HOPF ALGEBRAS OF SET SYSTEMS

CRISTIAN LENART AND NIGEL RAY

Abstract. Hopf algebras play a major rôle in such diverse mathematical areas as algebraic topology, formal group theory, and theoretical physics, and they are achieving prominence in combinatorics through the influence of G.-C. Rota and his school. Our primary purpose in this article is to build on work of W. Schmitt, and establish combinatorial models for several of the Hopf algebras associated with the universal formal group law and the Lazard ring. In so doing, we incorporate and extend certain invariants of simple graphs such as the umbral chromatic polynomial, and R. Stanley's recently introduced symmetric function. Our fundamental combinatorial components are finite set systems, together with a versatile generalization in which they are equipped with a group of automorphisms. Interactions with the Roman-Rota umbral calculus over graded rings of scalars which may contain torsion are a significant feature of our presentation.

1. Introduction

Binomial Hopf algebras and their duals, namely algebras of formal divided power series (or Hurwitz series), have recently come to provide a natural setting for the Roman-Rota umbral calculus [11]. Closely related are the divided power Hopf algebras and *their* duals, the algebras of formal power series. These objects also all arise in algebraic topology, and are central to the theory of formal groups. Moreover, their investigation has often exhibited a distinctly combinatorial flavor, and it has long been a challenge to develop the algebra in the context of appropriate discrete structures. The second author began such a program in [14], and one of our purposes here is to extend the constructions therein to a richer and more natural combinatorial setting, based on set systems both with, and without, automorphism groups.

We therefore introduce several Hopf algebras of set systems. They map onto the above Hopf algebras, and also onto related such algebras defined, for example, in terms of symmetric functions. Being larger, our examples exhibit richer algebraic structure which reflects their combinatorial origins; furthermore, we may interpret the projection maps as familiar algebraic invariants such as the partition and characteristic type polynomials, the umbral chromatic polynomial, and an extended version of Stanley's symmetric function generalization of the chromatic polynomial. One of the main themes of this program is that the passage from binomial to divided power Hopf algebras correponds, in our combinatorial setting, to the association of a group of automorphisms with a given set system.

We begin in §2 and §3 with background material, reformulated in a way which makes more effective use of the coalgebraic viewpoint. This is important preparation for our later sections, both to establish notation and because the information is not yet readily accessible to combinatorialists. In §4 we discuss set systems, their automorphism groups, and several algebraic invariants; we also offer a combinatorial proof of a familiar formal group law identity. In §5 we construct several Hopf algebras of set systems, and study associated Hopf algebra maps and analogues of the delta operators of umbral calculus, for which we offer two identities concerning their interaction with the product and antipode maps. In §6 we extend the constructions to set systems with a group of automorphisms, and investigate the corresponding problems; one of the main results is a first combinatorial model for the covariant bialgebra of the universal formal group law.

We refer the reader to Aigner [2] for general combinatorial terminology, to Hazewinkel [9] for an encyclopaedic description of the theory of formal groups, to Sweedler [20] and Nichols and Sweedler [11] for all information concerning Hopf algebras and their applications to umbral calculus, and to Bourbaki [3] for the concepts and notation of graded algebra (which we employ without further comment).

It is a pleasure to acknowledge the assistance of Peter Cameron, who provided Theorem 6.1. We are also grateful to William Schmitt for enjoyable and fruitful discussions on combinatorial Hopf algebras; the idea of the proof of Theorem 5.5 belongs to him.

2. Binomial and Divided Power Hopf Algebras

Throughout §2 and §3, we let A_* be a non-negatively graded commutative ring with identity, which we refer to as the ring of *scalars*. We emphasize that A_* is free of additive torsion when, and only when, it embeds in its rationalization $A\mathbb{Q}_* := A_* \otimes \mathbb{Q}$. All rings and algebras we consider are assumed graded by complex dimension, so that products commute without signs. We let C_* be a graded coalgebra over A_*, with comultiplication δ and counit ε; thus δ invests C_* with the structure of both left and right C_*-comodule. We usually assume that C_* is free, and of finite type, and write C^* for the graded dual $\mathrm{Hom}(C_*, A_*)$, which is naturally an A_*-algebra with identity.

We first recall how C^* may be interpreted as a ring of operators on C_*.

Let $\mathrm{Lop}(C_*)$ be the A_*-module consisting of those linear endomorphisms Γ of C_* which are left C_*-comodule maps, and so satisfy the condition

$$\delta \circ \Gamma = (I \otimes \Gamma) \circ \delta \tag{2.1}$$

(where I denotes the identity on C_*). We refer to these operators as *left-invariant*, and consider $\mathrm{Lop}(C_*)$ as an algebra under composition. Then C^* and $\mathrm{Lop}(C_*)$ are isomorphic as A_*-algebras under the map which assigns to each $f \in C^*$ the composition

$$C_* \xrightarrow{\delta} C_* \otimes C_* \xrightarrow{I\otimes f} C_* \otimes A_* \cong C_*\,, \tag{2.2}$$

which we denote by $\mathit{\Gamma}_f$. The inverse map associates to each linear operator $\mathit{\Gamma}$ satisfying (2.1) the linear functional $f_{\mathit{\Gamma}}$ defined by $f_{\mathit{\Gamma}}(z) = \varepsilon(\mathit{\Gamma} z)$, for all $z \in C_*$. We often use (2.2) to equate C^* and various of its subalgebras with their images in $\mathrm{Lop}(C_*)$, identifying f with $\mathit{\Gamma}_f$ and $\mathit{\Gamma}$ with $f_{\mathit{\Gamma}}$. In consequence, for any $\mathit{\Delta} \in \mathrm{Lop}(C_*)$ we may write

$$\langle \mathit{\Gamma} \cdot \mathit{\Delta} \mid z\rangle = \langle \mathit{\Gamma} \mid \mathit{\Delta} z\rangle\,, \tag{2.3}$$

where we follow the standard convention of expressing the duality map as $\langle f|z\rangle := f(z)$, for any $f \in C^*$ and $z \in C_*$. Indeed, 2.3 provides an alternative definition for the action of C^* on C_*.

For any A_*-coalgebra map $p : B_* \to C_*$, we note that

$$\mathit{\Gamma}_f \circ p = p \circ \mathit{\Gamma}_{p^* f} \tag{2.4}$$

in $\mathrm{Hom}(B_*, C_*)$, for all $f \in C^*$. We shall apply this formula in §5 and §6, for example, where C_* is a Hopf algebra, and p is either the product map or the antipode.

We remark that the algebra of *right*-invariant operators is defined by the obvious modification of (2.1), and that whenever C_* is cocommutative, the two concepts coincide. Otherwise, the map corresponding to (2.2) is actually an antiisomorphism.

Given a countable basis c_ω for C_*, we denote the dual pseudobasis for C^* by c^ω. We may then deduce directly from (2.2) that the comultiplication is given in terms of the action of c^ω on C_* by

$$\delta(z) = \sum_\omega c^\omega z \otimes c_\omega\,. \tag{2.5}$$

Whenever $f\colon C_* \to A_*$ is of degree -1 and $f(C_1)$ contains the identity of A_*, we refer to $\mathit{\Gamma}_f$ as a *delta operator*. We may define the category of coalgebras with delta operator by insisting that the morphisms are coalgebra maps which commute with the delta operators given on source and target respectively.

By way of example, consider the graded polynomial algebra $A_*[x]$, and the comultiplication, counit, and antipode maps specified by

$$\delta(x) = x \otimes 1 + 1 \otimes x\,, \qquad \varepsilon(x^i) = \delta_{i,0}\,, \qquad \text{and} \qquad S(x) = -x\,,$$

respectively. These maps invest $A_*[x]$ with the structure of a commutative, cocommutative Hopf algebra, which is known as the *binomial* Hopf algebra over A_* (in one variable). The standard basis consists of the powers x^n, for $n \geq 0$.

Note that δ may be rewritten as

$$\delta\colon A_*[x] \longrightarrow A_*[x, y]\,,$$

in which guise it is given by $\delta(x) = x + y$, and is known as the *shift* (by y). Then the notions of left- and right-invariant coincide, and are traditionally referred to as *shift invariant*. The most basic such operator is the derivative d/dx, which we abbreviate to D. Under the isomorphism of (2.2), it corresponds to the A_*-linear functional which annihilates all x^n for $n \neq 1$, and satisfies $\langle D \mid x \rangle = 1$ in A_0. Thus D is a delta operator on $A_*[x]$. In fact, the functionals defined by $\langle D_{(n)} \,|\, x^m \rangle := \delta_{n,m}$ form the pseudobasis dual to the standard basis (so $D_{(1)} = D$); by (2.5) they act on $A_*[x]$ such that $D_{(n)} x^m = \binom{n}{m} x^{m-n}$ for all non-negative integers n and m. Whenever A_* embeds in $A\mathbb{Q}_*$ we may rewrite $D_{(n)}$ as $D^n/n!$, and interpret (2.5) as the formal Taylor expansion

$$p(x+y) = \sum_n \frac{D^n}{n!} p(x)\, y^n$$

for any polynomial $p(x)$.

