TWO PROBLEMS IN TORIC TOPOLOGY

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Q1 (Lifting Problem): Let $\pi : M^n \longrightarrow P^n$ be a small cover over a simple convex polytope P^n. Is there a quasi-toric manifold $\widetilde{\pi} : M^{2n} \longrightarrow P^n$ over P^n such that M^n is the fixed point set of the *conjugation involution* on M^{2n}?

Remarks. The question is equivalent to asking the existence of the lifting $\widetilde{\lambda}$ of $\lambda : \mathcal{F}(P) \longrightarrow (\mathbb{Z}_2)^n$

$$\begin{tikzcd} & (\mathbb{Z})^n \arrow[d, "\text{mod } 2"] \\ \mathcal{F}(P) \arrow[ur, "\widetilde{\lambda}"] \arrow[r, "\lambda"'] & (\mathbb{Z}_2)^n \end{tikzcd}$$

where $\lambda : \mathcal{F}(P) \longrightarrow (\mathbb{Z}_2)^n$ is the characteristic function determined by $\pi : M^n \longrightarrow P^n$, and $\mathcal{F}(P)$ is the set of all facets of P^n. The problem is still open except for the cases of $n \leq 3$ and $m - n \leq 3$, where m is the number of all facets of P^n (see arXiv:1305.0136).

Q2: Let P be a simple convex polytope, and let $s(P)$ (resp. $s_{\mathbb{R}}(P)$) be the Buchstaber invariant (resp. real Buchstaber invariant) of P. Is it true that $s(P) = s_{\mathbb{R}}(P)$?

Remarks. This is true if $\dim P \leq 3$. In addition, it has been known that $s(P) \leq s_{\mathbb{R}}(P)$. However, in the more general case to simplicial complexes, as shown by Anton Ayzenberg recently, there exists an example of the simplicial complex K of dimension 3 such that $s(K) \neq s_{\mathbb{R}}(K)$. More recently, it is shown by Yi Sun that for the universal real simplicial complex $U^n_{\mathbb{R}}$ determined by $(\mathbb{Z}_2)^n$, when $n = 4$, the difference $\Delta(U^4_{\mathbb{R}}) = s_{\mathbb{R}}(U^4_{\mathbb{R}}) - s(U^4_{\mathbb{R}}) = 1$, and

$$1 = \Delta(U^4_{\mathbb{R}}) \leq \Delta(U^5_{\mathbb{R}}) \leq \cdots \leq \Delta(U^n_{\mathbb{R}}) \leq \cdots .$$