

CATEGORICAL ASPECTS of TORIC TOPOLOGY

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OVERVIEW

Aims: (i) *to describe categorical aspects of toric objects, and (ii) to give examples of useful calculations in this framework.*

1. THE CATEGORICAL VIEWPOINT

2. TORIC OBJECTS

3. HOMOTOPY THEORY

4. FORMALITY

This is why we are all here ..

$$\begin{array}{c} \mathcal{Z}_K \\ \downarrow T^{m-n} \\ M_1^{2n}, \dots, M_k^{2n} \\ \downarrow T^n \\ P^n \end{array}$$

Lower quotients are strict; $P^n = \text{Cone}(K')$.

$$\begin{array}{c} \mathcal{Z}_K \\ \downarrow T^{m-n} \\ M_1^{2n}, \dots, M_k^{2n} \\ \downarrow T^n \\ DJ(K) \end{array}$$

Lower quotients are homotopy quotients.

1. THE CATEGORICAL VIEWPOINT

Some people love categories, and others hate them; but they are here to stay!

A **category** \mathcal{C} has objects X , and a set of morphisms $\mathcal{C}(X, Y)$ between every pair of objects.

Some categories are large, such as top , the category of topological spaces and continuous maps. Others are small (finite, even!), such as the category $cat(K)$ of faces of a simplicial complex K and their inclusions.

Functors are morphisms between categories, such as the singular cochain algebra functor

$$C^*(-; R): top \longrightarrow dga_R$$

over a nice ring R .

Toric Topology exists within two categorical frameworks, which may seem independent . . . but they are deeply intertwined!

- (i) **Local:** many toric spaces admit natural decompositions into simpler subspaces; and these are often indexed by small categories such as $cat(K)$.

- (ii) **Global:** as problems vary, our spaces may lie in the category of smooth manifolds and diffeomorphisms; or CW-complexes and homotopy classes of maps; or

The local viewpoint considers toric spaces as **diagrams**, whereas the global viewpoint interprets their **invariants** as functors from geometric to algebraic categories.

As well as $cat(K)$, we like the small category Δ , with objects

$$(n) = \{0, 1, \dots, n\} \text{ for } n \geq 0,$$

and morphisms the non-decreasing maps. We denote their **opposites** by cat^{op} and Δ^{op} .

We like geometric categories such as

top_+ : pointed topological spaces
 $tmon$: topological monoids.

We also like algebraic categories such as

dga_R : differential graded algebras
 $cdga_{\mathbb{Q}}$: commutative dgas
 $dgc_{\mathbb{Q}}$: differential graded colagebras,

usually with (co)augmentations. Differentials go down in dga and dgc , and up in $cdga$.

Given a small indexing category a , we may view diagrams in c as functors $D: a \rightarrow c$; then the collection of all such diagrams also forms a category $[a, c]$.

If a is Δ , then $[\Delta, c]$ and $[\Delta^{op}, c]$ are the categories of **cosimplicial** and **simplicial objects** in c , often denoted by cc and sc respectively. For example:

(i) the **cosimplicial simplex**

$$\Delta^\bullet: \Delta \longrightarrow \text{top}$$

maps (n) to the standard n -simplex Δ^n ;

(ii) the **singular chain complex**

$$C_\bullet(X): \Delta \longrightarrow \text{set}$$

maps (n) to the set of continuous functions $f: \Delta^n \rightarrow X$ for any space X .

In nice categories, **pushouts** and **pullbacks** arise from diagrams of the form

$$\{1\} \longleftarrow \emptyset \longrightarrow \{2\} \quad \text{and} \quad \{1\} \longrightarrow \emptyset \longleftarrow \{2\},$$

and have certain universal properties. These diagrams are actually $cat(\bullet \bullet)$ and $cat^{op}(\bullet \bullet)$.

In $tmon$, the pushout of the diagram

$$T_1 \longleftarrow \{1\} \longrightarrow T_2$$

of circles is the free product $T_1 \star T_2 \rightarrow T_1 \times T_2$; in top_+ , the pushout of the diagram

$$BT_1 \longleftarrow * \longrightarrow BT_2$$

of classifying spaces is $BT_1 \vee BT_2 \subset BT \times BT$.

