special case where we assume that the entire reservoir contains only one region). Our numerical experiments have addressed the situation of reservoirs with two lithofacies and a hybrid arrangement of internal permeability profiles as well as reservoirs with more than two (here four) lithofacies assuming piecewise constant permeability profiles. The numerical results from synthetically created data (using a simulator completely independent from the simulator used in the reconstruction process and adding a sufficient amount of modelling and data noise) for realistic situations convince us that the algorithm provides very good estimates of structured reservoirs. In addition, we have tested here the use of topological perturbations for verifying the existence of certain geometrical objects (such as channels or barriers) in the reservoir. We believe that any combination of these basic building blocks can be applied without problems for characterizing real reservoirs and that, putting them all together, will provide the reservoir engineer with a powerful alternative reservoir characterization tool with very attractive additional features compared to the already existing classical tools. The algorithm can be extended easily to more realistic 3D situations as long as reservoir simulators for the forward and the adjoint problem are available. These simulators can be integrated in the reconstruction algorithm as black box solvers, such that the favourite tool of the reservoir engineer for simulating a given reservoir can be applied without problems during the characterization process.

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References


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