

## A topological calculus for formal power series

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ABSTRACT. We propose geometric models for performing computations with formal power series over a commutative ring, including reciprocation, substitution, reversion, and Lagrange inversion. The models are based on a family of complex Bott-Samelson varieties which may be realized as manifolds of flags satisfying appropriate restrictions. We discuss the relationship of the geometric computations with multiple complex cobordism theory, focussing on the dual of the Landweber-Novikov algebra and raising delicate issues concerning the construction of explicit cobordisms. We outline extensions of the calculus to Hurwitz series, appealing to the Faà di Bruno algebra of algebraic combinatorics.

### 1. Introduction

The methods of formal power series have permeated algebraic topology since the work of Hirzebruch in the 1950s, and have often centered around cobordism theory. The applications became more specific following the work of Novikov and Quillen on the relationship between formal group laws and complex cobordism theory in the 1960s, and have motivated many recent developments in stable homotopy theory [14]. At their heart lies the interpretation of Landweber-Novikov operations in terms of automorphisms of the universal formal group law.

Our purpose here is to develop geometrical aspects of the dual of the Landweber-Novikov algebra in this context. With the help of certain Bott-Samelson varieties, interpreted as *bounded flag manifolds* in [4], we introduce geometrical models which codify the substitution of one formal power series in another, as well as the computation of the substitutional inverse (or *reversion*) whenever it exists. One component of the models plays a corresponding rôle for products and reciprocals. Several of the underlying ideas are rooted in algebraic combinatorics, especially the coalgebraic framework of [16], and have been applied in [4] to the study of quantum doubling constructions [13] in algebraic topology.

Our project has been stimulated by growing interest amongst combinatorialists and theoretical physicists [12] in the Landweber-Novikov algebra, and by positive feedback to the author's recent lectures on the subject. The audience at the University of Wales 1996 Summer School in Gregynog deserve special mention in this

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regard, particularly Francis Clarke and Peter Johnson; the topologists at Northwestern offered similar encouragement, crucially augmented by conversations with André Joyal. Much of the material here extends work with Victor Buchstaber [4] and William Schmitt [16], to whom the author offers his sincerest thanks for the pleasure of their collaboration, the depth of their insight, and the permission to include joint ideas.

We have attempted to make our presentation reasonably self-contained by summarizing relevant results from [4] in §§3 and 4. For readers who seek background information in algebra, combinatorics, and geometry, we recommend the classic books by Kassel [11], Aigner [2], and Griffiths and Harris [8] respectively.

## 2. Formal power series

We begin by establishing notation and conventions for formal power series, basing our treatment on that of Henrici [9].

We assume given a commutative ring  $K$  with identity, known as the *scalars*, and consider the ring of formal power series  $K[[T]]$  in the variable  $T$  over  $K$ . Any such power series  $\sigma(T)$  consists of a sequence of scalars  $(\sigma_0, \sigma_1, \dots)$ , presented as the infinite sum

$$(2.1) \quad \sigma_0 + \sigma_1 T + \dots + \sigma_k T^k + \dots .$$

We emphasise that the coefficients  $\sigma_k$  may well be algebraically related in  $K$ , and that issues of convergence do not arise. Addition is taken termwise, and the product  $\rho(T) \cdot \sigma(T)$  is given by the Cauchy product of the corresponding sequences. In the context of (2.1) this amounts to multiplying the appropriate sums and collecting together powers of  $T$ , and is clearly commutative. The ring of scalars is embedded in  $K[[T]]$  as the subring of constant series, in which guise the identity 1 of  $K$  doubles as the identity of  $K[[T]]$ . In fact  $K[[T]]$  is a  $K$ -algebra, and contains the polynomial algebra  $K[T]$  as subalgebra.

We may represent each series  $\sigma(T)$  by a matrix  $C(\sigma)$ , whose  $(j, k)$  entry consists of  $\sigma_{k-j}$  whenever  $j \leq k$ , and 0 otherwise, for all  $j, k \geq 0$ . Any matrix of this form is known as *semicirculant*, and the set of all such matrices forms a subring  $\text{SC}(K)$  of the ring  $\text{UT}(K)$  of upper triangular matrices, with respect to matrix multiplication. The matrix  $C(\rho \cdot \sigma)$  of a product of power series is given by the product of matrices  $C(\rho)C(\sigma)$ , so the function  $\sigma(T) \mapsto C(\sigma)$  defines an isomorphism  $K[[T]] \rightarrow \text{SC}(K)$  of  $K$ -algebras.

Every diagonal element of  $C(\sigma)$  is  $\sigma_0$ , so that the matrix is invertible when and only when  $\sigma_0$  is invertible in  $K$ . In consequence,  $\sigma(T)$  admits a multiplicative inverse, or *reciprocal*, when and only when its constant term is invertible. The reciprocal  $1/\sigma(T)$  may therefore be obtained by the same recursive procedure as is used for inverting matrices.

We may choose  $K$  to be a polynomial algebra  $\mathbb{Z}[\alpha, \beta]$ , generated by two sequences of indeterminates  $(\alpha_0, \alpha_1, \dots)$  and  $(\beta_0, \beta_1, \dots)$ , where  $\alpha_0$  and  $\beta_0$  are both invertible and usually taken to be 1. We write  $\alpha(T) = \sum_{n \geq 0} \alpha_n T^n$  and  $\beta(T) = \sum_{n \geq 0} \beta_n T^n$ , noting that the coefficients  $\alpha_n$  and  $\beta_n$  are algebraically independent by construction. Then  $\mathbb{Z}[\alpha, \beta]$  becomes the *universal example* of a ring equipped with the product of two invertible formal power series, since computations with  $\alpha(T)$  and  $\beta(T)$  over  $\mathbb{Z}[\alpha, \beta]$  may be mapped to computations with  $\rho(T)$  and  $\sigma(T)$

over  $K$  under the ring homomorphism defined by  $\alpha_n \mapsto \rho_n$  and  $\beta_n \mapsto \sigma_n$ , irrespective of any relations which hold amongst the  $\rho_n$  and  $\sigma_n$ . If we are interested only in reciprocation, it suffices to consider  $\beta(T)$  over  $\mathbb{Z}[\beta]$ .

We now concentrate on those power series which lie in the principal ideal  $(T)$ , and therefore have  $\sigma_0$  zero; we denote such series generically by  $s(T)$ . For each  $p \geq 1$  we observe that the  $p$ th power again lies in the ideal, and write it as

$$(2.2) \quad s(T)^p = \sum_q s_{p,q} T^q,$$

observing that the  $s_{p,q}$  are polynomial expressions in  $s_1, \dots, s_{q-p}$  (closely related to the partial Bell polynomials [6]); in particular,  $s_{p,q}$  is zero for all  $q < p$ , and  $s_{p,p} = s_1^p$ . We may combine two such series by the operation of *substitution*, which we denote by  $r(s(T))$ , or  $r \circ s(T)$ . This is evaluated using the  $K$ -algebra operations in  $K[[T]]$ , and is not generally commutative. Clearly  $r(s(T))$  is again divisible by  $T$ , which itself acts as a two sided compositional identity.

We represent each  $s(T)$  by a matrix  $M(s)$ , whose  $(p, q)$  entry is  $s_{p,q}$  for all  $p, q \geq 1$ . Of course  $M(s)$  is upper triangular, and has increasing powers of  $s_1$  down the diagonal. Straightforward computation reveals that  $M(r \circ s)$  is the matrix product  $M(r)M(s)$ , so that the function  $s(T) \mapsto M(s)$  gives a faithful representation of the semigroup  $(T)$  in  $\text{UT}(K)$ ; this representation seems to be due to Jabotinski [10]. It immediately implies that substitution is associative, which is an awkward calculation from first principles, and that  $s(T)$  admits a substitutional inverse when and only when  $s_1$  is invertible in  $K$ . We describe such a series as *reversible*, and refer to its inverse as the *reversion* of  $s(T)$ , written  $s^{(-1)}(T)$ . The set of reversible formal power series over  $K$  therefore forms a nonabelian subgroup of  $\text{UT}(K)$ , in which  $M(s^{(-1)})$  is given by  $M(s)^{-1}$ .

We shall focus our attention on the subgroup  $\text{SR}(K)$  of *special* reversible series, for which  $s_1$  is always 1. In any such series we reindex  $s_{n+1}$  as  $s_n$  for all  $n \geq 0$ , so that  $s(T)$  assumes the form  $T + s_1 T^2 + \dots + s_n T^{n+1} + \dots$ . The *Lagrange inversion formula* originated in analytic function theory, and describes  $s^{(-1)}(T)$  in terms of the reciprocal of  $s(T)/T$  and its powers. It has spawned an enormous literature over the last 35 years, and been rediscovered many times.

Following our lead above, we may choose  $K$  to be a polynomial algebra  $\mathbb{Z}[a, b]$  generated by two sequences of indeterminates  $(a_0, a_1, \dots)$  and  $(b_0, b_1, \dots)$ , with identity  $a_0 = b_0 = 1$ ; we write  $a(T) = \sum_{n \geq 0} a_n T^{n+1}$  and  $b(T) = \sum_{n \geq 0} b_n T^{n+1}$ . The indeterminates  $a_n$  and  $b_n$  are algebraically independent by construction, and  $\mathbb{Z}[a, b]$  becomes the universal example of a ring equipped with one formal power series substituted in another. If we are interested only in reversion, we may restrict our attention to  $b(T)$  over  $\mathbb{Z}[b]$ .