Coproducts (or sums) are special cases of pushouts, which are themselves examples of **colimits** of arbitrary diagrams in a category. Similarly, **products** are special cases of pullbacks, which are examples of **limits**.

2. TORIC OBJECTS

We start with a simplicial complex K on vertices $V = \{v_1, \dots, v_m\}$, and construct two associated topological spaces:

- the *Davis-Januszkiewicz space* $DJ(K)$
- the *moment-angle complex* \mathcal{Z}_K .

The topologists amongst us are interested in their properties

up to **homotopy equivalence**,

so we have some freedom in making the constructions.

The vertices determine

- an m -torus T^V
- its classifying space $BT^V \simeq (\mathbb{C}P^\infty)^V$.

For any face $\sigma \subseteq V$ of K , there is

- a coordinate subtorus $T^\sigma \subseteq T^V$
- its classifying space $BT^\sigma \subseteq BT^V$
- the space $D_\sigma = (D^2)^\sigma \times T^{V \setminus \sigma}$.

So there are diagrams

- $T^K : \text{cat}(K) \longrightarrow \text{tmon}$
- $BT^K : \text{cat}(K) \longrightarrow \text{top}_+$
- $D^K : \text{cat}(K) \longrightarrow \text{top}_+$

which map an inclusion $\sigma \subseteq \tau$ of faces to

- the monomorphism $T^\sigma \subseteq T^\tau$
- the inclusion $BT^\sigma \subseteq BT^\tau$
- the inclusion $D_\sigma \subseteq D_\tau$

respectively.

To construct our first toric **spaces**, we take colimits of diagrams. We obtain

$$\operatorname{colim}^{tmon} T^K = \operatorname{Cir}(K^{(1)})$$

as topological groups; and

$$\operatorname{colim}^{top+} BT^K = \bigcup_{\sigma \in K} BT^\sigma \cong DJ(K)$$

and

$$\operatorname{colim}^{top+} D^K = \bigcup_{\sigma \in K} D_\sigma \cong \mathcal{Z}_K$$

as pointed topological spaces.

We can also define a diagram $T^{V \setminus K}$ by mapping $\sigma \subseteq \tau$ to the projection

$$T^{V \setminus \sigma} \longrightarrow T^{V \setminus \tau}.$$

In this case,

$$\operatorname{colim}^{top+} T^{V \setminus K} = \{1\}$$

is a single point.

For algebraic purposes, we write the vertices v_1, \dots, v_m as 2-dimensional variables; their desuspensions u_1, \dots, u_m are 1-dimensional. In either case, we denote the commutative monomials $\prod_{\alpha} w_i$ by w_{α} , for any multiset $\alpha: V \rightarrow \mathbb{N}$.

The symmetric algebra $S_R(V)$ is **polynomial** over R , with basis elements v^{α} . The symmetric algebra $\wedge_R(U)$ is **exterior**, with basis elements u^{α} for genuine subsets $\alpha \subseteq U$.

With $d = 0$, both are objects of *cdga*; and so is $\wedge(U) \otimes S(\sigma)$, with $du_i = v_i$ for all $v_i \in \sigma$.

The graded duals $S_R(V)'$ and $\wedge_R(U)'$ have dual basis elements v^{α} and u^{α} over R . In either case, their coproducts satisfy

$$\delta(w^{\alpha}) = \sum_{\alpha_1 \sqcup \alpha_2 = \alpha} w^{\alpha_1} \otimes w^{\alpha_2}$$

With $d = 0$, both are objects of *cdgc*.