We deduce that the graded dual of $A_*[x]$ (as a coalgebra) is the graded algebra $A^*\{\{D\}\}$ of formal divided power series (or *Hurwitz series*; see [6]) over A_*. For each n, we shall express the elements of $A^n\{\{D\}\}$ in the form

$$\alpha_{-n} I + \alpha_{1-n} D + \cdots + \alpha_{k-n} D_{(k)} + \cdots ,$$

where $\alpha_i \in A_i$ and $D_{(k)} D_{(l)} = \binom{k+l}{k} D_{(k+l)}$ for all $k, l > 0$.

We now select a sequence (or *umbra*) $\alpha = (1, \alpha_1, \alpha_2, \dots)$, again with $\alpha_i \in A_i$. Then the Hurwitz series

$$\alpha(D) := D + \alpha_1 D_{(2)} + \ldots + \alpha_{i-1} D_{(i)} + \ldots \tag{2.6}$$

lies in $A^1\{\{D\}\}$, and acts on $A_*[x]$ as a delta operator; in fact any delta operator on $A_*[x]$ is equal to $u\alpha(D)$ for some umbra α and some invertible element u in A_0. The Hurwitz series (2.6) form a group under substitution, with identity D, and we denote the umbra corresponding to the inverse of $\alpha(D)$ by $\overline{\alpha}$. We refer to $\overline{\alpha}(D)$ as the *conjugate* delta operator of $\alpha(D)$.

Note that the divided powers $\alpha(D)_{(n)}$ define a new pseudobasis for $A^*\{\{D\}\}$, and therefore that there is a dual basis of polynomials $B_n^\alpha(x)$ in $A_*[x]$. By definition, these polynomials satisfy $\langle \alpha(D)^n \mid B_m^\alpha(x) \rangle = n!\delta_{n,m}$, from which it follows that $B_n^\alpha(x)$ is monic of degree n (so that $B_0^\alpha(x) = 1$), and that

$$B_n^\alpha(0) = 0 \qquad \text{and} \qquad \alpha(D)\, B_n^\alpha(x) = n B_{n-1}^\alpha(x)\,,$$

for all $n > 0$.

The sequence of polynomials $B^\alpha = (1, B_1^\alpha(x), B_2^\alpha(x), \dots)$ is known as the *associated sequence* of $\alpha(D)$. Note that (2.5) immediately provides the formula

$$\delta(B_n^\alpha(x)) = \sum_{i=0}^{n} \binom{n}{i} B_i^\alpha(x) \otimes B_{n-i}^\alpha(x)\,, \tag{2.7}$$

which defines B^α to be a *binomial sequence*.

We recall some classic examples of delta operators and their associated sequences.

Examples 2.8.

(1) For any A_*, let θ be the umbra $(1,0,0,\dots)$; then $\theta(D)=D$ and $B^\theta_n(x)=x^n$.
(2) For scalars $k_*=\mathbb{Z}[u]$, where $u\in k_1$, let κ be the umbra $(1,u,u^2,\dots)$; then $\kappa(D)$ is the discrete derivative operator $(e^{uD}-1)/u$ and $B^\kappa_n(x)=x(x-u)\dots(x-(n-1)u)$. It follows that $\overline{\kappa}(D)=\ln(1+uD)/u$ and $B^{\overline{\kappa}}_n(x)=\sum_{i=1}^n u^{n-i}S(n,i)x^i$, where the $S(n,i)$ are Stirling numbers of the second kind. Thus $B^\kappa_n(x)$ and $B^{\overline{\kappa}}_n(x)$ are homogeneous versions of the *falling factorial* and *exponential polynomials*, respectively.
(3) For scalars $\varPhi_*=\mathbb{Z}[\phi_1,\phi_2,\dots]$, where $\phi_i\in\varPhi_i$, let ϕ be the umbra $(1,\phi_1,\phi_2,\dots)$. Then $B^\phi_n(x)$ and $B^{\overline{\phi}}_n(x)$ are the *conjugate Bell polynomials* and the *Bell polynomials*, respectively.

For basic information concerning the Bell polynomials we refer to [7]. The following simple property is taken from [13].

Theorem 2.9. *For any binomial Hopf algebra with delta operator $(A_*[x],\alpha(D))$, there is a unique ring homomorphism $g^\alpha\colon \varPhi_*\to A_*$, specified by $\phi_n\mapsto\alpha_n$, which induces a map $g^\alpha\colon(\varPhi_*[x],\phi(D))\to(A_*[x],\alpha(D))$ of graded Hopf algebras with delta operator. Thus $(\varPhi_*[x],\phi(D))$ is the universal example, and B^ϕ is the universal binomial sequence.*

In tandem with $A_*[x]$, and to cope with the situation when the scalars contain torsion, we must also consider the divided power algebra $R_*\{x\}$, where R_* is a similar ring of scalars. This is the free graded R_*-algebra with standard basis the divided powers $x_{(n)}\in R_n\{x\}$ for $n\geq 0$, where $x_{(0)}=1$, $x_{(1)}=x$, and $x_{(k)}x_{(l)}=\binom{k+l}{k}x_{(k+l)}$. Whenever R_* is torsion free, then $x_{(n)}$ becomes identified with $x^n/n!$ in $R\mathbb{Q}_*\{x\}$. We may specify comultiplication, counit, and antipode maps by

$$\delta(x_{(n)})=\sum_{i=0}^n x_{(i)}\otimes x_{(n-i)}\,,\qquad \varepsilon(x_{(n)})=\delta_{n,0}\,,\qquad\text{and}\qquad S(x_{(n)})=(-1)^n x_{(n)}\,,$$

respectively. These maps invest $R_*\{x\}$ with the structure of Hopf algebra. The R_*-linear map $j\colon R_*[x]\to R_*\{x\}$, defined by $x^n\mapsto n!x_{(n)}$ for all $n\geq 0$, is a map of Hopf algebras, and is monic whenever R_* is torsion free. The graded dual of $R_*\{x\}$ (as a coalgebra) is the graded algebra $R^*[[D]]$ of formal power series in the variable D, and the duality is expressed by $\langle D^n\mid x_{(m)}\rangle=\delta_{n,m}$. The action of D on $R_*\{x\}$ given by (2.2) is differentiation with respect to x, as before, whilst the dual map $j^*\colon R^*[[D]]\to R^*\{\{D\}\}$ is prescribed by $D^n\mapsto n!D_{(n)}$.

For any sequence $r=(1,r_1,r_2,\dots)$ with $r_i\in R_i$, the corresponding delta operator on $R_*\{x\}$ is

$$r(D):=D+r_1D^2+r_2D^3+\dots\,,$$

which lies in $R^1[[D]]$. As before, there is a conjugate delta operator $\overline{r}(D)$.

We have now established an important notational convention, to which we shall adhere below: a formal power series associated with an umbra denoted by a Greek or upper case Roman letter (or a lower case Roman letter) will be Hurwitz (or standard) respectively.

In $R_*\{x\}$ there is a basis of polynomials $\beta^r_n(x)$, dual to the alternative pseudobasis $r(D)^n$ of $R^*[[D]]$, where $n \geq 0$. These polynomials form a *divided power sequence*, in the sense that

$$\delta(\beta^r_n(x)) = \sum_{i=0}^{n} \beta^r_i(x) \otimes \beta^r_{n-i}(x)\,; \tag{2.10}$$

we refer to the sequence as β^r.

In order to construct the universal example, and relate it to Theorem 2.9, we take our inspiration from [1] and choose as scalars the poynomial algebra $H_* := \mathbb{Z}[b_1, b_2, \dots]$, where b_n has grading n. Let b be the sequence $(1, b_1, b_2, \dots)$, and observe that we may compatibly identify $\varPhi_*$ as a subalgebra of H_* by means of $\phi_n \mapsto (n+1)!b_n$. Thus there is a canonical inclusion $e\colon (\varPhi_*[x], \phi(D)) \to (H_*\{x\}, b(D))$ of Hopf algebras with delta operator, with respect to which $B^\phi_n(x) = n!\,\beta^b_n(x)$. The following analogue of Theorem 2.9 then holds.

Theorem 2.11. *For any divided power Hopf algebra with delta operator* $(R_*\{x\}, r(D))$, *there is a unique ring homomorphism* $g^r\colon H_* \to R_*$, *specified by* $b_n \mapsto r_n$, *which induces a map* $g^r\colon (H_*\{x\}, b(D)) \to (R_*\{x\}, r(D))$ *of graded Hopf algebras with delta operator. Thus* $(H_*\{x\}, b(D))$ *is the universal example, and* β^b *is the universal divided power sequence.*

Note that in the dual situation, the analogue of e is the algebra map

$$e^!\colon \varPhi^*\{\{D\}\} \to H^*[[D]] \tag{2.12}$$

defined by $\phi(D) \mapsto b(D)$.

3. Formal Group Laws

We now turn to the product structure on $A_*[x]$, and in particular to the question of expressing each $B^\alpha_i(x)B^\alpha_j(x)$ as a linear combination of the $B^\alpha_n(x)$. Inevitably, this is closely linked with the product structure on $A_*\{x\}$, and the expression of $\beta^a_i(x)\beta^a_j(x)$ in terms of the $\beta^a_n(x)$. The problem displays remarkable combinatorial complexity, and its investigation and interpretation are recurring themes below.