It is instructive to compute the upper left hand corners of the matrices  $M(b)$  and  $M(a \circ b)$ , obtaining

$$(2.3) \quad \begin{pmatrix} 1 & b_1 & b_2 & b_3 & \dots \\ 0 & 1 & 2b_1 & 2b_2 + b_1^2 & \dots \\ 0 & 0 & 1 & 3b_1 & \dots \\ 0 & 0 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

and

$$(2.4) \quad \begin{pmatrix} 1 & a_1 + b_1 & a_2 + 2a_1b_1 + b_2 & a_3 + 3a_2b_1 + a_1(2b_2 + b_1^2) + b_3 & \dots \\ 0 & 1 & 2(a_1 + b_1) & 2a_2 + a_1^2 + 6a_1b_1 + 2b_2 + b_1^2 & \dots \\ 0 & 0 & 1 & 3(a_1 + b_1) & \dots \\ 0 & 0 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

respectively; similarly,  $M(b^{\langle -1 \rangle})$  yields

$$(2.5) \quad \begin{pmatrix} 1 & -b_1 & 2b_1^2 - b_2 & -5b_1^3 + 5b_1b_2 - b_3 & \dots \\ 0 & 1 & -2b_1 & 5b_1^2 - 2b_2 & \dots \\ 0 & 0 & 1 & -3b_1 & \dots \\ 0 & 0 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

For general  $K$ , the matrices  $M(s)$ ,  $M(r \circ s)$ , and  $M(s^{\langle -1 \rangle})$  arise from (2.3), (2.4), and (2.5) by substituting  $r$  for  $a$  and  $s$  for  $b$ .

### 3. Bounded flag manifolds

We now turn to geometry, introducing bounded flag manifolds and their associated Boolean algebras  $\mathcal{X}(n)$  of nonsingular subvarieties.

We follow combinatorial convention by writing  $[n]$  for the set of natural numbers  $\{1, 2, \dots, n\}$ , which becomes a partially ordered set (or *poset*, for short) when equipped with the standard linear ordering  $<$ . Every *interval* in  $[n]$  has the form  $[v, w]$  for some  $1 \leq v \leq w \leq n$ , and consists of all integers  $m$  satisfying  $v \leq m \leq w$ ; our convention therefore dictates that  $[1, w]$  and  $[w]$  are synonymous. It is sometimes convenient to interpret  $[0]$  as the empty set, and  $[\infty]$  as the natural numbers.

We work with the *Boolean algebra*  $2^{[n]}$  of finite subsets  $Q \subseteq [n]$ , which is a poset when ordered by inclusion; we write  $|Q|$  for the cardinality of any such  $Q$ . For all  $n \leq m$ , the natural inclusions  $[n] \subseteq [m]$  induce a direct system of embeddings  $2^{[n]} \subseteq 2^{[m]}$ , whose existence ensures that a finite set  $Q \subseteq [\infty]$  may be assumed to lie in a specific interval  $[m]$ , for any  $m$  sufficiently large. For example, there are unique, mutually inverse order-preserving bijections  $\iota_Q: [\infty] \setminus Q \rightarrow [\infty]$  and  $\iota^Q: [\infty] \rightarrow [\infty] \setminus Q$ , which restrict to bijections  $[m] \setminus Q \rightarrow [m - |Q|]$  and  $[m - |Q|]$  into  $[m] \setminus Q$  respectively, for any  $m$  sufficiently large. We decompose each  $Q$  into maximal subintervals of the form  $I(1) \cup \dots \cup I(t)$ , where  $I(j) = [v(j), w(j)]$  for  $1 \leq j \leq t$ , and the decomposition is valid in any interval  $[m] \supseteq Q$ . It is occasionally convenient to let  $I(j)^+$  denote the interval  $[v(j), w(j) + 1]$ .

We work in an ambient complex inner product space  $Z_{n+1}$ , which we assume to be invested with a preferred orthonormal basis  $z_1, \dots, z_{n+1}$ , and write  $Z_E$  for the subspace spanned by the vectors  $\{z_e : e \in E\}$ , given any  $E \subseteq [n + 1]$ . We abbreviate  $Z_{[w]}$  to  $Z_w$  for each  $1 \leq w \leq n + 1$ , and write  $CP(Z_E)$  for the projective space of lines in  $Z_E$ . We let  $U - T$  denote the orthogonal complement of  $T$  in  $U$  for any subspaces  $T < U$  of  $Z_{n+1}$ , and abuse notation by writing  $0$  for the subspace which consists only of the zero vector. A *bounded flag*  $U$  in  $Z_{n+1}$  is a sequence of proper subspaces

$$0 = U_0 < U_1 < \dots < U_i < \dots < U_n < U_{n+1} = Z_{n+1},$$

in which  $U_i$  contains the subspace  $Z_{i-1}$ , for every  $1 \leq i \leq n+1$ . The *standard flag*  $Z_0 < \cdots < Z_i < \cdots < Z_{n+1}$  is therefore bounded.

We define the *bounded flag manifold*  $B(Z_{n+1})$  to consist of all bounded flags in  $Z_{n+1}$ , for each  $n \geq 0$ ; it is a nonsingular complex projective algebraic variety, and an  $n$ -dimensional subvariety of the usual flag manifold  $F(Z_{n+1})$ . We refer to [4] for further details, remarking that  $B(Z_{n+1})$  was originally introduced by Bott and Samelson [3] as a homogeneous space, and subsequently described in [15] as an iterated sphere bundle. By way of example we note that  $B(Z_1)$  consists solely of the trivial flag, whilst  $B(Z_2)$  is isomorphic to the projective line  $CP(Z_2)$  with standard complex structure.

To each bounded flag  $U$  in  $Z_{n+1}$  we assign the *support*  $S(U)$ , given by  $\{j \in [n] : U_j \neq Z_j\}$ , and consider the subspace

$$(3.1) \quad e_Q = \{U \in B(Z_{n+1}) : S(U) = Q\}$$

for each subset  $Q \subseteq [n]$ ; for example,  $e_\emptyset$  consists solely of the standard flag. Then  $e_Q$  is an open topological  $2|Q|$ -cell, whose closure in  $B(Z_{n+1})$  is

$$X_Q = \{U \in B(Z_{n+1}) : S(U) \subseteq Q\} = \bigcup_{R \subseteq Q} e_R.$$

In fact  $X_Q$  is the image of an algebraic embedding  $i_Q : \times_j B(Z_{I(j)^+}) \rightarrow B(Z_{n+1})$ , defined by choosing the  $k$ th component of  $i_Q(U(1), \dots, U(t))$  as

$$\begin{cases} Z_{v(j)-1} \oplus U(j)_i & \text{if } k = v(j) + i - 1 \text{ in } I(j) \\ Z_k & \text{otherwise,} \end{cases}$$

for  $1 \leq k \leq n+1$ . We deduce the following result of [4].

**PROPOSITION 3.1.** *For any  $Q \subseteq [n]$ , the subvariety  $X_Q$  is nonsingular and isomorphic to the cartesian product  $\times_j B(Z_{I(j)^+})$ .*

The cells  $e_Q$  provide a CW decomposition for  $B(Z_{n+1})$  with  $2^n$  cells, and define a basis for the integral homology groups  $H_*(B(Z_{n+1}))$ . We may interpret

$$\mathcal{X}(n) = \{X_Q : Q \subseteq [n]\}$$

as a Boolean algebra of nonsingular subvarieties of  $B(Z_{n+1})$ , ordered by inclusion, on which the support function  $S$  induces a bijection of Boolean algebras with  $2^{[n]}$ . For any  $n \leq m$ , the natural inclusions  $[n] \subseteq [m]$  induce a direct system of embeddings  $\mathcal{X}(n) \subseteq \mathcal{X}(m)$ , and Proposition 3.1 ensures that the isomorphism class of each  $X_Q$  is independent of  $m$ . These embeddings are compatible with those of  $2^{[n]}$  in  $2^{[m]}$  described above.

To every bounded flag  $U$  in  $Z_{n+1}$  we may associate the lines  $U_i - Z_{i-1}$  and  $U_{i+1} - U_i$  in  $Z_{[i, n+1]}$ , for all  $1 \leq i \leq n$ . These define complex line bundles  $\gamma_i$  and  $\rho_i$  over  $B(Z_{n+1})$ , whose classifying maps combine to embed  $B(Z_{n+1})$  algebraically into a product of complex projective spaces. Defining  $\gamma_0$  and  $\rho_0$  as 0 and  $\gamma_1$  respectively, we obtain

$$(3.2) \quad \gamma_i \oplus \rho_i \oplus \rho_{i+1} \oplus \cdots \oplus \rho_n \cong \mathbb{C}^{n-i+2}$$

for all  $0 \leq i \leq n$ .

As described in [15], the tangent bundle  $\tau$  of  $B(Z_{n+1})$  satisfies  $\tau \oplus \mathbb{R} \cong (\bigoplus_{i=2}^{n+1} \gamma_i) \oplus \mathbb{R}$ , so that (3.2) leads to isomorphisms

$$(3.3) \quad \nu \cong -\left(\bigoplus_{i=2}^{n+1} \gamma_i\right) \cong \bigoplus_{i=2}^n (i-1)\rho_i$$

for the stable normal bundle  $\nu$ . We follow [4] in referring to the resulting  $U$ -structure on  $B(Z_{n+1})$  as *basic*. We emphasise that these isomorphisms are of real bundles only, and therefore that the basic  $U$ -structure does not arise from the underlying complex algebraic variety. On  $B(Z_2)$ , for example, the basic  $U$ -structure is that of a 2-sphere  $S^2$ , rather than  $CP^1$ . Up to diffeomorphism, we may identify  $B(Z_{n+1})$  with the total space of the sphere bundle of  $\gamma_1 \oplus \mathbb{R}$  over  $B(Z_{[2, n+1]})$ , and the basic  $U$ -structure then extends over the associated 3-disc bundle for all values of  $n$ ; as a result,  $B(Z_{n+1})$  represents zero in the complex cobordism ring  $\Omega_*^U$ , and all Chern numbers of the basic  $U$ -structure vanish.