We can define a diagram $cat(K) \rightarrow dga$ by:

- \wedge^K maps $\sigma \subseteq \tau$ to the monomorphism

$$\wedge(\sigma) \longrightarrow \wedge(\tau),$$

and diagrams $cat^{op}(K) \rightarrow cdga$ by:

- S_K maps $\tau \supseteq \sigma$ to the epimorphism

$$S(\tau) \longrightarrow S(\sigma),$$

- $\wedge \otimes S_K$ maps $\tau \supseteq \sigma$ to the epimorphism

$$\wedge(U) \otimes S(\tau) \longrightarrow \wedge(U) \otimes S(\sigma).$$

... and a diagram $cat(K) \rightarrow cdgc$ by:

- $(S^K)'$ maps $\tau \subseteq \sigma$ to the monomorphism

$$S(\tau)' \longrightarrow S(\sigma)'$$

To define **algebraic** toric objects, we consider

$$\operatorname{colim}^{dga} \wedge^K \cong T(u_1, \dots, u_m) / I,$$

$$\text{where } I = \left(u_h^2, [u_i, u_j] : \forall h, \{u_i, u_j\} \in K \right);$$

also

$$\lim^{cdga} S_K \cong R[K],$$

the **Stanley-Reisner algebra** of K ; and

$$\lim^{cdga} \wedge \otimes S_K \cong (\wedge \otimes R[K], d),$$

where $du_i = v_i$ for $1 \leq i \leq m$; and

$$\operatorname{colim}^{cdgc} (S^K)' \cong R\langle K \rangle,$$

the **Stanley-Reisner coalgebra** of K .

We can also define a diagram $\wedge_{U \setminus K}$ by mapping $\tau \supseteq \sigma$ to the monomorphism

$$\wedge(U \setminus \tau) \longrightarrow \wedge(U \setminus \sigma).$$

In this case,

$$\lim^{cdga} \wedge_{U \setminus K} \cong R$$

is simply the ground ring in dimension 0.

3. HOMOTOPY THEORY

Classical homotopy theory does not interact well with limits and colimits!

Taking colimits in top_+ , we have that

$$\operatorname{colim} D^K = \mathcal{Z}_K \quad \text{and} \quad \operatorname{colim} T^{V \setminus K} = \{1\}$$

However, the projections

$$D_\sigma = (D^2)^\sigma \times T^{V \setminus \sigma} \longrightarrow T^{V \setminus \sigma},$$

induce a morphism $D^K \rightarrow T^{V \setminus K}$, which is a **homotopy equivalence** for each face σ of K .

The simplest example of this case is $P^1 = \Delta^1$, so $K = \bullet \bullet$. Then D^K is the pushout diagram

$$T_1 \times D_2^2 \longleftarrow T_1 \times T_2 \longrightarrow D_1^2 \times T_2,$$

and $\operatorname{colim} D^K \cong S^3$.

But $T^{V \setminus K}$ is the pushout

$$T_1 \longleftarrow T_1 \times T_2 \longrightarrow T_2,$$

and $\operatorname{colim} T^{V \setminus K} = \{1\}$.

Algebraically, we take limits in *cdga* and find:

$$\begin{aligned} \lim \wedge \otimes S_K &= \wedge \otimes R[K] \\ \text{and } \lim \wedge_{U \setminus K} &= R \end{aligned}$$

However, the monomorphisms

$$\wedge(U \setminus \sigma) \longrightarrow \wedge(U) \otimes S(\sigma)$$

induce a morphism $\wedge_{U \setminus K} \rightarrow \wedge \otimes S_K$, which is a **quasi-isomorphism** for each face σ of K . Both have cohomology $\wedge(U \setminus \sigma)$.

Again, the simplest example is $K = \bullet \bullet$; then $\wedge \otimes S_K$ is the pullback diagram

$$\begin{array}{ccc} \wedge(u_1, u_2) \otimes S(v_2) & \longrightarrow & \wedge(u_1, u_2) \\ & & \longleftarrow \wedge(u_1, u_2) \otimes S(v_1), \end{array}$$

and $\lim \wedge \otimes S_K = \wedge(u_1, u_2) \otimes S(v_1, v_2)$ (with $du_i = v_i$). It has cohomology $\wedge(u_1 v_2)$.

But $\wedge_{U \setminus K}$ is the pullback

$$\wedge(u_1) \longrightarrow \wedge(u_1, u_2) \longleftarrow \wedge(u_2),$$

and $\lim \wedge_{U \setminus K} = R$.

In both geometric and algebraic contexts, we learn that *objectwise weak equivalences do not preserve colimits or limits*.