Let us begin with $A_*[x]$. The dual of the product is a *formal comultiplication*

$$\delta\colon A^*\{\{D\}\} \longrightarrow A^*\{\{D\}\}\widehat{\otimes} A^*\{\{D\}\}\,,$$

in which we use a suitably completed tensor product $\widehat{\otimes}$. Equivalently, and more naturally, we may interpret δ as the A^*-algebra map

$$\delta\colon A^*\{\{X\}\} \to A^*\{\{X, Y\}\} \tag{3.1}$$

specified by $\delta(X) = X + Y$. This is tantamount to writing X and Y for the respective shift invariant operators $\partial/\partial x$ and $\partial/\partial y$. Clearly δ is coassociative, cocommutative, and an algebra map, whilst $X \mapsto 0$ defines a counit.

To address our problem we first consider its dual, which asks for a description of $\delta(\alpha(X)) = \alpha(X + Y)$ in terms of $\alpha(X)$ and $\alpha(Y)$.

Proposition 3.2. *There are Hurwitz series*

$$F^\alpha(X, Y) \in A^1\{\{X, Y\}\} \qquad \textit{and} \qquad [-1]^\alpha(X) \in A^1\{\{X\}\}$$

such that

$$F^\alpha(\alpha(X), \alpha(Y)) = \alpha(X + Y) \qquad \textit{and} \qquad F^\alpha(X, [-1]^\alpha(X)) = 0\,.$$

Proof. Choose $F^\alpha(X, Y) := \alpha(\overline{\alpha}(X) + \overline{\alpha}(Y))$ and $[-1]^\alpha(X) := \alpha(-\overline{\alpha}(X))$. ☐

It is important to observe that the dual of S on $A_*[x]$ is the algebra endomorphism induced by $X \mapsto -X$, which may equally well be described by $\alpha(X) \mapsto [-1]^\alpha(\alpha(X))$.

For each positive integer m, we shall write

$$F^\alpha(X, Y)_{(m)} = \sum_{i,j} F^{\alpha,m}_{i,j} X_{(i)} Y_{(j)} \qquad \text{and} \qquad ([-1]^\alpha(X))_{(m)} = \sum_k \iota^{\alpha,m}_k X_{(k)}$$

(omitting the superscript m whenever it takes the value 1). We shall make regular use of the abbreviations

$$X +_\alpha Y := F^\alpha(X, Y) \qquad \text{and} \qquad -_\alpha X := [-1]^\alpha(X)\,;$$

these are extremely convenient, and suitably graphic. The first notation may be iterated, whence it makes sense to write $\sum^\alpha$.

So long as A_* is torsion free, we may reinterpret $X +_\alpha Y$ as the 1-dimensional commutative formal group law

$$\sum_{i,j} F^\alpha_{i,j}/(i!j!)\, X^i Y^j \tag{3.3}$$

over $A\mathbb{Q}_*$. This interpretation holds more generally, over any extension $A_* \subseteq A'_*$ which contains the appropriately divided coefficients. We therefore refer to $X +_\alpha Y$ as a *Hurwitz group law* over A_*.

A crucial example of such an extension for arbitrary A_* occurs when there is an embedding $e\colon A_* \to {}^+\!A_*$ into some ring ${}^+\!A_*$ which contains a sequence a such that $\alpha_n = (n+1)!a_n$ for all $n \geq 0$. Then ${}^+\!A_*$ does indeed contain elements $f^a_{i,j}$ for which $F^\alpha_{i,j} = i!j!f^a_{i,j}$, as required, and we denote the formal group law $\sum_{i,j} f^a_{i,j} X^i Y^j$ in ${}^+\!A^*[[X, Y]]$ by $f^a(X, Y)$, or $X +_a Y$. By analogy with the universal case, e extends to an embedding $e\colon (A_*[x], \alpha(D)) \to ({}^+\!A_*\{x\}, a(D))$ of Hopf algebras with delta operator, with respect to which $B^\alpha_n(x) = n!\, \beta^a_n(x)$. The dual analogue of e is

$$e^!\colon A^*\{\{D\}\} \to {}^+\!A^*[[D]]\,,$$

induced by $\alpha(D) \mapsto a(D)$.

Of course, the product on ${}^+\!A_*\{x\}$ dualizes to the formal comultiplication (3.1) on ${}^+\!A^*[[X]]$; then $e^!$ identifies $X +_\alpha Y$ with $X +_a Y$, and, in addition, rewrites 3.2 as

$$f^a(a(X), a(Y)) = a(X+Y) \qquad \text{and} \qquad f^a(X, [-1]^a(X)) = 0\,.$$

Thus $X +_a Y$ has *exp series* $a(X)$, *log series* $\overline{a}(X)$, and $[-1]$ series $[-1]^a(X) = \sum_k i_k^a X^k$, where e provides the identification $\iota_k^\alpha = k!\, i_k^a$ in ${}^+\!A_*$. The covariant bialgebra $U(f^a)_*$ of f^a (see [9]) lies in the chain of Hopf algebra maps

$$A_*[x] \longrightarrow U(f^a)_* \longrightarrow {}^+\!A_*\{x\}\,, \tag{3.4}$$

and is the free ${}^+\!A_*$-module spanned by the polynomials $\beta_n^a(x)$.

With reference to our examples (2.8), both $F^\theta(X,Y)$ and $f^\theta(X,Y)$ are the *additive* group law $X+Y$, whilst both $F^\kappa(X,Y)$ and $f^\kappa(X,Y)$ are the *multiplicative* group law $X+Y+uXY$. Moreover, $F^\phi(X,Y)$ is the *universal* Hurwitz group law, by virtue of Theorem 2.9. The minimal ring with an embedding of the form $e\colon \varPhi_* \to {}^+\varPhi_*$ is clearly H_*, whilst the minimal extension of the form $\varPhi_* \subseteq \varPhi'_*$ is the Lazard ring L_* (which is a subalgebra of H_*, and is again polynomial [9]). The formal group law $f^b(X,Y)$ restricts to L_* by construction, over which it is also universal, and its covariant bialgebra $U(f^b)_*$ is the free L_*-algebra spanned by $\beta_n^b(x)$, where $n \geq 0$. The main thrust of our work in §6 is to provide combinatorial models for these universal examples of 3.4, and for related Hopf algebras.

Since $F^\alpha(X,Y)$ and $[-1]^\alpha(X)$ respectively encode the action of δ and the antipode in terms of the pseudobasis $\alpha(X)_{(m)}$, we may immediately dualize to obtain

$$B_i^\alpha(x)B_j^\alpha(x) = \sum_{m=1}^{i+j} F_{i,j}^{\alpha,m} B_m^\alpha(x) \qquad \text{and} \qquad B_k^\alpha(-x) = \sum_{m=1}^{k} \iota_k^{\alpha,m} B_m^\alpha(x)\,. \tag{3.5}$$

These formulae answer our original question in terms of the Hurwitz group law; they may neatly be summarized as

$$B(X)B(Y) = B(X +_\alpha Y) \qquad \text{and} \qquad S(B(X)) = B(-_\alpha X) \tag{3.6}$$

in $A_*[x]\{\{X,Y\}\}$, where $B(X) = \sum_i B_i^\alpha(x) X_{(i)}$. By iterating the former, we conclude that $\prod B(X_i) = B(\sum_\alpha X_i)$ for any finite sequence of variables X_i; we write $F^{\alpha,m}_{n_1,\dots,n_r}$ for the coefficient of $\prod (X_i)_{(n_i)}$ in $(\sum_\alpha X_i)_{(m)}$.

Applying (2.4) to the product map and the antipode of $A_*[x]$ respectively, we further deduce that

$$\alpha(D)_{(m)}\,(p(x)q(x)) = \sum_{i,j\geq 0} F_{i,j}^{\alpha,m} \bigl(\alpha(D)_{(i)}\, p(x)\bigr)\bigl(\alpha(D)_{(j)}\, q(x)\bigr) \qquad \text{and} \tag{3.7}$$

$$\alpha(D)_{(m)}\, S(p(x)) = \sum_{k\geq 1} \iota_k^{\alpha,m}\, S\bigl(\alpha(D)_{(k)}\, p(x)\bigr)\,, \tag{3.8}$$

for arbitrary $p(x)$ and $q(x) \in A_*[x]$. We refer to (3.7) as the *Leibnitz rule* for $\alpha(D)$.

Returning to our examples (2.8), we note that, in the case of $\theta(D)$, the formulae (3.5) are trivial, and the Leibnitz formula is the standard one for D. In the case of $\kappa(D)$, the product formula (3.5) becomes the *Vandermonde convolution*, and the Leibnitz formula reduces to the well-known action of the discrete derivative on a product. The universal case $\phi(D)$ is considerably more mysterious, and is discussed in later sections.

Formulae (3.5), (3.6), (3.7), and (3.8) may easily be rewritten in terms of $a(D)$ in ${}^{+}\!A_*\{x\}$, in which context the first two are well-known. Note that

$$m!F^{\alpha,m}_{n_1,\dots,n_r} = n_1!\dots n_r! f^{a,m}_{n_1,\dots,n_r} \tag{3.9}$$

in A_*, under the identification provided by e.

