Bernoulli convolutions and beta-expansions

Workshop abstracts

Boris Solomyak (Seattle), *The transversality method, an overview*

Abstract. The "transversality method" was developed by several authors in the last 10 years, to study families of fractal sets and measures. In particular, it was used to prove the absolute continuity of Bernoulli convolutions for a.e. parameter (in the natural region) and estimate the dimension of the exceptional set. I will survey the highlights of this development.

Hajnal Toth (Budapest), *The absolute continuity of the distribution of random sums with digits* $\{0, 1, \ldots, m-1\}$

Abstract. Let $m \geq 2$ be a natural number. Let ν_λ^m be the distribution of the random sum $\sum_{n=0}^{\infty} \theta_n \lambda^n$, where θ_n are i.i.d. and for every n the random variable θ_n takes value in the set $\{0, \ldots, m-1\}$ with equal probabilities. As a generalization of Solomyak Theorem we prove that for Lebesgue a.e. $\lambda \in (1/m, 1)$ the measure ν_λ^m is absolute continuous w.r.t. the Lebesgue measure. (For smaller λ, the measure ν_λ^m is supported by a Cantor set, so if $\lambda < 1/m$ then ν_λ^m is singular.)

Alain Thomas (Marseille), *Some multifractal properties of the Erdős measure and generalisations*

Abstract. The singularity spectrum and the scale spectrum of a measure are image - the one from the other - by the Legendre transform. We give some properties of analyticity, or piecewise linearity, of both maps in the case of the Erdős measure.

Nikita Sidorov (Manchester), *Multidimensional beta-expansions and golden Sierpiński gaskets*

Abstract. Let $\boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{p}_3$ be three points in $\mathbb{R}^2$. We consider the following generalisation of the one-dimensional beta-expansions:

$$\boldsymbol{x} = \sum_{n=1}^{\infty} \lambda^n \boldsymbol{a}_n,$$

where $\lambda \in (0, 1)$ and $\boldsymbol{a}_n$ is one of the vertices $\boldsymbol{p}_i$. Denote by $\mathcal{S}_\lambda$ the set of all points $\boldsymbol{x}$ which admit at least one representation of this form. (Notice that for $\lambda = \frac{1}{2}$ it is just the Sierpiński gasket.)

The aim of my talk is to study some geometric properties of the set $\mathcal{S}_\lambda$ for $\lambda \in \left(\frac{1}{2}, \frac{2}{3}\right)$–*i.e.*, the "interesting region". This work is joint with D. Broomhead and J. Montaldi.

Thomas Jordan (Manchester) *Fat Sierpiński carpets*

Abstract. We study a generalisation of the well known Sierpinski carpets where the images of the similarities overlap. We show how transversality combined with a theorem of Marstrand can be used to obtain limited results on the Lebesgue measure and the Hausdorff dimension of these sets. This is joint work with my PhD supervisor Mark Pollicott.

Paola Loreti (Rome), *On unique expansions in non-integer bases*

Abstract. This talk deals with a result obtained in collaboration with V. Komornik (Amer. Math. Monthly 1998).

Let $1 < q < 2$ be a real number. We consider sequences (ε_i) of zeros and ones leading to an expansion of the number 1:

$$1 = \sum_{i=1}^{\infty} \varepsilon_i q^{-i}.$$

For some particular values of q there is only one such expansion. We look for the smallest such number q. Thanks to an algebraic characterization of the sequences leading to unique expansions, this corresponds to the lexicographically smallest sequence satisfying (in the lexicographic order)

$$\varepsilon_{n+1}\varepsilon_{n+2}\cdots < \varepsilon_1\varepsilon_2\ldots \quad \text{whenever} \quad \varepsilon_n = 0$$

and

$$\overline{\varepsilon_{n+1}\varepsilon_{n+2}\ldots} < \varepsilon_1\varepsilon_2\ldots \quad \text{whenever} \quad \varepsilon_n = 1,$$

where $\overline{\varepsilon_i}$ denotes $1 - \varepsilon_i$.