#### 4. Algebraic constructions

In this section we perform various key algebraic constructions with the posets  $\mathcal{X}(n)$ ; the crucial information is captured by an infinite matrix  $M_B$ , whose properties we explore in detail.

We begin by considering the free abelian group  $\mathbb{Z}X_*$ , generated by the isomorphism classes of the varieties  $X_Q$  as  $Q$  ranges over all finite subsets of  $[\infty]$ . To avoid elaborate notation we shall not distinguish between a variety and its isomorphism class, except in the case of  $B(Z_{n+1})$  itself, where we write  $B_n$  for the class of  $X_{[v, v+n-1]}$  given any positive integer  $v$ . Cartesian product invests  $\mathbb{Z}X_*$  with the structure of commutative ring, and Proposition 3.1 implies that the class of  $X_Q$  is the monomial  $\prod_{j=1}^t B_{[w(j)-v(j)+1]}$ , in the notation of §3. In other words  $\mathbb{Z}X_*$  is a polynomial algebra  $\mathbb{Z}[B]$ , generated by the sequence of isomorphism classes  $(B_0, B_1, \dots)$ , whose identity  $B_0 = 1$  is represented by the singleton  $X_\emptyset$ .

Alternatively, we may consider the additive semigroup  $[\infty]^*$ , consisting of infinite sequences  $\omega = (\omega_1, \omega_2, \dots)$  of nonnegative, eventually zero integers. As  $\omega$  ranges over  $[\infty]^*$ , the monomials  $B^\omega = \prod_j B_j^{\omega_j}$  range over a set of additive generators for  $\mathbb{Z}X_*$ . For every  $\omega$  there are many possible finite sets  $Q(\omega) \subseteq [\infty]$  with  $\omega_j$  maximal intervals of cardinality  $j$ , for all  $j \geq 0$ ; the isomorphism class of any  $X_{Q(\omega)}$  represents the monomial  $B^\omega$  in  $\mathbb{Z}X_*$ . By assigning the *dimension*  $|\omega| = 2 \sum j\omega_j$  to each monomial  $B^\omega$ , we invest  $\mathbb{Z}X_*$  with the structure of a graded ring, and in this parlance  $|\omega|$  is the dimension of  $X_{Q(\omega)}$  as a smooth manifold.

In order to record salient features of the Boolean algebras  $\mathcal{X}(n)$ , we introduce the infinite matrix  $M_B$  over the polynomial algebra  $\mathbb{Z}X_*$ . For all  $p, q \geq 0$ , the  $(p, q)$  entry consists of the formal sum of isomorphism classes  $\sum X_Q$ , taken over those subsets  $Q \subseteq [p]$  of cardinality  $p - q$ . When expressed in terms of the  $B_n$  we obtain a homogeneous polynomial of dimension  $2(p - q)$ , confirming that  $M_B$  is lower triangular, and lies in the group  $LT(\mathbb{Z}X_*)$ . For example, when  $p = 3$  and  $q = 1$ , the subsets in question are  $\{1, 2\}$ ,  $\{1, 3\}$ , and  $\{2, 3\}$ , yielding isomorphism classes  $B_2$ ,  $B_1^2$ , and  $B_2$  respectively, and the  $(3, 1)$  entry of  $M_B$  is  $2B_2 + B_1^2$ . Continuing

in this vein, we compute the upper left hand corner of  $M_B$  to be

$$(4.1) \quad \begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ B_1 & 1 & 0 & 0 & \dots \\ B_2 & 2B_1 & 1 & 0 & \dots \\ B_3 & 2B_2 + B_1^2 & 3B_1 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Readers may enjoy comparing this matrix with  $M(b)$  in (2.3).

The next step is to consider *pairs* of finite subsets  $P \subseteq Q \subseteq [n]$ , and to let  $X(P, Q)$  denote the subvariety  $X_{Q \setminus P}$  of  $B(Z_{[n+1] \setminus P})$ . With the assistance of the bijection  $\iota_P$  we may identify  $X_{Q \setminus P}$  with  $X_{\iota_P(Q \setminus P)}$  in  $B(Z_{n+1-|P|})$ , and deduce that the isomorphism class of  $X(P, Q)$  is independent of  $n$ , for  $n$  sufficiently large. In this notation  $X(\emptyset, Q)$  is  $X_Q$ , whilst  $X(Q, [n])$  is isomorphic to  $B(Z_{n+1-|Q|})$ .

In similar spirit we consider the free abelian group generated by *pairs* of isomorphism classes  $(W_P, X_Q)$ , where  $P$  and  $Q$  range independently over finite subsets of  $[\infty]$ . This group is isomorphic to  $\mathbb{Z}X_* \otimes \mathbb{Z}X_*$ , and is a commutative ring with respect to cartesian product in each coordinate. As such it is a graded polynomial algebra  $\mathbb{Z}[A, B]$ , where  $A_m$  and  $B_n$  denote the classes of the pairs  $(B(Z_{m+1}), \cdot)$  and  $(\cdot, B(Z_{n+1}))$  respectively.

We define an infinite matrix  $M_{A \circ B}$  in  $\text{LT}(\mathbb{Z}[A, B])$  by combining these constructions. For all  $p, q \geq 0$ , the  $(p, q)$  entry consists of the formal sum of isomorphism classes

$$(4.2) \quad \sum_{P \subseteq Q} (W(\emptyset, P), X(P, Q)),$$

taken over all  $P \subseteq Q \subseteq [p]$  for which  $|Q| = p - q$ . For example, when  $p = 3$  and  $q = 1$ , the appropriate subsets  $Q$  are  $\{1, 2\}$ ,  $\{1, 3\}$ , and  $\{2, 3\}$ , as before. For  $\{1, 2\}$ , the intermediate subset  $P$  ranges over  $\{1, 2\}$ ,  $\{1\}$ ,  $\{2\}$ , and  $\emptyset$ , which contribute monomials  $A_2$ ,  $A_1B_1$ ,  $A_1B_1$  and  $B_2$  respectively. Repeating for the other values of  $Q$ , we deduce that the  $(3, 1)$  entry of  $M_{A \circ B}$  is  $2A_2 + A_1^2 + 6A_1B_1 + 2B_2 + B_1^2$ ; similar computations reveal the upper left hand corner of  $M_{A \circ B}$  to be

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ A_1 + B_1 & 1 & 0 & 0 & \dots \\ A_2 + 2A_1B_1 + B_2 & 2(A_1 + B_1) & 1 & 0 & \dots \\ A_3 + (A_1^2 + 2A_2)B_1 + 3A_1B_2 + B_3 & 2A_2 + A_1^2 + 6A_1B_1 + 2B_2 + B_1^2 & 3(A_1 + B_1) & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Readers may compare *this* matrix with  $M(a \circ b)$  in (2.4).

**THEOREM 4.1.** *The matrix  $M_{A \circ B}$  is the product  $M_A M_B$  in  $\text{LT}(\mathbb{Z}[A, B])$ .*

**PROOF.** For every  $p \geq q \geq 0$ , the  $(p, q)$  entry of  $M_A M_B$  is given by

$$(4.3) \quad \sum_{p \geq k \geq q} \left( \sum_P W_P \right) \left( \sum_R X_R \right),$$

where  $P$  ranges over subsets of  $[p]$  of cardinality  $p - k$  and  $R$  ranges over subsets of  $[k]$  of cardinality  $k - q$ . But every variety  $W(\emptyset, P)$  appearing in (4.2) may be paired bijectively with  $W_P$  in (4.3), and every variety  $X(P, Q)$  may be paired with  $X_R$ ,

where  $R = \iota_P(Q \setminus P)$ ; the latter pairing is also bijective, by setting  $Q = P \cup \iota^P(R)$ . The varieties so paired are isomorphic, making the sums (4.2) and (4.3) equal.  $\square$

We refer to a sequence of subvarieties  $X_{Q_0} \subset X_{Q_1} \subset \cdots \subset X_{Q_\ell}$  in  $\mathcal{X}(n)$  as a *chain of length  $\ell$*  from  $X_{Q_0}$  to  $X_{Q_\ell}$ ; every such chain corresponds uniquely to the chain  $Q_0 \subset Q_1 \subset \cdots \subset Q_\ell$  in  $2^{[n]}$ , written as  $\gamma: Q_0 \rightsquigarrow Q_\ell$ . By analogy with  $M_B$ , we construct an infinite matrix  $N_B$  which displays information concerning chains in the  $\mathcal{X}(n)$ . For each  $p, q \geq 0$ , the  $(p, q)$  entry consists of the formal sum of monomials

$$(4.4) \quad \sum (-1)^\ell \prod_{i=1}^{\ell} X(Q_{i-1}, Q_i),$$

taken over chains in  $2^{[p]}$  from  $\emptyset$  to  $Q$ , where  $Q$  ranges over subsets of cardinality  $p - q$ . The resulting matrix clearly lies in  $\text{LT}(\mathbb{Z}[B])$ . For example, when  $p = 3$  and  $q = 1$  the relevant subsets  $Q \subseteq [3]$  are  $\{1, 2\}$ ,  $\{1, 3\}$ , and  $\{2, 3\}$ . The first of these yields chains

$$\emptyset \subset \{1\} \subset \{1, 2\}, \quad \emptyset \subset \{2\} \subset \{1, 2\}, \quad \text{and} \quad \emptyset \subset \{1, 2\},$$

in  $2^{[3]}$ , which contribute  $B_1^2$ ,  $B_1^2$  and  $-B_2$  respectively to (4.4); the second yields chains which contribute  $B_1^2$ ,  $B_1^2$  and  $-B_1^2$ , and the third contributes  $B_1^2$ ,  $B_1^2$  and  $-B_2$ . The  $(3, 1)$  entry of  $N_B$  is therefore  $5B_1^2 - 2B_2$ . Continuing, we compute the upper left hand corner of  $N_B$  as

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ -B_1 & 1 & 0 & 0 & \cdots \\ 2B_1^2 - B_2 & -2B_1 & 1 & 0 & \cdots \\ -5B_1^3 + 5B_1B_2 - B_3 & 5B_1^2 - 2B_2 & -3B_1 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

which compares with  $M(b^{(-1)})$  in (2.5).