4. Set Systems and Their Invariants

Our primary aim in §5 and §6 is to construct combinatorial models for the universal examples $(\varPhi_*[x], \phi(D))$ and $(H_*\{x\}, b(D))$, as described in Theorems 2.9 and 2.11. In this section we introduce the necessary ingredients; they include set systems and automorphisms thereof, Möbius functions, and partition and characteristic type polynomials. Purely as a matter of algebra, our constructions may be carried over to $(A_*[x], \alpha(D))$ and $({}^{+}\!A_*\{x\}, a(D))$ via the maps g^α and g^a. This does not, of course, preclude the eventual emergence of more appropriate models in certain special cases (p-typical examples spring immediately to mind), but is nonetheless our justification for concentrating on the universal cases. A further notational consequence also deserves comment; since the rings of scalars $\varPhi_*$ and H_* are both torsion free, we now (and henceforth) choose to rewrite divided powers such as $\phi(D)_{(n)}$ and $x_{(n)}$ in the more explicit forms $\phi(D)^n/n!$ and $x^n/n!$, respectively.

Throughout the remainder of our work, we shall write $|V|$ for the cardinality of a given set V, and $[n]$ for the set of integers $\{1, 2, \dots, n\}$.

Given any finite set V of *vertices* (possibly empty), we refer to a collection of subsets $\mathcal{S} \subseteq 2^V$ as a *set system* if $\emptyset \in \mathcal{S}$ and $V = \bigcup_{W\in\mathcal{S}} W$; since $\mathcal{S}$ uniquely determines the vertices, we denote V by $V(\mathcal{S})$ whenever $\mathcal{S}$ is in doubt. We shall always assume that a set system contains all the singleton sets.

We shall regularly consider the set systems

$$\mathcal{N}_V := \{\{x\} \,:\, x \in V\} \cup \{\emptyset\}, \qquad \mathcal{K}_V := 2^V, \qquad \text{and} \qquad \mathcal{K}_\pi := \bigcup_{B\in\pi} B,$$

where π is a partition of V. If $V = [n]$, we denote $\mathcal{N}_V$ by $\mathcal{N}_n$, and $\mathcal{K}_V$ by $\mathcal{K}_n$; more generally, if $(n_1, n_2, \dots, n_r)$ lies in $\mathbb{N}^r$, and $\pi = \{[n_1], n_1+[n_2], \dots, \sum_{i=1}^{r-1} n_i+[n_r]\}$ (where $n + [m] := \{n+1, \dots, n+m\}$), we denote $\mathcal{K}_\pi$ by $\mathcal{K}_{n_1,\dots,n_r}$. We define the *complement* of $\mathcal{S}$ by

$$\overline{\mathcal{S}} := (\mathcal{K}_{V(\mathcal{S})} \setminus \mathcal{S}) \cup \mathcal{N}_{V(\mathcal{S})}.$$

Two set systems $\mathcal{S}$ and $\mathcal{T}$ are isomorphic if there is a bijection $f\colon V(\mathcal{S}) \to V(\mathcal{T})$ such that $\{f(U) : U \in \mathcal{S}\} = \mathcal{T}$; we write $\mathbb{S}$ for the set of all isomorphism classes. We shall not attempt to distinguish notationally between a set system and its isomorphism class, since in those cases where it matters, we have taken care to ensure that the context is clear.

Given a group G acting on a set X, we follow convention by writing the set of orbits under the action of G by $G \backslash\backslash X$, the orbit of $x \in X$ by $G(x)$, and the stabilizer of x by G_x. We refer to a pair $(\mathcal{S}, G)$, consisting of a set system $\mathcal{S}$ and a group G of automorphisms of $\mathcal{S}$, as a *set system with automorphism group.* We decree that two such pairs $(\mathcal{S}_1, G_1)$ and $(\mathcal{S}_2, G_2)$ are isomorphic if there is an isomorphism $f\colon V(\mathcal{S}_1) \to V(\mathcal{S}_2)$ of $\mathcal{S}_1$ and $\mathcal{S}_2$ such that G_1 is isomorphic to G_2 via the map $g \mapsto f \circ g \circ f^{-1}$; we then write $\mathbb{SG}$ for the set of isomorphism classes, with the same notational proviso as above. Every set system may be considered as a pair, with respect to the trivial automorphism group consisting of the identity map on the vertices.

We now fix a set system $\mathcal{S}$ for the rest of this section. Let U be a subset of $V(\mathcal{S})$, let σ be a partition of U (employing the non-standard convention that the empty set has a unique partition $\{\emptyset\}$), and let $\widetilde{\sigma} := \sigma \cup \{\{x\} : x \in V(\mathcal{S}) \setminus U\}$. We define the set system $\mathcal{S}|\sigma$ to be

$$\mathcal{S}|\sigma := \{W \in \mathcal{S} : W \subseteq B \text{ for some } B \in \sigma\},$$

and call it the *restriction* of $\mathcal{S}$ to σ. If σ is the single block $\{U\}$, we abbreviate the restriction to $\mathcal{S}|U$. The restriction of the partition σ to a union of its blocks is defined similarly. We also define the *(strong) contraction* of $\mathcal{S}$ through σ to be the following set system with vertex set $\widetilde{\sigma}$:

$$\mathcal{S}//\sigma := \{\widetilde{\sigma}|W : W \in (\mathcal{S} \cup \sigma) \cap \mathrm{Bool}(\widetilde{\sigma})\};$$

here $\mathrm{Bool}(\widetilde{\sigma})$ denotes the Boolean algebra of subsets of $V(\mathcal{S})$ consisting of arbitrary unions of blocks of $\widetilde{\sigma}$. Once more, we abbreviate $\mathcal{S}//\{U\}$ to $\mathcal{S}//U$. For instance,

$$\{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,2,3\}\}//\{2,3\} = \{\emptyset, \{\overline{1}\}, \{\overline{2}\}, \{\overline{1}, \overline{2}\}\},$$

where $\overline{1} := \{1\}$ and $\overline{2} := \{2,3\}$. Given a group G of automorphisms of $\mathcal{S}$, we let $G|\sigma$ denote the image of the group $\bigcap_{B \in \sigma} G_B$ under the projection $S_{V(\sigma)} \times S_{V(\mathcal{S}) \setminus V(\sigma)} \to S_{V(\sigma)}$. The restriction of $(\mathcal{S}, G)$ to σ is naturally defined by $(\mathcal{S}, G)|\sigma := (\mathcal{S}|\sigma, G|\sigma)$. Restriction to $\{U\}$ is abbreviated as before.

Given set systems $\mathcal{S}_1$ and $\mathcal{S}_2$, we shall write their *disjoint union* as $\mathcal{S}_1 \cdot \mathcal{S}_2$, and define their *join* by

$$\mathcal{S}_1 \vee \mathcal{S}_2 := \{U_1 \sqcup U_2 : U_1 \in \mathcal{S}_1, U_2 \in \mathcal{S}_2\},$$

where $\sqcup$ denotes disjoint union of sets. Then

$$\mathcal{S}_1 \odot \mathcal{S}_2 := \overline{\overline{\overline{\mathcal{S}_1} \cdot \overline{\mathcal{S}_2}}}$$

is the operation complementery to disjoint union, which we shall find useful below. We say that a set system is *connected* if it is not isomorphic to a non-trivial disjoint union, and write $\mathbb{S}_\circ$ for the subset of $\mathbb{S}$ consisting of isomorphism classes of connected set systems.

In the case of set systems with automorphism group, we define complement and disjoint union by

$$\overline{(\mathcal{S},G)} := (\overline{\mathcal{S}},G)\,, \qquad \text{and} \qquad (\mathcal{S}_1,G_1)\cdot(\mathcal{S}_2,G_2) := (\mathcal{S}_1\cdot\mathcal{S}_2, G_1\times G_2) \tag{4.1}$$

respectively. Join and $\odot$ are defined analogously, as are the notion of connected, and the subset $\mathbb{SG}_\circ \subset \mathbb{SG}$.

We refer to any partition σ of $V(\mathcal{S})$ which satisfies $\sigma \subseteq \mathcal{S}$ as a *division* by $\mathcal{S}$, and denote the set of such divisions by $\Pi(\mathcal{S})$. It is partially ordered, as usual, by refinement, and the partition of $V(\mathcal{S})$ into singletons is the minimum element $\widehat{0}$ (or $\widehat{0}_{\Pi(\mathcal{S})}$ if the context is unclear). In particular, $\Pi(\mathcal{K}_V)$ is the lattice of partitions of V, and is usually denoted by $\Pi(V)$, or Π_n when $V = [n]$. A *preferential arrangement* is a pair (σ,λ) consisting of a partition σ and a linear ordering λ on the blocks of σ. We write $A(\mathcal{S})$ for the set of all preferential arrangements (σ,λ) with $\sigma \in \Pi(\mathcal{S})$. Following Wagner [21], we define a *coloring* of $\mathcal{S}$ with colors A to be a map $f\colon V(\mathcal{S}) \to A$ whose kernel is a division by $\mathcal{S}$; we denote the set of such colorings by $\Xi_A(\mathcal{S})$. If $A = \mathbb{N}$ or $A = [n]$, we simply call them colorings and colorings with at most n colors, respectively.

Given a set system with automorphism group $(\mathcal{S},G)$, the group acts in an obvious way on $\Pi(\mathcal{S})$ and $A(\mathcal{S})$. It also acts on the set of colorings via the map $(g,f) \mapsto f\circ g^{-1}$, where $g \in G$ and f is a coloring.