Vilmos Komornik (Strasbourg), *On the topological structure of univoque sets*

Abstract. We present a joint work with Paola Loreti. Erdős, Horváth and Joó discovered some years ago that for some real numbers $1 < q < 2$ there exists only one sequence c_i of zeros and ones such that $\sum c_i q^{-i} = 1$. Subsequently, the set U of these numbers was characterized algebraically. We establish an analogous characterization of the closure $\overline{U}$ of U. This allows us to clarify the topological structure of these sets: $\overline{U} \backslash U$ is a countable dense set of $\overline{U}$, so the latter set is perfect. Moreover, since U is known to have zero Lebesgue measure, $\overline{U}$ is a Cantor set.

Marco Pedicini (Rome), *Greedy expansions and sets with deleted digits*

Abstract. We generalize a result of Daróczy and Kátai, on the characterization of univoque numbers with respect to a non-integer base by relaxing the digits alphabet to a generic set of real numbers. We apply the result to derive the construction of a Büchi automata accepting all and only the greedy sequences for a given base and digit set.

Paul Glendinning (Manchester), *Dynamics and expansions in non-integer bases*

Abstract. To appear.

Boris Solomyak (Seattle), *A family of IFS in the plane and complex Bernoulli convolutions*

Abstract. We consider the iterated function system $\{\lambda z - 1, \lambda z + 1\}$ in the complex plane, where λ is a complex number less than one in modulus. We study the dimension and topological properties of the attractor, which is precisely the support of the complex Bernoulli convolution with parameter λ. The connectedness locus for this family (sometimes called the "Mandelbrot set" for the pair of linear maps) has been studied by several authors, among them Barnsley, Bousch, and Bandt. We will describe recent progress in the study of this set; in particular, we answer a question of Bandt on local asymptotic similarity of attractors to the connectedness locus near certain points (the analogs of Misiurewicz points from complex dynamics).

Karma Dajani (Utrecht), *Measures of maximal entropy for random beta-expansions*

Abstract. We consider expansions of real numbers in non-integer bases, and digits generated by means of a random map K. We study the dynamics of K, and show that all expansions, in a given non-integer base, can be obtained by means of the map K. Furthermore, we prove that K has a unique measure of maximal entropy with marginal measure a generalization of the well-known Erdős measure. Furthermore, under the measure of maximal entropy the sequence of digits is a uniform Bernoulli process. For a certain class of bases, we show that the measure of maximal entropy is a Markov measure. This is joint work with Martijn de Vries.

Martijn de Vries (Utrecht) *Invariant densities for random beta-expansions*

Abstract. Let $\beta > 1$ be a non-integer. We consider β-expansions of the form $\sum_{i=1}^{\infty} \frac{d_i}{\beta^i}$, where the digits $(d_i)_{i\geq 1}$ are generated by means of a random map K_β defined on $\{0,1\}^{\mathbb{N}} \times [0, \lfloor \beta \rfloor/(\beta-1)]$. We show existence and uniqueness of an absolutely continuous K_β-invariant probability measure w.r.t. $m_p \times \lambda$, where m_p is the Bernoulli measure on $\{0,1\}^{\mathbb{N}}$ with parameter p $(0 < p < 1)$ and λ is the normalized Lebesgue measure on $[0, \lfloor \beta \rfloor/(\beta-1)]$. Furthermore this measure is of the form $m_p \times \mu_{\beta,p}$, where $\mu_{\beta,p}$ is equivalent with λ. In the special case that 1 has a finite greedy expansion with positive coefficients we will show that $m_p \times \mu_{\beta,p}$ and the measure of maximal entropy for K_β are mutually singular. The results are based on joint work with Karma Dajani.

Dave Broomhead (Manchester) *Digital communications and the Bernoulli convolution*

Abstract. We analyse a simple model of a digital communications channel. This model proves to be closely related to an iterated function system (IFS) related to the Bernoulli convolution. We derive it from a randomly forced first-order ordinary differential equation. This allows the parameter of the Bernoulli convolution—the contraction rate, λ—to be related to the rate at which symbols are input to the channel. It is shown that for a channel with equiprobable binary inputs the mutual information between input and output distributions is the stationary measure of the complement of the overlap region of the IFS. We show that the mutual information is Hölder continuous with respect to λ and decreases hyper-exponentially as $\lambda \to 1$. We also study the case of non-equiprobable binary inputs and show that the maximum of the mutual information—the channel capacity—does not always correspond to equiprobable inputs. This talk is based on a joint work with Nikita Sidorov.