In order to understand this situation properly, we follow Quillen's inspired ideas for axiomatising categories in which we can “do homotopy theory”.

This is the world of **model category** theory. In any such category, three classes of special morphism are defined; the **fibrations**, the **cofibrations**, and the **weak equivalences**. They obey axioms that are suggested by the properties of *top*, and allow us to pass to a **homotopy category**, where the weak equivalences are invertible.

The beauty of the axioms is that many algebraic categories also admit natural model structures, as well as more obvious geometric examples such as *top₊* and *sset*.

Given a model category mc and a nice indexing category a , the category $[a, mc]$ admits a canonical model structure, and

a weak equivalence of diagrams is an objectwise weak equivalence.

Recent results show that \lim and colim may always be replaced by more robust functors **holim** and **hocolim**: $[a, mc] \rightarrow mc$.

To construct $\operatorname{hocolim}^{mc} D$ for any diagram $D: a \rightarrow mc$ we:

- (i) replace the objects $D(a)$ by nicer $D'(a)$
- (ii) replace the diagram D' by nicer D''
- (iii) form $\operatorname{colim}^{mc} D''$.

Then $\operatorname{hocolim}^{mc} D$ is preserved (up to weak equivalence in mc) by weak equivalences of diagrams; and by certain functors which do not preserve colim .

Some diagrams are *given* in the form D'' , so \lim and colim are weakly equivalent to holim and $\operatorname{hocolim}$. Examples are BT^K and S_K , for which there is also an isomorphism

$$H^*(\operatorname{colim}^{\operatorname{top}_+} BT^K; R) \cong \lim^{\operatorname{cdga}} S_K.$$

This is better known as

$$H^*(DJ(K); R) \cong R[K] !$$

The source of our weak equivalence $D^K \rightarrow T^{V \setminus K}$ is of the form D'' , but the target is not. So we have a zig-zag

$$\mathcal{Z}_K \simeq \operatorname{colim} D^K \longrightarrow \dots \longleftarrow \operatorname{hocolim} T^{V \setminus K}$$

of weak equivalences in top_+ .

The target of our weak equivalence $\wedge_{U \setminus K} \rightarrow \wedge \otimes S_K$ is also of the form D'' , but the source is not. So we have a zig-zag

$$\operatorname{holim} \wedge_{U \setminus K} \longrightarrow \dots \longleftarrow \lim \wedge \otimes S_K \simeq C^*(\mathcal{Z}_K; \mathbb{Q})$$

of weak equivalences in $\operatorname{cdga}_{\mathbb{Q}}$.

Now consider the problem of describing the loop space $\Omega DJ(K)$ as a colimit.

There are homomorphisms $T^\sigma \rightarrow \text{colim}^{tmon} TK$, which combine to give a homotopy homomorphism

$$\Omega \text{colim}^{top+} BT^K \longrightarrow \text{colim}^{tmon} TK.$$

When K is flag, this is a weak equivalence

$$\Omega DJ(K) \xrightarrow{\cong} Cir(K^{(1)});$$

but not in general.

There is also a homotopy homomorphism

$$\Omega DJ(K) \xrightarrow{\cong} \text{hocolim}^{tmon} TK,$$

which is a weak equivalence for *all* K . So

looping preserves homotopy colimits.

Almost all our algebraic categories admit model structures, in which **weak equivalences are the quasi-isomorphisms**.

So we hope we can describe structures like the Pontrjagin ring $H_*(\Omega DJ(K); R)$ as the homology of an appropriate hocolim in *dga*.

When K is flag, there is a zig-zag of weak equivalences

$$C_*(\Omega DJ(K); \mathbb{Q}) \xrightarrow{\simeq} \cdots \xleftarrow{\simeq} \operatorname{colim}^{dga_{\mathbb{Q}}} \wedge^K;$$

in general, there is a zig-zag

$$C_*(\Omega DJ(K); \mathbb{Q}) \xrightarrow{\simeq} \cdots \xleftarrow{\simeq} \operatorname{hocolim}^{dga_{\mathbb{Q}}} \wedge^K.$$

The proof uses Adams's cobar construction

$$\Omega_* : dgc_{\mathbb{Q}} \longrightarrow dga_{\mathbb{Q}},$$

which is the algebraic analogue of taking loops.