We now introduce the Möbius type function (as considered in [10], for example) for a poset P of partitions of a finite set V, ordered by refinement. For such P, the *incidence algebra* $\Phi_*(P)$ is the free Φ_*-module spanned by all functions from the collection of intervals in P to the ring Φ_*, with multiplication specified by the *convolution* formula

$$(f_1 * f_2)(\pi,\sigma) := \sum_{\pi\le\omega\le\sigma} f_1(\pi,\omega)\, f_2(\omega,\sigma)\,.$$

The identity of $\Phi_*(P)$ is the function $\delta(\pi,\sigma) := \delta_{\pi,\sigma}$. For each partition σ of a given set, we define its *type* $\tau^\phi(\sigma) := \phi_1^{k_1}\phi_2^{k_2}\ldots$, where k_i is the number of blocks of σ with $i+1$ elements. The function ζ^ϕ in $\Phi_*(P)$ is defined by $\zeta^\phi(\pi,\sigma) := \tau^\phi(\sigma/\pi)$, where σ/π is the *induced partition* on the blocks of π. The type of the chain $\gamma = \{\pi = \pi_0 < \pi_1 < \ldots < \pi_m = \sigma\}$ is defined by

$$\tau^\phi(\gamma) := (-1)^{l(\gamma}\zeta^\phi(\pi_0,\pi_1)\,\ldots\,\zeta^\phi(\pi_{m-1},\pi_m)\,,$$

where γ has *length* $l(\gamma) := m$. We write $C(P)$ for the set of chains between a minimal and a maximal element of P. The function ζ^ϕ has a convolution inverse, which is denoted by μ^{ϕ} (or μ^{ϕ}_P if there is possible ambiguity); it is called the *Möbius type function* of P, and is given by

$$\mu^{\phi}(\pi,\sigma)=\sum_{\gamma}\tau^{\phi}(\gamma)\,, \tag{4.2}$$

where the summation ranges over $C(\pi,\sigma)$. Observe that both $\zeta^{\phi}(\pi,\sigma)$ and $\mu^{\phi}(\pi,\sigma)$ lie in $\varPhi_{|\pi|-|\sigma|}$.

The classical Möbius function $\mu(\pi,\sigma)$ (or $\mu_P(\pi,\sigma)$) is obtained by setting each ϕ_i to 1. This suggests that we might generalize certain standard properties of $\mu(\pi,\sigma)$ to $\mu^{\phi}(\pi,\sigma)$. Thus we establish the following lemma by direct adaptation of the proof in [18].

Lemma 4.3. *Consider a subposet Q of the lattice of partitions of W, where $W\cap V=\emptyset$, and identify the pair $(\sigma,\sigma')\in P\times Q$ with $\sigma\cup\sigma'$; then*

$$\mu^{\phi}_{P\times Q}(\pi\cup\pi',\sigma\cup\sigma')=\mu^{\phi}_P(\pi,\sigma)\,\mu^{\phi}_Q(\pi',\sigma')$$

in $\varPhi_*$.

Let us now label the vertices of $\mathcal{S}$ by $v_1,v_2,\dots,v_d$. Given another set system $\mathcal{T}$ with $V(\mathcal{T})=V(\mathcal{S})$, and a partition $\sigma\in\varPi(\mathcal{S})$, we define the *Möbius type* of σ with respect to $\mathcal{T}$ by $\nu^{\phi}_{\mathcal{T}}(\sigma):=-\mu^{\phi}_{\varPi(\mathcal{T})\cup\{\sigma\}}(\widehat{0},\sigma)$. We associate with $\mathcal{S}$ the three polynomials

$$\rho^{\phi}(\mathcal{S}\,;x):=\sum_{\sigma\in\varPi(\mathcal{S})}\tau^{\phi}(\sigma)\,x^{|\sigma|}\,,\qquad c^{\phi}(\mathcal{S}\,;x):=\sum_{\sigma\in\varPi(\mathcal{S})}\mu^{\phi}_{\varPi(\mathcal{S})}(\widehat{0},\sigma)\,x^{|\sigma|}\,,$$
$$\text{and}\qquad \chi^{\phi}(\mathcal{S}\,;x):=\sum_{\sigma\in\varPi(\mathcal{S})}\nu^{\phi}_{\mathcal{S}}(\sigma)\,B^{\phi}_{|\sigma|}(x)\,,$$

in $\varPhi_d[x]$, and the degree d homogeneous symmetric function

$$X(\mathcal{S}\,;x):=\sum_{f}x_{f(v_1)}x_{f(v_2)}\dots x_{f(v_d)}\,, \tag{4.4}$$

where the summation ranges over colorings of $\mathcal{S}$, in the $\mathbb{Z}$-algebra $\Lambda_{\mathbb{Z}}$ of symmetric functions in $x=(x_1,x_2,\dots)$. We call $\rho^{\phi}(\mathcal{S}\,;x)$ the *partition type polynomial* of $\mathcal{S}$, $c^{\phi}(\mathcal{S}\,;x)$ the *characteristic type polynomial* of $\mathcal{S}$, and $\chi^{\phi}(\mathcal{S}\,;x)$ the *umbral chromatic polynomial* of $\mathcal{S}$.

We may extend (4.4) to a set system with automorphism group $(\mathcal{S},G)$ by summing only over a transversal of the orbits of G on the colorings; we shall denote this function by $X(\mathcal{S},G\,;x)$.

Special cases of these invariants are well-known. For instance, the partition polynomial of $\mathcal{S}$ investigated in [21] can be retrieved by substituting ϕ_i with 1 in $\rho^{\phi}(\mathcal{S}\,;x)$. The characteristic type polynomial of a poset of partitions of a finite set appears in [15]; if $\mathcal{S}$ is a simplicial complex (so $U\in\mathcal{S}$ whenever $U\subseteq W\in\mathcal{S}$), and all the maximal partitions of $\varPi(\mathcal{S})$ have cardinality m, then the substitution $\phi_i\mapsto 1$ maps $c^{\phi}(\mathcal{S}\,;x)$ to the characteristic polynomial of the poset $\Pi(\mathcal{S})$, up to a factor x^m. The umbral chromatic polynomial of a set system is investigated in [10], and has the property that after *umbral substitution* by $n\phi$, it enumerates by type the *factorized colorings* of $\mathcal{S}$ with at most n colors. If $\mathcal{S}$ is a simplicial complex, then $\nu^{\phi}_{\mathcal{S}}(\sigma) = \tau^{\phi}(\sigma)$ for all $\sigma \in \Pi(\mathcal{S})$; hence, after umbral substitution, $\chi^{\phi}(\mathcal{S}\,;x)$ enumerates ordinary colorings by type. Whenever $\mathcal{S}$ is the independence complex $\mathcal{I}(G)$ of a graph G, the polynomial $\chi^{\phi}(\mathcal{S}\,;x)$ reduces to the umbral chromatic polynomial $\chi^{\phi}(G\,;x)$ (as documented in [15], for example); the classical chromatic polynomial of the graph may be retrieved by replacing each ϕ_i with 1. The symmetric function $X(\mathcal{I}(G)\,;x)$ was defined in [19] as another generalization of the chromatic polynomial of G; as Stanley points out, this function encodes the same information as the umbral chromatic polynomial of G.

By way of simple examples, we remark that

$$\rho^{\phi}(\mathcal{N}_n\,;x) = c^{\phi}(\mathcal{N}_n\,;x) = x^n\,, \quad \rho^{\phi}(\mathcal{K}_n\,;x) = B^{\overline{\phi}}_n(x)\,, \quad c^{\phi}(\mathcal{K}_n\,;x) = B^{\phi}_n(x)\,. \tag{4.5}$$

We also have

$$\overline{\phi}_n = \mu^{\phi}_{\Pi_{n+1}}(\widehat{0},\widehat{1})\,. \tag{4.6}$$

If we combine the standard isomorphism $\Pi(\mathcal{S}_1\cdot\mathcal{S}_2) \cong \Pi(\mathcal{S}_1)\times\Pi(\mathcal{S}_2)$ with Lemma 4.3, we may immediately deduce the following result.

Proposition 4.7. *For any set systems $\mathcal{S}_1$ and $\mathcal{S}_2$, we have*

$$\rho^{\phi}(\mathcal{S}_1\cdot\mathcal{S}_2\,;x) = \rho^{\phi}(\mathcal{S}_1\,;x)\,\rho^{\phi}(\mathcal{S}_2\,;x)\,, \qquad c^{\phi}(\mathcal{S}_1\cdot\mathcal{S}_2\,;x) = c^{\phi}(\mathcal{S}_1\,;x)\,c^{\phi}(\mathcal{S}_2\,;x)\,.$$

We also record a simple property of $X(\mathcal{S},G\,;x)$, directly from the definition.

Proposition 4.8. *Given set systems with automorphism group $(\mathcal{S}_1,G_1)$ and $(\mathcal{S}_2,G_2)$, then*

$$X(\mathcal{S}_1\vee\mathcal{S}_2, G_1\times G_2\,;x) = X(\mathcal{S}_1,G_1\,;x)\,X(\mathcal{S}_2,G_2\,;x)\,;$$

in particular, $X(\mathcal{S}_1\vee\mathcal{S}_2\,;x) = X(\mathcal{S}_1\,;x)\,X(\mathcal{S}_2\,;x)$.