**THEOREM 4.2.** *The matrices  $M_B$  and  $N_B$  are mutually inverse in the group  $\text{LT}(\mathbb{Z}[B])$ .*

**PROOF.** To demonstrate that  $N_B M_B$  is the identity matrix, we need only check that its  $(p, q)$  entry is 0 whenever  $p > q > 0$ . The terms contributing to this entry consist of a chain from  $X_\emptyset$  to  $X_Q$ , where  $Q \subseteq [p]$  has cardinality  $p - k$ , together with a subvariety  $X_R$ , where  $R \subseteq [k]$  has cardinality  $k - q$ , for some  $p \geq k \geq q$ . We partition these pairs into subsets  $\Gamma_0$  and  $\Gamma_1$ , where all the former have  $k = 0$  (and therefore  $X_R = \cdot$ ), and all the latter have  $k > 0$ . Each pair in  $\Gamma_0$  is therefore determined solely by a chain of the form  $X_\emptyset \subset \cdots \subset X_{Q_{\ell-1}} \subset X_{Q_\ell}$  in  $\mathcal{X}(p)$ , to which we associate the pair in  $\Gamma_1$  consisting of the chain  $X_\emptyset \subset \cdots \subset X_{Q_{\ell-1}}$  in  $\mathcal{X}(p)$ , and the subvariety  $X_P$  in  $\mathcal{X}(p - |Q_{\ell-1}|)$ , where  $P = \iota_{Q_{\ell-1}}(Q_\ell \setminus Q_{\ell-1})$ . This association is bijective, since an inverse may be constructed by judicious application of  $\iota^{Q_{\ell-1}}$ .

Since  $X(Q_{\ell-1}, Q_\ell)$  and  $X_P$  are isomorphic, we deduce that

$$(-1)^\ell \prod_{i=1}^{\ell} X(Q_{i-1}, Q_i) = (-X_P) \cdot (-1)^{\ell-1} \prod_{i=1}^{\ell-1} X(Q_{i-1}, Q_i)$$

in  $\mathbb{Z}X_*$  and therefore that the combined contribution of the elements of  $\Gamma_0$  and  $\Gamma_1$  to the  $(p, q)$  entry is 0.

The same methods confirm that  $M_B N_B = I$ .  $\square$

For future reference, we note that the  $(p, q)$  entries of  $M_{A \circ B}$  and  $N_B$  are homogeneous polynomials of dimension  $2(p - q)$ , in the variables  $A_m$  and  $B_n$ , and  $B_n$  respectively. We pay particular attention to the cases  $q = 0$ , and write the corresponding polynomials as  $M_p(A, B)$  and  $N_p(B)$ , for all  $p \geq 0$ .

### 5. Reconciliation and reformulation

We now reconcile the constructions of §§2 and 4 by means of their matrix representations, and reformulate both in the context of Hopf algebras.

We consider the universal special reversible series  $b(T)$  in  $\text{SR}(\mathbb{Z}[b])$ , and its representing matrix  $M(b)$  in  $\text{UT}(\mathbb{Z}[b])$ . The multiplicative isomorphism  $\mathbb{Z}[b] \rightarrow \mathbb{Z}[B]$ , induced by  $b_n \mapsto B_n$  for each  $n \geq 0$ , defines a special reversible series  $B(T) = \sum_{q \geq 0} B_q T^{q+1}$  over  $\mathbb{Z}X_*$  (which serves equally well as the universal example). It is represented by the matrix  $M(B)$  in  $\text{UT}(\mathbb{Z}[B])$ , whose transpose turns out to be familiar! To analyze why, it is convenient to interpret transposition as an antiisomorphism  $\text{UT}(\mathbb{Z}[B]) \rightarrow \text{LT}(\mathbb{Z}[B])$ .

PROPOSITION 5.1. *We have that  $M(B)^t = M_B$  in  $\text{LT}(\mathbb{Z}[B])$ .*

PROOF. For all  $p, q \geq 1$ , the  $(p, q)$  entry of  $M(B)^t$  consists of the coefficient of  $T^p$  in  $(T + B_1 T^2 + \dots + B_k T^{k+1} \dots)^q$ . All monomials in this expansion have the form  $B_{i_1} T^{i_1+1} \dots B_{i_q} T^{i_q+1}$ , where one factor arises from each bracket, and we collect together those which satisfy  $i_1 + \dots + i_q = p - q$ . We may express every such monomial as  $X_Q T^p$ , where  $Q \subseteq [p]$  is obtained by partitioning  $[p]$  into successive subintervals of lengths  $i_1 + 1, i_2 + 1, \dots, i_q + 1$ , and deleting the last (or only) element of each. Note that  $Q$  never contains the integer  $p$ , and that every subset of  $[p - 1]$  arises from one and only one monomial in the expansion; moreover, the cardinality of  $Q$  is  $p - q$ . The coefficient of  $T^p$  is therefore  $\sum X_Q$ , taken over all subsets  $Q \subseteq [p - 1]$  of cardinality  $(p - 1) - (q - 1)$ , as required.  $\square$

The reindexing in this proof arises because we chose  $p, q \geq 1$  and  $p, q \geq 0$  respectively, to index the entries of  $M(b)$  and  $M_B$ .

COROLLARY 5.2. *In  $\text{LT}(\mathbb{Z}[A, B])$ , we also have that*

$$M(B \circ A)^t = M_{A \circ B} \quad \text{and} \quad M(B^{(-1)})^t = N_B.$$

PROOF. We know that  $M(B \circ A)^t = M(A)^t M(B)^t$ , and it follows from Proposition 5.1 that we may rewrite the right hand side as  $M_A M_B$ ; applying Theorem 4.1, we replace this by  $M_{A \circ B}$  in turn, as required. The proof for  $M(B^{(-1)})$  is similar, applying Theorem 4.2.  $\square$

These results exemplify our central theme, that computations with substitution and reversion of universal formal power series may be expressed in terms of the posets  $\mathcal{X}(n)$ , through the medium of the matrices the  $M_{A \circ B}$  and  $N_B$ . The homogeneous polynomials  $M_p(A, B)$  and  $N_p(B)$  of §3 play a fundamental rôle, as ratified by the expressions

$$(5.1) \quad B(A(T)) = \sum_{p \geq 0} M_p(A, B) T^{p+1} \quad \text{and} \quad B^{(-1)}(T) = \sum_{p \geq 0} N_p(B) T^{p+1}.$$

For more general scalars, we make simple modifications to cope with power series  $s(T)$  in  $\text{SR}(K)$ . Specifically, we record the isomorphism class of each subvariety

$X_Q$  as a monomial  $\prod_{j=1}^t s_{[w(j)-v(j)+1]}$  in  $K$ , and write  $M_s$  for the resulting matrix in  $\text{LT}(K)$ , obtained from the  $\mathcal{X}(n)$  by the procedure corresponding to that of (4.1). Of course  $M_s$  is obtained from the universal example  $M_B$  by the replacement of  $B$  with  $s$ , irrespective of any relations which may hold amongst the  $s_n$ . We deduce from Proposition 5.1 and Corollary 5.2 that

$$(5.2) \quad M(s)^t = M_s, \quad M(s \circ r)^t = M_{r \circ s}, \quad \text{and} \quad M(s^{(-1)})^t = N_s$$

in  $\text{LT}(K)$ . We may harmonize the order of composition (and eliminate transpositions) in these formulae by electing to substitute power series on the *right*, and to represent them by *lower* triangular matrices accordingly.

In the universal case, (5.1) and (5.2) may be translated into a more elegant algebraic framework. We define homomorphisms of abelian groups

$$\delta: \mathbb{Z}[B] \rightarrow \mathbb{Z}[A, B], \quad \chi: \mathbb{Z}[B] \rightarrow \mathbb{Z}[B], \quad \text{and} \quad \epsilon: \mathbb{Z}[B] \rightarrow \mathbb{Z}$$

by means of the respective equations

$$(5.3) \quad \begin{aligned} \delta(X_Q) &= \sum_{\emptyset \subseteq P \subseteq Q} (W(\emptyset, P), X(P, Q)), \\ \chi(X_Q) &= \sum_{\gamma: \emptyset \rightsquigarrow Q} (-1)^\ell \prod_{i=1}^{\ell} X(Q_{i-1}, Q_i), \quad \text{and} \\ \epsilon(X_Q) &= \begin{cases} X_Q & \text{if } Q = \emptyset \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

on additive generators, and interpret  $\mathbb{Z}[A, B]$  as  $\mathbb{Z}[B] \otimes \mathbb{Z}[B]$  by rewriting  $A$  as  $B \otimes 1$  and  $B$  as  $1 \otimes B$ .