3. FORMALITY

Sullivan's PL-forms functor $A^*: \mathit{top} \rightarrow \mathit{cdga}_{\mathbb{Q}}$ provides a very good representation of the rational homotopy category as an algebraic model category. Understanding $A^*(X)$ in $\mathit{dga}_{\mathbb{Q}}$ is as good as understanding $X_{\mathbb{Q}}$ in $\mathit{top}_{\mathbb{Q}}$.

Certain nice spaces X (such as Eilenberg-Mac Lane spaces, or classifying spaces of Lie groups) are **formal**, because there exists a zig-zag of quasi-isomorphisms

$$(A^*(X), d) \longrightarrow \cdots \longleftarrow (H^*(X; \mathbb{Q}), 0)$$

in $\mathit{cdga}_{\mathbb{Q}}$. In fact $DJ(K)$ is formal for every K .

Some X are also **integrally formal**, because a similar zig-zag of quasi-isomorphisms

$$(C^*(X; \mathbb{Z}), d) \longrightarrow \cdots \longleftarrow (H^*(X; \mathbb{Z}), 0)$$

exists in dga , for the singular cochain algebra $C^*(X; \mathbb{Z})$. This is also true for every $DJ(K)$.

Now we can discuss **toric manifolds** M^{2n} !

We let K be the boundary of a simplicial convex n -polytope, and search for a **linear system of parameters** t_1, \dots, t_n in $\mathbb{Z}[K]$. For any such system t , an M is defined (up to weak equivalence) as the pullback of

$$DJ(K) \xrightarrow{t} BT^n \xleftarrow{p} ET^n,$$

in *top*. So T^n acts freely on M .

Moreover, t induces $t_*: T^V \rightarrow T^n$, and each subtorus $t_*(T^\sigma) < T^n$ is an isotropy subgroup $T(\sigma)$. So there is a *cat*(K)-diagram T^n/K , which maps each $\sigma \subseteq \tau$ to the projection

$$T^n/T(\sigma) \longrightarrow T^n/T(\tau).$$

Then M is hocolim^{*top*} T^n/K , and the orbit space M/T^n is the **nerve** of *cat*(K), or P^n .

We may verify that $A^*(M)$ is weakly equivalent to the homotopy colimit of

$$A^*(DJ(K)) \longleftarrow A^*(BT^n) \longrightarrow A^*(ET^n).$$

Similarly, $H^*(M; \mathbb{Q})$ is weakly equivalent to the homotopy colimit of

$$H^*(DJ(K); \mathbb{Q}) \longleftarrow H^*(BT^n; \mathbb{Q}) \longrightarrow \mathbb{Q},$$

because t is a linear system of parameters.

As $DJ(K)$ is formal and ET^n is contractible, the two diagrams are related by a zig-zag of weak equivalences.

So their homotopy colimits are related by a zig-zag of weak equivalences in $cdga_{\mathbb{Q}}$, and M is formal.

Quillen studies $X_{\mathbb{Q}}$ by means of the model category $dgl_{\mathbb{Q}}$ of differential graded Lie algebras. The homology of $(L_*(X_{\mathbb{Q}}), d)$ gives $\pi_*(\Omega X) \otimes \mathbb{Q}$, equipped with the Whitehead product bracket.

Then X is **coformal** if there is a zig-zag of quasi-isomorphisms

$$(L_*(X_{\mathbb{Q}}), d) \xrightarrow{\cong} \dots \xleftarrow{\cong} (\pi_*(\Omega X) \otimes \mathbb{Q}, 0)$$

in $dgl_{\mathbb{Q}}$.

By restricting to primitive elements, we find that such a zig-zag exists for $DJ(K)$ when and only when there is a zig-zag

$$\Omega_*\mathbb{Q}\langle K \rangle \xrightarrow{\cong} \dots \xleftarrow{\cong} H_*(\Omega DJ(K); \mathbb{Q}).$$

So $DJ(K)$ is coformal if and only if K is flag.