We comment in passing that the operation $\vee$ corresponds to disjoint union of graphs, when we identify a graph with its independence complex.

For later use, we must now recall certain results from [10]; those concerning the partition type polynomial do not appear there explicitly, but they either follow immediately (like (4.11)), or else are entirely similar to the corresponding results for the characteristic type polynomial (like Theorem 4.13).

Theorem 4.9. *If $U \in \mathcal{S}$ has cardinality > 1, then the deletion/contraction formula*

$$c^{\phi}(\mathcal{S}\,;x) = c^{\phi}(\mathcal{S}\setminus U\,;x) + \mu^{\phi}_{\Pi(\mathcal{S}|U)}(\widehat{0},\{U\})\,c^{\phi}(\mathcal{S}/\!/U\,;x) \tag{4.10}$$

holds for $c^{\phi}(\mathcal{S}\,;x)$. If U is also a maximal element in $(\mathcal{S},\subseteq)$, then

$$\rho^{\phi}(\mathcal{S}\,;x) = \rho^{\phi}(\mathcal{S}\setminus U\,;x) + \phi_{|U|-1}\,\rho^{\phi}(\mathcal{S}/\!/U\,;x)\,. \tag{4.11}$$

Note that we have abbreviated $\mathcal{S}\setminus\{U\}$ to $\mathcal{S}\setminus U$.

Theorem 4.12. *For any set system* $\mathcal{S}$*, we have that*

$$\chi^{\phi}(\mathcal{S}\,;x)=c^{\phi}(\overline{\mathcal{S}}\,;x)\,.$$

Theorem 4.13. *For any set system with automorphism group* $(\mathcal{S},G)$*, the polynomials* $\rho^{\phi}(\mathcal{S}\,;x)/|G|$*,* $c^{\phi}(\mathcal{S}\,;x)/|G|$*, and* $\chi^{\phi}(\mathcal{S}\,;x)/|G|$ *all lie in* $H_{|V(\mathcal{S})|}\{x\}$ *.*

We demonstrated in [10] that these polynomials can be expressed by summing over the orbits of G on the poset $A(\mathcal{S})$, rather than over $\varPi(\mathcal{S})$. According to the above theorem, it makes sense to denote them by $\rho^{b}(\mathcal{S},G\,;x)$, $c^{b}(\mathcal{S},G\,;x)$, and $\chi^{b}(\mathcal{S},G\,;x)$, respectively.

Before moving on to our central constructions, we offer a combinatorial interpretation for the coefficients of the universal Hurwitz group law (4.16) as an application of our setup so far. This enables us to give a combinatorial proof for a familiar formal group law identity (Theorem 4.17), which is usually proved by formal power series manipulations, as in [12], for example.

Proposition 4.14. *The elements* $F^{\phi,m}_{n_1,\dots,n_r}\in\varPhi_*$*, defined in §2, may be written as*

$$F^{\phi,m}_{n_1,\dots,n_r}=\sum_{\sigma}\nu^{\phi}_{\mathcal{K}_{n_1,\dots,n_r}}(\sigma)\,,\tag{4.15}$$

where the summation ranges over those divisions by the complement of $\mathcal{K}_{n_1,\dots,n_r}$ *of cardinality* m*; in particular,*

$$F^{\phi}_{n,m}=\nu^{\phi}_{\mathcal{K}_{n,m}}(\{[n+m]\})\,.\tag{4.16}$$

Proof. Using the iterated version of (3.5), (4.5), Proposition 4.7, and Theorem 4.12 successively, we have

$$\begin{aligned}F^{\phi,m}_{n_1,\dots,n_r}&=\langle\phi(D)^m/m!\mid\prod_{i=1}^{r}B^{\phi}_{n_i}(x)\rangle=\langle\phi(D)^m/m!\mid\prod_{i=1}^{r}c^{\phi}(\mathcal{K}_{n_i}\,;x)\rangle\\&=\langle\phi(D)^m/m!\mid c^{\phi}(\mathcal{K}_{n_1,\dots,n_r}\,;x)\rangle=\langle\phi(D)^m/m!\mid\chi^{\phi}(\overline{\mathcal{K}_{n_1,\dots,n_r}}\,;x)\rangle\,.\end{aligned}$$

This establishes 4.15, directly from the definition of $\chi^{\phi}(\cdot\,;x)$. □

We write $\overline{\phi}\,'(X)$ and $\frac{\partial F^{\phi}}{\partial Y}(X,Y)$ for the formal derivatives of the corresponding formal power series.

Theorem 4.17. *In* $\varPhi^*\{\{X\}\}$*, we have that*

$$\overline{\phi}\,'(X)=\left(\frac{\partial F^{\phi}}{\partial Y}(X,0)\right)^{-1}\,,$$

where $(\,\cdot\,)^{-1}$ *denotes the multiplicative inverse.*

Proof. The given formula may easily be seen to be equivalent to the set of identities

$$\overline{\phi}_n = -\sum_{i=0}^{n-1} \binom{n}{i} \overline{\phi}_i \, F^{\phi}_{1,n-i}\,, \qquad n > 0\,. \tag{4.18}$$

Fix $n > 0$. By (4.6) and (4.2), we have

$$\overline{\phi}_n = \sum_{\gamma \in C(\mathit{\Pi}_{n+1})} \tau^{\phi}(\gamma)\,,$$

so it suffices to establish that the right-hand side of (4.18) also enumerates by type the chains in $C(\mathit{\Pi}_{n+1})$.

We first partition these chains into classes $\mathcal{C}(A)$, with $\{1\} \subseteq A \subsetneqq [n+1]$, by assigning the chain $\{\widehat{0} < \sigma_1 < \ldots < \sigma_r < \widehat{1}\}$ to $\mathcal{C}(A)$ if and only if $A \in \sigma_r$. As $1 \le |A| \le n$, and there are $\binom{n}{k-1}$ ways of choosing A of cardinality k, we see that it is enough to prove that if $|A| = k$ then $\sum_{\gamma \in \mathcal{C}(A)} \tau^{\phi}(\gamma) = -\overline{\phi}_{k-1}\, F^{\phi}_{1,n-k+1}$. We do this as follows, by using (4.2), Lemma 4.3 twice, (4.6), and (4.16); for clarity, it helps to set $\overline{A} := [n+1] \setminus A$, and $\omega^{\phi}_{\mathcal{S}}(\sigma) := \mu^{\phi}_{\mathit{\Pi}(\mathcal{S})}(\widehat{0}, \sigma)$ for any set system $\mathcal{S}$ and $\sigma \in \mathit{\Pi}(\mathcal{S})$, thereby yielding

$$\begin{aligned}
\sum_{\gamma \in \mathcal{C}(A)} \tau^{\phi}(\gamma) &= -\sum_{\sigma \in \mathit{\Pi}(\mathcal{K}_{\overline{A}})} \Big(\sum_{\gamma \in C(\mathit{\Pi}(\mathcal{K}_{\sigma \cup \{A\}}))} \tau^{\phi}(\gamma) \Big)\, \zeta^{\phi}(\sigma \cup \{A\}, \{[n+1]\}) \\
&= -\sum_{\sigma \in \mathit{\Pi}(\mathcal{K}_{\overline{A}})} \omega^{\phi}_{\mathcal{K}_{\sigma \cup \{A\}}}(\sigma \cup \{A\})\, \zeta^{\phi}(\sigma \cup \{A\}, \{[n+1]\}) \\
&= -\sum_{\sigma \in \mathit{\Pi}(\mathcal{K}_{\overline{A}})} \omega^{\phi}_{\mathcal{K}_A}(\{A\})\, \omega^{\phi}_{\mathcal{K}_\sigma}(\sigma)\, \zeta^{\phi}(\sigma \cup \{A\}, \{[n+1]\}) \\
&= -\overline{\phi}_{k-1} \sum_{\pi \in \mathit{\Pi}(\mathcal{K}_1 \cdot \mathcal{K}_{\overline{A}})} \omega^{\phi}_{\mathcal{K}_1 \cdot \mathcal{K}_{\overline{A}}}(\pi)\, \zeta^{\phi}(\pi, \{\{1\} \cup \overline{A}\}) \\
&= -\overline{\phi}_{k-1}\, \nu^{\phi}_{\mathcal{K}_1 \cdot \mathcal{K}_{\overline{A}}}(\{\{1\} \cup \overline{A}\}) = -\overline{\phi}_{k-1}\, F^{\phi}_{1,n+1-k}\,,
\end{aligned}$$

as sought. $\square$

5. Hopf Algebras of Set Systems

Consider the free $\mathit{\Phi}_*$-module $\mathit{\Phi}_*\langle \mathbb{S} \rangle$ spanned by the set $\mathbb{S}$. In this section we define several graded Hopf algebra structures on $\mathit{\Phi}_*\langle \mathbb{S} \rangle$, following the general methods of [16] and [17] for constructing the *incidence Hopf algebra* of a *hereditary family of posets* with a *Hopf relation*. The resulting Hopf algebras map onto $\mathit{\Phi}_*[x]$ in a variety of ways, and so provide combinatorial generalizations of the algebraic phenomena associated with the universal Hurwitz group law.