**PROPOSITION 5.3.** *The polynomial algebra  $\mathbb{Z}[B]$  is a Hopf algebra with respect to coproduct  $\delta$ , antipode  $\chi$ , and counit  $\epsilon$ .*

**PROOF.** We first we show that  $\delta$  and  $\chi$  are ring homomorphisms. As usual, we assume the decomposition of  $Q$  into maximal subintervals  $I(j)$  for  $1 \leq j \leq t$ , and write  $P(j)$  for  $P \cap I(j)$ , given any subset  $P \subseteq Q$ . Then we have

$$\sum_{\emptyset \subseteq P \subseteq Q} (W(\emptyset, P), X(P, Q)) = \prod_{j=1}^t \left( \sum_{\emptyset \subseteq P(j) \subseteq I(j)} (W(\emptyset, P(j)), X(P(j), I(j))) \right),$$

which confirms that  $\delta(X_Q) = \prod_{j=1}^t \delta(X_{I(j)})$ , and

$$\sum_{\gamma: \emptyset \rightsquigarrow Q} (-1)^\ell \prod_{i=1}^{\ell} X(Q_{i-1}, Q_i) = \prod_{j=1}^t \left( \sum_{\gamma(j): \emptyset \rightsquigarrow I(j)} (-1)^\ell \prod_{i=1}^{\ell} X(Q_{i-1}(j), Q_i(j)) \right),$$

which confirms that  $\chi(X_Q) = \prod_{j=1}^t \chi(X_{I(j)})$ , as required;  $\epsilon$  is multiplicative by definition.

To check that  $\chi$  is indeed an antipode, we observe that

$$\mu \cdot \chi \otimes 1 \cdot \delta(X_Q) = \sum_{\emptyset \subseteq P \subseteq Q} \left( \sum_{\gamma: \emptyset \rightsquigarrow P} (-1)^\ell \prod_{i=1}^{\ell} X(P_{i-1}, P_i) \right) X(P, Q)$$

in  $\mathbb{Z}[B]$ , where  $\mu$  denotes the polynomial product. As in the proof of Theorem 4.2, and so long as  $Q \neq \emptyset$ , the contribution to this sum of each chain  $\emptyset \subset \cdots \subset P_{\ell-1} \subset$

$Q$  with  $P = Q$  cancels that of  $\emptyset \subset \cdots \subset P_{\ell-1}$  with  $P = P_{\ell-1}$ . We deduce that  $\mu \cdot \chi \otimes 1 \cdot \delta(X_Q)$  is 0 when  $Q \neq \emptyset$ , and is 1 when  $Q = \emptyset$ , as required.  $\square$

Now (5.1) and (5.2) encode the structure maps of  $\mathbb{Z}[B]$ , as

$$(5.4) \quad \begin{aligned} \delta(B(T)) &= 1 \otimes B(B(T) \otimes 1), & \chi(B(T)) &= B^{(-1)}(T), \\ \text{and} \quad \epsilon(B(T)) &= T \end{aligned}$$

in terms of generating functions, and as

$$(5.5) \quad \delta(M_B) = M_{1 \otimes B \circ B \otimes 1}, \quad \chi(M_B) = N_B, \quad \text{and} \quad \epsilon(M_B) = I$$

in terms of matrices, respectively.

When emphasizing these structures (as opposed to others which emerge below) we write  $\mathbb{Z}^\circ X_*$  and  $\mathbb{Z}^\circ[B]$ , to acknowledge their origins in substitution. Cognoscenti will recognize  $\mathbb{Z}^\circ[B]$  as the dual of the Landweber-Novikov algebra; for others, we provide an explanation in §6.

The expressions (5.3) are closely related to formulae of [16] and [17].

## 6. Complex cobordism

In this section we enrich our algebraic and geometric constructions, explaining how the correspondence between formal power series and the geometry of the varieties  $X_Q$  extends to the context of multiple complex cobordism theory.

Complex cobordism is, of course, based on the Thom spectrum  $MU$ . We refer to [4] for details of the double complex cobordism functor  $\Omega_{2U}^*(\ )$ , based on  $2U = MU \wedge MU$ , and (by extrapolation) of treble complex cobordism  $\Omega_{3U}^*(\ )$ , based on  $3U = MU \wedge MU \wedge MU$ . The corresponding coefficient rings are  $\Omega_*^{2U}$  and  $\Omega_*^{3U}$ , defined by cobordism classes of manifolds whose stable normal bundles split as  $\nu \cong \nu_1 \oplus \nu_2$  and  $\nu \cong \nu_1 \oplus \nu_2 \oplus \nu_3$  respectively; we refer to the ordered pair  $(\nu_1, \nu_2)$  as a  $2U$ -structure and the ordered triple  $(\nu_1, \nu_2, \nu_3)$  as a  $3U$ -structure on the underlying manifold.

In §3 we invested each bounded flag manifold  $B(Z_{n+1})$  with the basic, and bounding,  $U$ -structure  $\nu \cong -(\oplus_{i=2}^{n+1} \gamma_i)$ . So  $B(Z_{n+1})$  also admits the basic  $2U$ -structure  $(\gamma, \gamma_1)$ , where  $\gamma = -(\oplus_{i=1}^{n+1} \gamma_i)$ , and the basic  $3U$ -structures

$$(\gamma, \gamma_1, 0), \quad (\gamma, 0, \gamma_1) \quad \text{and} \quad (0, \gamma, \gamma_1),$$

which we distinguish by the respective labels (1, 2), (1, 3), and (2, 3). Applying Proposition 3.1, we extend these structures to each subvariety  $X_Q$  by cartesian product, writing  $\gamma_Q$  for the  $t$ -plane bundle  $\gamma_1 \times \cdots \times \gamma_1$  over  $X_Q$ . We continue to refer to the resulting structures as basic; although the  $U$ -cobordism class of each  $X_Q$  is 0, the same is emphatically false in  $2U$ - and  $3U$ -cobordism, as we shall see.

It follows from the definitions that there are complex orientation classes  $x_1^{2U}$ ,  $x_2^{2U}$  in  $\Omega_{2U}^2(CP^\infty)$  and  $x_1^{3U}$ ,  $x_2^{3U}$ ,  $x_3^{3U}$  in  $\Omega_{3U}^2(CP^\infty)$ , and therefore expansions of the form

$$x_2^{2U} = \sum_{n \geq 0} b_n^U (x_1^{2U})^{n+1},$$

where  $b_n^U$  lies in  $\Omega_{2n}^{2U}$ , and

$$x_2^{3U} = \sum_{n \geq 0} b_{n,2}^U (x_1^{3U})^{n+1}, \quad x_3^{3U} = \sum_{n \geq 0} b_{n,3}^U (x_1^{3U})^{n+1},$$

$$\text{and} \quad x_3^{3U} = \sum_{n \geq 0} b_{n,4}^U (x_2^{3U})^{n+1},$$

where  $b_{n,2}^U$ ,  $b_{n,3}^U$ , and  $b_{n,4}^U$  lie in  $\Omega_{2n}^{3U}$ , for each  $n \geq 0$ . With these elements to hand, we may describe the  $2U$ - and  $3U$ -cobordism rings by embellishing results of [4].

**THEOREM 6.1.** *As  $\Omega_*^U$ -algebras, we have that*

$$\Omega_*^{2U} \cong \Omega_*^U[b^U] \quad \text{and} \quad \Omega_*^{3U} \cong \Omega_*^U[b_i^U, b_j^U],$$

where  $i \neq j$  is any pair such that  $2 \leq i, j \leq 4$ ; each  $b_n^U$  is represented by  $B(Z_{n+1})$  with basic  $2U$ -structure, and each  $b_{n,i}^U$  by  $B(Z_{n+1})$  with the basic  $3U$ -structures  $(1, 2)$ ,  $(1, 3)$ , and  $(2, 3)$  respectively.

Considering  $\Omega_*^{2U}$  as  $\pi_*(MU \wedge MU)$  leads to the more orthodox identification with  $A_*^U$ , the Hopf algebroid of homology cooperations in  $\Omega_*^U$ -theory (for further details, see [14], for example). The structural morphisms of coproduct, antiautomorphism, and counit are

$$(6.1) \quad \delta: A_*^U \rightarrow A_*^U \otimes A_*^U, \quad \chi: A_*^U \rightarrow A_*^U \quad \text{and} \quad \epsilon: A_*^U \rightarrow \Omega_*^U$$

respectively, each of which restricts to the subalgebra  $\mathbb{Z}[b^U]$  and turns it into a sub-Hopf algebra of  $A_*^U$ . This Hopf algebra is usually denoted by  $S_*$ , and is the integral dual of the Landweber-Novikov algebra; as such, the basis of monomials  $(b^U)^\omega$  is dual to the basis of Landweber-Novikov cohomology operation  $s_\omega$  in  $A_U^{2|\omega|}$ , as  $\omega$  ranges over  $[\infty]^*$ .

By unravelling the definitions, we may interpret the morphisms of (6.1) as

$$\delta: \Omega_{2U}^* \rightarrow \Omega_{3U}^*, \quad \chi: \Omega_{2U}^* \rightarrow \Omega_{2U}^* \quad \text{and} \quad \epsilon: \Omega_{2U}^* \rightarrow \Omega_*^U,$$

and confirm that they are induced on the cobordism class of any  $2U$ -manifold  $(M; \nu_1, \nu_2)$  by

$$(6.2) \quad \delta(M; \nu_1, \nu_2) = (M; \nu_1, 0, \nu_2), \quad \chi(M; \nu_1, \nu_2) = (M; \nu_2, \nu_1),$$

$$\text{and} \quad \epsilon(M; \nu_1, \nu_2) = (M; \nu_1 \oplus \nu_2)$$

respectively. These restrict to the multiplicative generators  $b_n^U$  of  $S_*$  by the corresponding operations on the basic  $2U$ -structures on  $B(Z_{n+1})$ , and to additive generators by extension to the  $X_Q$ . We may now clarify the relationship between  $S_*$  and the Hopf algebra  $\mathbb{Z}X_*$  of Theorem 5.3.