Let $\mathfrak{P}$ be the hereditary family consisting of all finite products of intervals from the posets $(\mathcal{K}_{V(\mathcal{S})}, \subseteq)$, for arbitrary set systems $\mathcal{S}$. Intervals corresponding to different set systems are considered distinct, even if they consist of identical sets, so we index by $\mathcal{S}$ the elements of $\mathcal{K}_{V(\mathcal{S})}$ determining an interval. We define a map from $\mathfrak{P}$ to $\mathbb{S}$ as follows: let $\mathcal{S}_i$ be typical set systems for $i \in [n]$, let $[U_{\mathcal{S}_i}, W_{\mathcal{S}_i}]$ be typical intervals in $\mathcal{K}_{V(\mathcal{S}_i)}$, and map $[U_{\mathcal{S}_1}, W_{\mathcal{S}_1}] \times \ldots \times [U_{\mathcal{S}_n}, W_{\mathcal{S}_n}]$ to the isomorphism class of the set system $\mathcal{S}_1|(W_{\mathcal{S}_1} \setminus U_{\mathcal{S}_1}) \cdot \ldots \cdot \mathcal{S}_n|(W_{\mathcal{S}_n} \setminus U_{\mathcal{S}_n})$. Let $\sim$ be the kernel of this map. The order compatibility of $\sim$ follows from the fact that disjoint union interacts with restriction in such a way that $(\mathcal{S}_1 \cdot \mathcal{S}_2)|(U_1 \sqcup U_2) = (\mathcal{S}_1|U_1) \cdot (\mathcal{S}_2|U_2)$, where $U_i \subseteq V(\mathcal{S}_i)$ for $i = 1, 2$. Moreover, since isomorphism of set systems is a congruence with respect to disjoint union, the relation $\sim$ is a Hopf relation.

Let $H(\mathfrak{P})$ be the Φ_*-incidence Hopf algebra of the family $\mathfrak{P}$ modulo the Hopf relation $\sim$, as defined in [17]. The bijection from $\mathfrak{P}/{\sim}$ to $\mathbb{S}$ induced by the map above can be extended by linearity to a bijection from $H(\mathfrak{P})$ to $\Phi_*\langle\mathbb{S}\rangle$. We use this bijection to transfer the Hopf algebra structure of $H(\mathfrak{P})$ to $\Phi_*\langle\mathbb{S}\rangle$. Thus the comultiplication in $\Phi_*\langle\mathbb{S}\rangle$ is specified by

$$\delta(\mathcal{S}) = \sum_{W \subseteq V(\mathcal{S})} \mathcal{S}|W \otimes \mathcal{S}|\overline{W}\,,$$

where $\overline{W} := V(\mathcal{S}) \setminus W$. The counit is determined by

$$\varepsilon(\mathcal{S}) = \begin{cases} 1 & \text{if } \mathcal{S} = \{\emptyset\} \\ 0 & \text{otherwise}\,. \end{cases}$$

Multiplication is disjoint union, and the unit is the map η specified by $\eta(1) = \{\emptyset\}$. The antipode is determined by

$$S(\mathcal{S}) = \sum_{\sigma \in \Pi(V(\mathcal{S}))} (-1)^{|\sigma|}\, |\sigma|!\, \mathcal{S}|\sigma\,. \tag{5.1}$$

Clearly, the Hopf algebra $\Phi_*\langle\mathbb{S}\rangle$ is commutative and cocommutative. It is also graded, by setting the degree of $\mathcal{S}$ equal to $|V(\mathcal{S})|$, and it has finite type. The indecomposables in $\Phi_*\langle\mathbb{S}\rangle$ are the isomorphism classes of connected set systems, so that $\Phi_*\langle\mathbb{S}\rangle$ is isomorphic, as an algebra, to the polynomial algebra $\Phi_*[\mathbb{S}_\circ]$. Since each poset in the family $\mathfrak{P}$ is a Boolean algebra, we can apply Theorem 10.2 in [17] to obtain further information about the structure of $\Phi_*\langle\mathbb{S}\rangle$, as in Theorem 5.2 below. For this purpose, we recall from [17] the projection λ of $\Phi_*\langle\mathbb{S}\rangle$ onto its primitive elements, specified by

$$\lambda(\mathcal{S}) := \sum_{\sigma \in \Pi(V(\mathcal{S}))} (-1)^{|\sigma|-1}\, (|\sigma|-1)!\, \mathcal{S}|\sigma\,.$$

Theorem 5.2. *The Hopf algebra $\Phi_*\langle\mathbb{S}\rangle$ is isomorphic to the polynomial Hopf algebra $\Phi_*[\lambda(\mathbb{S}_\circ)]$, having primitive indeterminates.*

We can define similar Hopf algebra structures on $\Phi_*\langle\mathbb{S}\rangle$ by basing the multiplication on $\vee$ or $\odot$, rather than disjoint union; this is possible because both operations interact with restriction in similar fashion to $\cdot$, and isomorphism of set systems is still a congruence. The resulting Hopf algebras are likewise polynomial, and have primitive indeterminates. Complementation of set systems induces a Hopf algebra isomorphism from $(\Phi_*\langle\mathbb{S}\rangle, \cdot)$ to $(\Phi_*\langle\mathbb{S}\rangle, \odot)$.

We now consider the graded dual of the coalgebra $\Phi_*\langle\mathbb{S}\rangle$, which we write as $\Phi^*\langle\!\langle D_{\mathbb{S}}\rangle\!\rangle$ after selecting the pseudobasis $D_{\mathbb{S}} := \{D_{\mathcal{S}} : \mathcal{S} \in \mathbb{S}\}$ dual to $\mathbb{S}$. As a left-invariant operator, $D_{\mathcal{S}}$ acts on $\Phi_*\langle\mathbb{S}\rangle$ according to the rule

$$D_{\mathcal{S}}\,\mathcal{T} = \sum_{W} \mathcal{T}|\overline{W}\,,$$

where the summation ranges over those subsets of $V(\mathcal{T})$ for which $\mathcal{T}|W = \mathcal{S}$, and $\overline{W} := V(\mathcal{T}) \setminus W$. The multiplication in $\Phi^*\langle\!\langle D_{\mathbb{S}}\rangle\!\rangle$ is given by

$$D_{\mathcal{S}_1}\,D_{\mathcal{S}_2} = \sum_{\mathcal{S}\in\mathbb{S}} (\mathcal{S};\mathcal{S}_1,\mathcal{S}_2)\,D_{\mathcal{S}}\,,$$

where $(\mathcal{S};\mathcal{S}_1,\mathcal{S}_2)$ denotes the coefficient of $\mathcal{S}_1 \otimes \mathcal{S}_2$ in $\delta(\mathcal{S})$. If we utilize Theorem 5.2 to view $\Phi_*\langle\mathbb{S}\rangle$ as a polynomial algebra with primitive indeterminates, and write $D_{\lambda(\mathcal{S})}$ for the partial differentiation operator with respect to the variable $\lambda(\mathcal{S})$ in $\Phi_*[\lambda(\mathbb{S}_\circ)]$, then $\Phi^*\langle\!\langle D_{\mathbb{S}}\rangle\!\rangle$ is isomorphic to the algebra $\Phi^*\{\{D_{\lambda(\mathbb{S}_\circ)}\}\}$ of Hurwitz series in the variables $D_{\lambda(\mathcal{S})}$, where $\mathcal{S}$ ranges over $\mathbb{S}_\circ$.

Perceptive readers may have noticed that, thus far, we could have restricted our scalars to $\mathbb{Z}$. We now impose deletion/contraction relations on $\Phi_*\langle\mathbb{S}\rangle$ which involve the scalars Φ_* in an essential manner. Let R_* be the graded submodule of $\Phi_*\langle\mathbb{S}\rangle$ spanned by all elements of the form

$$\mathcal{S} - \mathcal{S}\setminus U - \phi_{|U|-1}\,\mathcal{S}/\!/U\,, \tag{5.3}$$

where U is a maximal element of the poset $(\mathcal{S}, \subseteq)$ with $|U| > 1$, and let Q_* be the graded submodule spanned by all elements of the form

$$\mathcal{S} - \mathcal{S}\setminus U - \mu^{\phi}_{\Pi(\mathcal{S}|U)}(\widehat{0}, \{U\})\,\mathcal{S}/\!/U\,,$$

with U as before. Then writing ρ^ϕ, c^ϕ, and χ^ϕ for the Φ_*-linear maps $\Phi_*\langle\mathbb{S}\rangle \to \Phi_*[x]$ obtained by respectively assigning $\rho^\phi(\mathcal{S}\,;x)$, $c^\phi(\mathcal{S}\,;x)$, and $\chi^\phi(\mathcal{S}\,;x)$ to $\mathcal{S}$, it is not difficult to check, using Proposition 4.9, that $\ker\rho^\phi = R_*$ and $\ker c^\phi = Q_*$. Our first realization result for polynomial invariants can now be proven.

Theorem 5.4. *The maps* ρ^ϕ, $c^\phi\colon (\Phi_*\langle\mathbb{S}\rangle, \cdot) \to \Phi_*[x]$ *and* $\chi^\phi\colon (\Phi_*\langle\mathbb{S}\rangle, \odot) \to \Phi_*[x]$ *are surjective maps of graded Hopf algebras.*

Proof. Surjectivity follows from (4.5), whilst ρ^ϕ and c^ϕ are algebra maps by Proposition 4.7. It therefore suffices to prove that they are coalgebra maps as well, since any bialgebra map of Hopf algebras is a Hopf algebra map. We will show that R_* and Q_* are coideals in the corresponding coalgebras, concentrating on the former.