**THEOREM 6.2.** *The homomorphism  $\mathbb{Z}X_* \rightarrow \Omega_*^{2U}$ , induced by associating the basic  $2U$ -cobordism class to each isomorphism class  $X_Q$ , is an isomorphism of Hopf algebras onto  $S_*$ .*

**PROOF.** The homomorphism maps  $B_n$  to  $b_n^U$  for each  $n \geq 0$ , by Theorem 6.1, and so is an isomorphism of rings and preserves counits. Moreover, as explained in [1] for example,  $\delta$  and  $\chi$  are described on the generating function  $b^U(T) = \sum_{n \geq 0} b_n^U T^{n+1}$  by

$$\delta(b^U(T)) = 1 \otimes b^U(b^U(T) \otimes 1) \quad \text{and} \quad \chi(b^U(T)) = (b^U)^{\langle -1 \rangle}(T).$$

Comparing with (5.4), we conclude that the isomorphism also preserves  $\delta$  and  $\chi$ .  $\square$

Theorem 6.2 provides a new perspective on our geometric calculus. The isomorphism relation defining  $\mathbb{Z}X_*$  is extremely strict, and we have little freedom to vary the representative of each equivalence class  $X_Q$  (although that freedom makes a crucial contribution to our initial description of  $\delta$  and  $\chi$ ). The cobordism relation is much weaker and provides a rich supply of representatives for each equivalence class, leading to immediate simplifications such as (6.2). Moreover, addition and subtraction of cobordism classes may be described with equal ease in terms of connected sums and negated orientations, offering straightforward geometrical interpretation in terms of basic  $3U$ - and  $2U$ -structures on appropriate  $X_Q$  for the polynomials  $M_p(A, B)$  and  $N_p(B)$  of §3. We are therefore presented with the delicate problem of reconciling the two geometric viewpoints, which we pose in terms of the following corollary to Theorem 6.2.

**COROLLARY 6.3.** *There is a  $3U$ -cobordism between the bounded flag manifold  $B(Z_{n+1})$  with  $3U$ -structure  $(\gamma, 0, \gamma_1)$  and the polynomial  $M_n(A, B)$ , where the  $A$ s and  $B$ s are invested with the  $3U$ -structures (1, 2) and (2, 3) respectively. Similarly, there is a  $2U$ -cobordism between  $B(Z_{n+1})$  with  $2U$ -structure  $(\gamma_1, \gamma)$  and the polynomial  $N_n(B)$  with basic  $2U$ -structure.*

**PROOF.** Combine Theorem 6.2 with the structure maps for  $\mathbb{Z}X_*$  as described in (5.5), and for  $S_*$  as described in (6.2).  $\square$

We refer to the cobordisms of Corollary 6.3 as the *substitution cobordism* and the *reversion cobordism* respectively. We have succeeded in making them explicit only in the case  $n = 1$  (an instructive exercise for readers), and it is a fascinating challenge to do better. A result of [4] asserts that the  $(p, q)$  entry of  $M_B$  may be represented in  $\Omega_{2(p-q)}^{2U}$  by the cobordism class of  $(X_{[q, p-1]}; \gamma - (q-1)\gamma_1, q\gamma_1)$  for any  $p \geq q \geq 1$ , reducing  $M_p(A, B)$  to a quadratic polynomial and simplifying the substitution cobordism. Both cobordisms represent a deeper layer of geometric structure in our calculus; they are invisible in  $\mathbb{Z}X_*$ , which offers little more than a novel framework for the traditional procedures of substitution and reversion.

## 7. Products and the Grassmannian

Although we began the study of formal power series with the straightforward commutative product, we have yet to discuss its geometric interpretation. We now make good this omission, drawing parallels with all aspects of our treatment of substitution and reversion, and employing the infinite complex Grassmannian  $BU$ .

For each poset  $\mathcal{X}(n)$  we focus on a subset  $\mathcal{C}(n)$ , consisting of the elements of the chain  $X_\emptyset \subset X_{[1]} \subset \cdots \subset X_{[n]}$  of length  $n$ ; the support map defines an isomorphism of  $\mathcal{C}(n)$  with the chain  $\emptyset \subset [1] \subset \cdots \subset [n]$  in  $2^{[n]}$ . We develop the theory of these chains by analogy with that of the  $\mathcal{X}(n)$  in earlier sections, allowing familiarity to breed substantial compression. We note that the direct system of inclusions  $[n] \subseteq [m]$  induces compatible embeddings  $\mathcal{C}(n) \subseteq \mathcal{C}(m)$ , and let  $\delta: [j_0] \rightsquigarrow [j_\ell]$  denote a generic chain of the form  $[j_0] \subset [j_1] \subset \cdots \subset [j_\ell]$  in some suitably large  $2^{[m]}$ . The isomorphism class of each  $X_{[j]}$  is  $B_j$ , and the  $X_{[j]}$  therefore yield a set of polynomial generators for  $\mathbb{Z}X_*$  as  $[j]$  ranges over all  $\mathcal{C}(n)$ . If we wish to obtain *additive* generators for  $\mathbb{Z}X_*$  in similar fashion, we have to enlarge our pool of posets to include products of the form  $\mathcal{C}(n_1) \times \cdots \times \mathcal{C}(n_k) \subset \mathcal{X}(n_1 + \cdots + n_k)$ .

We record the elements of the  $\mathcal{C}(n)$  by means of a lower triangular matrix  $C_B$  in  $\text{LT}(\mathbb{Z}[B])$ , whose  $(j, k)$  entry is  $X_{[j-k]}$  for all  $j \geq k \geq 0$ . Similarly, we introduce

$C_{A \cdot B}$  in  $\text{LT}(\mathbb{Z}[A, B])$  and  $D_B$  in  $\text{LT}(\mathbb{Z}[B])$  with  $(j, k)$  entries

$$\sum_{h=0}^{j-k} (W([0], [h]), X([h], [j-k])) \quad \text{and} \quad \sum_{\delta: [0] \rightsquigarrow [j-k]} (-1)^\ell \prod_{i=1}^{\ell} X([j_{i-1}], [j_i])$$

respectively. These expressions depend only on  $j-k$ , and consist of homogeneous polynomials  $C_{j-k}(A, B)$  and  $D_{j-k}(B)$  of dimension  $2(j-k)$ , in the generators  $A_m$  and  $B_n$ . For example,  $C_{j-k}(A, B) = \sum_h A_h B_{j-k-h}$ .

**PROPOSITION 7.1.** *The matrices  $C_A$  and  $C_B$  commute in  $\text{LT}(\mathbb{Z}[A, B])$ , and  $C_A C_B = C_{A \cdot B}$ ; the matrices  $C_B$  and  $D_B$  are mutually inverse in  $\text{LT}(\mathbb{Z}[B])$ .*

**PROOF.** The  $(j, k)$  entry of both  $C_A C_B$  and  $C_B C_A$  is  $\sum_{h=0}^{j-k} (W_{[h]}, X_{[j-k-h]})$ , as required. To confirm that  $D_B C_B = I$ , we adapt the proof of Theorem 4.2 by restricting attention to chains in the subposets  $\mathcal{C}(n)$ .  $\square$

To relate the matrices  $C_B$  to formal power series, we recall the universal invertible series  $\beta(T)$  over  $\mathbb{Z}[\beta]$ , and its representing semicirculant  $C(\beta)$  in  $\text{UT}(\mathbb{Z}[\beta])$ . The multiplicative isomorphism  $\mathbb{Z}[\beta] \rightarrow \mathbb{Z}[B]$ , induced by  $\beta_n \mapsto B_n$  for each  $n \geq 0$ , defines the invertible series  $\tilde{B}(T) = \sum_{k \geq 0} B_k T^k$  over  $\mathbb{Z}X_*$  (which serves equally well as the universal example). We remark that

$$(7.1) \quad \tilde{B}(T) = B(T)/T$$

in  $\mathbb{Z}[B][[T]]$ , where  $B(T)$  is defined in §5, and that  $\tilde{B}(T)$  is represented by the semicirculant  $C(\tilde{B})$  in  $\text{UT}(\mathbb{Z}[B])$ . In fact  $C(\tilde{B})^t = C_B$  by construction, which leads to the following corollary to Proposition 7.1.

**COROLLARY 7.2.** *In  $\text{LT}(\mathbb{Z}[A, B])$ , we have that*

$$C(\tilde{B})^t = C_B, \quad C(\tilde{A} \cdot \tilde{B})^t = C_{B \cdot A}, \quad \text{and} \quad C(1/\tilde{B})^t = D_B.$$

The commutativity of the product of power series therefore mirrors the relation  $C_B C_A = C_{A \cdot B}$ , and confirms that we may replace  $C(\tilde{B})$  by its transpose from the start; we then acquire an isomorphism of  $\mathbb{Z}[B][[T]]$  with the semicirculant lower triangular matrices, and duly eliminate transpositions from Corollary 7.2.

These results allow us to compute products and reciprocals of the universal power series in terms of the posets  $\mathcal{C}(n)$ ; in particular, the polynomials  $C_k(A, B)$  and  $D_k(B)$  appear in the expressions

$$(7.2) \quad \tilde{A}(T)\tilde{B}(T) = \sum_{k \geq 0} C_k(A, B)T^k \quad \text{and} \quad 1/\tilde{B}(T) = \sum_{k \geq 0} D_k(B)T^k.$$

For power series  $\sigma(T)$  over more general scalars  $K$ , we record the isomorphism class of each subvariety  $X_{[j]}$  as  $\sigma_j$ , and write  $C_\sigma$  for the resulting matrix in  $\text{LT}(K)$ . We may then generalize Corollary 7.2 to

$$C(\sigma)^t = C_\sigma, \quad C(\rho \cdot \sigma)^t = C_{\rho \cdot \sigma}, \quad \text{and} \quad C(1/\sigma)^t = D_\sigma.$$

The universal case is again linked to the theory of Hopf algebras. We define ring homomorphisms

$$\delta^\bullet: \mathbb{Z}[B] \rightarrow \mathbb{Z}[A, B], \quad \chi^\bullet: \mathbb{Z}[B] \rightarrow \mathbb{Z}[B], \quad \text{and} \quad \epsilon: \mathbb{Z}[B] \rightarrow \mathbb{Z}$$

by means of the respective equations

$$(7.3) \quad \begin{aligned} \delta^\bullet(X_{[n]}) &= \sum_{i=0}^n (W([0], [i]), X([i], [n])), \\ \chi^\bullet(X_{[n]}) &= \sum_{\delta: [0] \rightsquigarrow [n]} (-1)^\ell \prod_{i=1}^{\ell} X([j_{i-1}], [j_i]) \quad \text{and} \\ \epsilon(X_{[n]}) &= \begin{cases} X_{[n]} & \text{if } n = 0 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

on polynomial generators, and replace  $\mathbb{Z}[A, B]$  by  $\mathbb{Z}[B] \otimes \mathbb{Z}[B]$ , as before.

**PROPOSITION 7.3.** *The polynomial algebra  $\mathbb{Z}[B]$  is a Hopf algebra with respect to the coproduct  $\delta^\bullet$ , antipode  $\chi^\bullet$ , and counit  $\epsilon$ .*

**PROOF.** Since  $\delta^\bullet$ ,  $\chi^\bullet$ , and  $\epsilon$  are extended to monomials by multiplicativity, it remains only to establish that  $\chi^\bullet$  is an antipode. This parallels the proof for  $\chi$  in Proposition 5.3.  $\square$

We denote this Hopf algebra by  $\mathbb{Z}^\bullet X_*$  or  $\mathbb{Z}^\bullet[B]$ , whenever we wish to contrast it with  $\mathbb{Z}^\circ X_*$  or  $\mathbb{Z}^\circ[B]$  of Proposition 5.3; as algebras they are, of course, the same. Appealing to (7.2) and Corollary 7.2, the structure maps of Proposition 7.3 may be encoded in terms of generating functions by

$$(7.4) \quad \begin{aligned} \delta^\bullet(\tilde{B}(T)) &= \tilde{B}(T) \otimes \tilde{B}(T), \quad \chi^\bullet(\tilde{B}(T)) = 1/\tilde{B}(T), \\ &\text{and } \epsilon(\tilde{B}(T)) = 1, \end{aligned}$$

and in terms of matrices by

$$\delta^\bullet(C_B) = C_{B \otimes 1 \cdot 1 \otimes B}, \quad \chi^\bullet(C_B) = D_B, \quad \text{and} \quad \epsilon(C_B) = I.$$

It is possible to extend the entire analysis to additive generators by observing that  $C_{B \cdot m} = C_B^m$  is obtained from  $m$ -fold cartesian products of posets  $\mathcal{C}(n)$  by concentrating on chains which are themselves  $m$ -fold products.

We now relate the Hopf algebra  $\mathbb{Z}^\bullet[B]$  to cobordism theory. The Whitney sum map  $\oplus: BU \times BU \rightarrow BU$  turns  $\Omega_*^U(BU_+)$  into an  $\Omega_*^U$ -algebra, which is a polynomial algebra  $\Omega_*^U[\beta^U]$  on generators  $\beta_n^U$ , where  $\beta_0^U = 1$ . The additive basis of monomials  $(\beta^U)^\omega$  is dual to the basis of cobordism Chern classes  $c_\omega$  in  $\Omega_U^*(BU_+)$ , as  $\omega$  ranges over  $[\infty]^*$ . The Thom isomorphism  $\Omega_*^U(MU) \rightarrow \Omega_*^U(BU_+)$  is a map of  $\Omega_*^U$ -algebras, and equates  $\Omega_*^U(BU_+)$  with both  $\Omega_*^{2U}$  and  $A_*^U$  by virtue of §5.

**LEMMA 7.4.** *The Thom isomorphism  $\Omega_*^{2U} \rightarrow \Omega_*^U(BU_+)$  identifies the  $2U$ -cobordism class of any  $2U$ -manifold  $(M : \nu_1, \nu_2)$  with the  $U$ -cobordism class of  $\nu_2: M \rightarrow BU$ , where the latter  $M$  has the  $U$ -structure  $\nu_1 \oplus \nu_2$ .*

This lemma is proven in [4]; when combined with the result of [15] that  $\beta_n$  is represented by the classifying map  $\gamma_1: B(Z_{n+1}) \rightarrow BU$ , it confirms that the Thom isomorphism acts by  $b_n^U \mapsto \beta_n^U$ .

The diagonal, orthogonal complement, and collapse maps

$$\Delta: BU \rightarrow BU \times BU, \quad \perp: BU \rightarrow BU, \quad \text{and} \quad e: BU \rightarrow \cdot$$

induce  $\Omega_*^U$ -algebra maps

$$\begin{aligned} \Delta_* : \Omega_*^U(BU_+) &\rightarrow \Omega_*^U(BU_+) \otimes \Omega_*^U(BU_+), & \perp_* : \Omega_*^U(BU_+) &\rightarrow \Omega_*^U(BU_+), \\ \text{and } e_* : \Omega_*^U(BU_+) &\rightarrow \Omega_*^U. \end{aligned}$$

Moreover, the composition  $\oplus \cdot \perp \times 1 \cdot \Delta$  agrees with the projection  $e$ , so that  $\Omega_*^U(BU_+)$  becomes an  $\Omega_*^U$ -Hopf algebra with respect to the coproduct  $\Delta_*$ , antipode  $\perp_*$  and counit  $e_*$ . The subalgebra  $\mathbb{Z}[\beta_U]$ , which we abbreviate to  $R_*$  hereafter, is closed with respect to  $\Delta_*$  and  $\perp_*$ , whilst  $e_*$  restricts as  $R_* \rightarrow \mathbb{Z}$ ; in consequence,  $R_*$  is a sub-Hopf algebra over  $\mathbb{Z}$  (isomorphic to the integral homology algebra  $H_*(BU)$ ). The Thom isomorphism  $A_*^U \rightarrow \Omega_*^U(BU_+)$  restricts to an isomorphism  $t : S_* \rightarrow R_*$  of rings, but preserves neither coproducts nor antipodes.

**THEOREM 7.5.** *The homomorphism  $\mathbb{Z}^\bullet X_* \rightarrow \Omega_*^U(BU_+)$ , induced by associating the  $U$ -cobordism class  $\gamma_Q : X_Q \rightarrow BU$  to each isomorphism class  $X_Q$ , is an isomorphism of Hopf algebras onto  $R_*$ .*

**PROOF.** The homomorphism maps  $B_n$  to  $\beta_n^U$  for each  $n \geq 0$ , and so is an isomorphism of rings and preserves counits. Moreover, as explained in [1],  $\Delta_*$  and  $\perp_*$  are defined on the generating function  $\beta^U(T) = \sum_{n \geq 0} \beta_n^U T^n$  by

$$\Delta_*(\beta^U(T)) = \beta^U(T) \otimes \beta^U(T) \quad \text{and} \quad \perp_*(\beta^U(T)) = 1/\beta^U(T).$$

Comparing with (7.4), we conclude that the isomorphism also maps  $\delta^\bullet$  to  $\Delta_*$  and  $\chi^\bullet$  to  $\perp_*$ .  $\square$

The isomorphisms of Theorems 6.2 and 7.5 combine to produce a commutative diagram

$$(7.5) \quad \begin{array}{ccc} \mathbb{Z}^\circ X_* & \xrightarrow{\cong} & S_* < \Omega_*^U(MU) \\ 1 \downarrow & & \downarrow t \\ \mathbb{Z}^\bullet X_* & \xrightarrow{\cong} & R_* < \Omega_*^U(BU_+). \end{array}$$

This compares the two Hopf algebra structures on  $\mathbb{Z}X_*$ , and the cobordism formulation of Lagrange inversion arises from the interplay between the two antipodes  $\chi$  and  $\chi^\bullet$ .

**PROPOSITION 7.6.** *As elements of  $\Omega_*^U(BU)$ , there is a cobordism between  $n+1$  times  $\gamma : B(Z_{n+1}) \rightarrow BU$  and  $-\gamma_1^{\oplus(n+1)} : B(Z_{n+1}) \rightarrow BU$ .*

**PROOF.** By (5.1), Corollary 6.3, and Lemma 7.4, we may chase the coefficient of  $T^{n+1}$  in  $B(T)$  around Diagram 7.5 to the  $U$ -cobordism class of  $\gamma : B(Z_{n+1}) \rightarrow BU$ ; on the other hand, the Lagrange inversion formula [9] identifies this coefficient with  $1/(n+1)$  times the coefficient of  $T^n$  in  $(1/\tilde{B})^{n+1}$ . We may chase the latter around to  $-\gamma_1^{\oplus(n+1)} : B(Z_{n+1}) \rightarrow BU$  by Theorem 7.5, and the result follows.  $\square$

We may reverse the argument of Proposition 7.6, and prove the Lagrange inversion formula by exhibiting a suitable *Lagrange cobordism*; this is simple when  $n = 1$ , but becomes more problematic as  $n$  increases. Nevertheless, manipulations with Chern classes confirm that such a cobordism does exist, and so provide some sort of hybrid proof. In the spirit of our calculus, we would prefer to probe the underlying geometry by making the Lagrange cobordism explicit. We note in passing

that the Chern class calculations have purely algebraic ramifications, allowing Lagrange inversion to be reformulated in the cohomology ring of  $B(Z_{n+1})$ ; we discuss these matters elsewhere, together with more general forms of Lagrange inversion.

### 8. Faà di Bruno algebra

We conclude by summarizing modifications to the theory which deal with formal *divided* power series, otherwise known as *Hurwitz series* [5].

We assume (for notational convenience) that our ring of scalars  $K$  is free of additive torsion, and consider the  $K$ -algebra  $K\{\{T\}\}$  of Hurwitz series over  $K$ , writing a typical element  $\zeta(T)$  as  $\sum_{k \geq 0} \zeta_k T^k / k!$ , and assuming henceforth that  $\zeta_0 = 1$ . We emphasize that no scalar  $\zeta_k$  need be divisible by  $k!$  in  $K$ . Addition is taken termwise, and the product  $\varepsilon(T) \cdot \zeta(T)$  is defined by the Cauchy product of the corresponding sequences, taking account of the extra factorials which arise from the denominators  $k!$ ; clearly  $K[[T]]$  is a subalgebra, and the reciprocal  $1/\zeta(t)$  is again a Hurwitz series.

We concentrate on special reversible Hurwitz series which lie in the principal ideal  $(T) < K\{\{T\}\}$ , and therefore have  $\zeta_0$  zero; we denote such series generically by  $f(T) = T + f_1 T^2 / 2! + \cdots + f_n T^{n+1} / (n+1)! + \cdots$ . For each  $p \geq 1$  we observe that the  $p$ th divided power is again in  $(T)$ , and write it as

$$(8.1) \quad f(T)^p / p! = \sum_p f_{p,q} T^q / q!,$$

where the  $f_{p,q}$  are partial Bell polynomials in  $f_1, \dots, f_{q-p}$ . Given two such series  $e(T)$  and  $f(T)$ , we obtain a third by substitution, written as  $e(f(T))$  or  $e \circ f(T)$ . The reversion  $f^{(-1)}(T)$  also lies in  $(T)$  which therefore becomes a nonabelian group, with subgroup  $\text{SR}(K)$ .

The universal ring of scalars is a polynomial algebra  $\mathbb{Z}[g, h]$ , generated by sequences of indeterminates  $(g_0, g_1, \dots)$  and  $(h_0, h_1, \dots)$ , with  $g_0 = h_0 = 1$ ; it admits Hurwitz series  $g(T)$  and  $h(T)$  and their reversions, plus the substitution of one in another. The ring homomorphism  $\mathbb{Z}[g, h] \rightarrow \mathbb{Z}[a, b]$ , which perceives the standard power series  $a(T)$  and  $b(T)$  as degenerate Hurwitz series, is therefore induced by  $g_n \mapsto (n+1)!a_n$  and  $h_n \mapsto (n+1)!b_n$ .

To extend our geometric calculus to Hurwitz series, we construct a family of algebraic varieties  $H(U)$ , indexed by finite subsets  $U$  of  $[\infty]$ . Given  $U = \{u(1), \dots, u(c)\}$  of cardinality  $c$ , we consider a  $c$ -fold cartesian product  $(CP^1)^{\times c}$ , where the factors are indexed by the elements of  $U$ , and write  $\eta_{u(j)}$  for the Hopf line bundle on the  $u(j)$  factor, where  $1 \leq j \leq c$ . Then  $H(U)$  is the Milnor hypersurface of degree  $(1, 1, \dots, 1)$ , dual to the tensor product  $\otimes_{j=1}^{n+1} \eta_{u(j)}$ ; to algebraic geometers, it is a hyperplane section and a nonsingular complex algebraic variety of dimension  $c - 1$ . We continue to write the restriction of each Hopf bundle to  $H(U)$  as  $\eta_{u(j)}$ , and abbreviate  $H([n+1])$  to  $H_n$  in order to emphasize its dimension. Clearly  $H_1$  is a singleton.

For each nonnegative integer  $n$  we consider the poset  $\Pi(n)$  of partitions of the set  $[n]$ , ordered by refinement [2]; for each  $n < m$  there is an inclusion  $\Pi(n) \subset \Pi(m)$ , obtained by adjoining singleton blocks  $\{n+1\}, \dots, \{m\}$  to all partitions of  $[n]$ . For every  $\pi$  in  $\Pi(n+1)$  we define a further subvariety  $P_\pi$  of  $(CP^1)^{\times(n+1)}$  as follows. We list the blocks of  $\pi$  as  $\pi(j)$  for  $1 \leq j \leq b(\pi)$ , where each  $\pi(j)$  is a subset of  $[n+1]$  and  $b(\pi)$  is the number of blocks, and construct  $P_\pi$  by successively restricting and

dualizing the tensor products  $\otimes_{\pi(j)} \eta_u$ , for  $1 \leq j \leq b(\pi)$ . The result is independent of the order of dualization and has complex dimension  $n + 1 - b(\pi)$ ; its structure is immediate.

PROPOSITION 8.1. *For any partition  $\pi$  of  $[n + 1]$ , the subvariety  $P_\pi$  is nonsingular and isomorphic to the cartesian product  $\times_j H(\pi(j))$ .*

Singleton blocks contribute trivial cartesian factors to this decomposition, and the isomorphism type of  $P_\pi$  is therefore independent of large  $n$ .

We obtain posets  $\mathcal{P}(n + 1)$  of subvarieties of  $(CP^1)^{\times n+1}$ , ordered by  $P_\nu < P_\pi$  whenever  $\nu < \pi$  in  $\Pi(n + 1)$ , and forming a direct system with respect to  $n$ . The *type* of  $\pi$  is the monomial  $\prod_{j \geq 0} \phi_j^{w(j)}$ , where  $w(j)$  is the number of blocks of cardinality  $j + 1$  and  $\phi_0 = 1$ ; by Proposition 8.1, it may be read off from the poset isomorphism  $\Pi(n + 1) \cong \mathcal{P}(n + 1)$ .

We now consider the free abelian group  $\mathbb{Z}P_*$  generated by the isomorphism classes of all varieties  $P_\pi$ . Cartesian product invests  $\mathbb{Z}P_*$  with the structure of commutative ring, and Proposition 8.1 implies that the class of  $P_\pi$  is the monomial  $\prod_{j=1}^{b(\pi)} H_{|\pi(j)|-1}$ . Thus  $\mathbb{Z}P_*$  is a polynomial algebra  $\mathbb{Z}[H]$ , generated by the sequence of isomorphism classes  $(H_0, H_1, \dots)$ , with identity  $H_0 = 1$ ; it admits the Hurwitz series  $H(T) = \sum_{q \geq 0} H_q T^{q+1} / (q + 1)!$ . We extend to the product  $\mathbb{Z}P_* \otimes \mathbb{Z}P_*$  by considering *pairs* of isomorphism classes  $(O_\nu, P_\pi)$ . The result is isomorphic to  $\mathbb{Z}[G, H]$ , and admits Hurwitz series  $G(T)$  and  $H(T)$  which may be taken as universal examples.

By analogy with (5.3) we turn  $\mathbb{Z}P_*$  into a Hopf algebra, whose structure maps encode the combinatorics of the posets  $\mathcal{P}(n)$ . In turn, we may mimic (5.4) and (5.5) by describing  $\delta$ ,  $\chi$ , and  $\epsilon$  in terms of exponential generating functions or in terms of matrices; the former relates explicitly to computations with Hurwitz series, and yields

$$\begin{aligned} \delta(H(T)) &= 1 \otimes H(H(T) \otimes 1), & \chi(H(T)) &= H^{(-1)}(T) \\ \text{and} & & \epsilon(H(T)) &= T. \end{aligned}$$

As a Hopf algebra,  $\mathbb{Z}P_*$  is isomorphic to the Faà di Bruno algebra [7].

The ring homomorphism  $\mathbb{Z}[H] \rightarrow \mathbb{Z}[B]$  of universal examples is induced by  $H_n \mapsto (n + 1)!B_n$ , and embeds the Faà di Bruno algebra as a sub-Hopf algebra of the dual of the Landweber-Novikov algebra; sadly, we have no geometric interpretation for this map. Worse, there can be no  $2U$ -structure on  $H_n$  which renders it cobordant to  $(n + 1)!$  copies of  $B_n$  with basic  $2U$ -structure, since a simple calculation reveals that  $H_n$  can never be a  $U$ -boundary (except when  $n = 1$  or  $2$ ). On the other hand, if we collapse  $\Omega_*^{2U}$  to  $H_*(MU)$  under the Thom map (which is an isomorphism on  $S_*$ ), then  $H_n$  and  $(n + 1)!B_n$  may at least be made homologous, giving some degree of credence to the model. Nevertheless, our calculus for Hurwitz series remains at a rudimentary stage, and it is a stimulating challenge to find alternative realizations which lead to a full integration with the theory of §§5, 6 and 7. Usually, it is a problem to find natural representatives for a *divided* cobordism class; in this case, we are working with *multiples*.

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