Obviously, $\varepsilon(R_*) = 0$. Now consider an element of the form (5.3), and $W \subseteq V(\mathcal{S})$. If $U \subseteq W$, then $(\mathcal{S}\setminus U)|\overline{W} = \mathcal{S}|\overline{W}$. If $W \subseteq \overline{U}$, then $(\mathcal{S}\setminus U)|W = \mathcal{S}|W$. If none of the above hold, then $(\mathcal{S}\setminus U)|W = \mathcal{S}|W$ and $(\mathcal{S}\setminus U)|\overline{W} = \mathcal{S}|\overline{W}$. Finally, since deletion commutes with restriction, we have

$$\delta(\mathcal{S}-\mathcal{S}\setminus U) = \sum_{W\supseteq U}(\mathcal{S}|W-(\mathcal{S}|W)\setminus U)\otimes\mathcal{S}|\overline{W}+\sum_{W\subseteq\overline{U}}\mathcal{S}|W\otimes(\mathcal{S}|\overline{W}-(\mathcal{S}|\overline{W})\setminus U)\,.$$

Since U is a maximal element of $(\mathcal{S},\subseteq)$, then $\mathcal{S}//U$ is isomorphic to $\mathcal{N}_1\cdot\mathcal{S}|\overline{U}$ and

$$\begin{aligned}\delta(\mathcal{S}//U) &= (\mathcal{N}_1\otimes\{\emptyset\}+\{\emptyset\}\otimes\mathcal{N}_1)\cdot\sum_{W\subseteq\overline{U}}\mathcal{S}|W\otimes\mathcal{S}|(\overline{U}\setminus W)\\ &= \sum_{W\supseteq U}(\mathcal{S}|W)//U\otimes\mathcal{S}|\overline{W}+\sum_{W\subseteq\overline{U}}\mathcal{S}|W\otimes(\mathcal{S}|\overline{W})//U\,.\end{aligned}$$

These relations show that $\delta(R_*)\subseteq R_*\otimes\Phi_*\langle\mathbb{S}\rangle+\Phi_*\langle\mathbb{S}\rangle\otimes R_*$, whence R_* is a coideal.

The proof for Q_* is similar, and the result for χ^ϕ follows from that for c^ϕ, using Theorem 4.12. ☐

According to [8], the algebra $\Lambda_{\mathbb{Z}}$ of symmetric functions has a Hopf algebra structure which may be described as follows. Given $A\subseteq\mathbb{N}$ and $t(x)\in\Lambda_{\mathbb{Z}}$, let $t(x_A)$ denote the symmetric function obtained from $t(x)$ by substituting 0 for x_i whenever $i\notin A$. If $t(x)$ has degree n, then its image under comultiplication can be computed by examining the image of $t(x_{[2n]})$ under the natural isomorphism $\mathbb{Z}[x_1,\dots,x_{2n}]\cong\mathbb{Z}[x_1,\dots,x_n]\otimes\mathbb{Z}[x_{n+1},\dots,x_{2n}]$. This enables us to establish an analogous result to Theorem 5.4 for the $\mathbb{Z}$-linear map $X\colon\mathbb{Z}\langle\mathbb{S}\rangle\to\Lambda_{\mathbb{Z}}$ specified by $\mathcal{S}\mapsto X(\mathcal{S}\,;x)$.

Theorem 5.5. *The map $X\colon(\mathbb{Z}\langle\mathbb{S}\rangle,\vee)\to\Lambda_{\mathbb{Z}}$ is a map of graded Hopf algebras.*

Proof. According to Proposition 4.8, we have only to prove that X is a coalgebra map. Given any coloring f of a set system $\mathcal{S}$, let $x^f:=\prod_{v\in V(\mathcal{S})}x_{f(v)}$. Consider a set system $\mathcal{S}$ with d vertices. It suffices to show that

$$X(\mathcal{S}\,;x_{[2d]})=\sum_{W\subseteq V(\mathcal{S})}X(\mathcal{S}|W\,;x_{[d]})\,X(\mathcal{S}|\overline{W}\,;x_{d+[d]})\,.$$

Now there is an obvious bijection from $\Xi_{[2d]}(\mathcal{S})$ to $\bigcup_{W\subseteq V(\mathcal{S})}\Xi_{[d]}(\mathcal{S}|W)\times\Xi_{d+[d]}(\mathcal{S}|\overline{W})$, namely $f\mapsto(f',f'')$, where $f'=f|f^{-1}([d])$ and $f''=f|f^{-1}(d+[d])$; moreover, we clearly have $x^f=x^{f'}x^{f''}$, from which the formula follows. ☐

Inspection of the comultiplication in $\Phi_*\langle\mathbb{S}\rangle$ reveals that the sequences $(\mathcal{N}_n)$ and $(\mathcal{K}_n)$ are binomial in the sense of (2.7), and according to 4.5 they map (as they must) to familiar binomial sequences in $\Phi_*[x]$ under ρ^ϕ, c^ϕ, and χ^ϕ. It is therefore of interest to determine *all* binomial sequences in $\Phi_*\langle\mathbb{S}\rangle$, and especially those whose elements lie entirely within the generating set $\mathbb{S}$. For this purpose, we define the set system

$$\mathcal{K}_n(A):=\{U\subseteq[n]\;:\;|U|\in A\}$$

for each $n \in \mathbb{N} \cup \{0\}$, and each set $A \subseteq \mathbb{N} \cup \{0\}$ containing 0 and 1. Obviously, $\mathcal{K}_n(\{0,1\}) = \mathcal{N}_n$, and $\mathcal{K}_n(\mathbb{N} \cup \{0\}) = \mathcal{K}_n$, for all n.

Proposition 5.6. *The only binomial sequences whose elements lie in* $\mathbb{S}$ *are those of the form* $(\mathcal{K}_n(A))$ *for* $\{0,1\} \subseteq A \subseteq \mathbb{N} \cup \{0\}$.

Proof. Clearly, all sequences $(\mathcal{K}_n(A))$ are binomial. So let $(\mathcal{B}_n)$ be a binomial sequence, and let C_* be the subcoalgebra of $\varPhi_*\langle\mathbb{S}\rangle$ spanned by the elements $\mathcal{B}_n$. Assume for induction that there is a set $A_n \subseteq [n] \cup \{0\}$ such that $\mathcal{B}_i = \mathcal{K}_i(A_n)$ for all $0 \leq i \leq n$; this certainly holds for $n = 0$. Then define

$$A_{n+1} := \begin{cases} A_n \cup \{n+1\} & \text{if } V(\mathcal{B}_{n+1}) \in \mathcal{B}_{n+1} \\ A_n & \text{otherwise}. \end{cases}$$

Choose an arbitrary subset U of $V(\mathcal{B}_{n+1})$ with $1 \leq |U| \leq n$, and $x \in V(\mathcal{B}_{n+1}) \setminus U$. The binomial property implies that $\mathcal{B}_{n+1}|(V(\mathcal{B}_{n+1}) \setminus \{x\})$ is isomorphic to $\mathcal{B}_n$. It follows by the inductive hypothesis that $U \in \mathcal{B}_{n+1}$ if and only if $|U| \in A_n \subseteq A_{n+1}$. The proof is completed by setting $A := \bigcup_{n \geq 0} A_n$. ☐

It is now time to embellish our Hopf algebras with delta operators.

Let $\theta\colon \varPhi_*\langle\mathbb{S}\rangle \to \varPhi_*[x]$ be a surjective map of graded coalgebras. Then its dual $\theta^*\colon \varPhi^*\{\{D\}\} \to \varPhi^*\langle\langle D_{\mathbb{S}}\rangle\rangle$ is an injective algebra map. Hence θ^* induces an isomorphism from $\varPhi^*\{\{D\}\}$ to the subalgebra $\varPhi^*\{\{D^\theta\}\}$ of $\varPhi^*\langle\langle D_{\mathbb{S}}\rangle\rangle$, where $D^\theta := \theta^*(D)$. Therefore, given a delta operator $\psi(D) \in \varPhi^1\{\{D\}\}$, we may write the delta operator $\theta^*(\psi(D))$ as $\psi(D^\theta)$.

Proposition 5.7. *With the above notation, the map* $\theta\colon (\varPhi_*\langle\mathbb{S}\rangle, \psi(D^\theta)) \to (\varPhi_*[x], \psi(D))$ *is a map of coalgebras with delta operator.*

Proof. By definition, we have $\langle \psi(D^\theta) \mid \cdot \rangle = \langle \psi(D) \mid \theta(\cdot) \rangle$. So (2.2) implies that $\psi(D^\theta) = (I \otimes (\langle \psi(D) \mid \cdot \rangle \circ \theta)) \circ \delta$. The result now follows from the commutativity of the diagram

$$\begin{tikzcd}
\varPhi_*\langle\mathbb{S}\rangle \arrow[r, "\delta"] \arrow[d, "\theta"'] & \varPhi_*\langle\mathbb{S}\rangle \otimes \varPhi_*\langle\mathbb{S}\rangle \arrow[r, "I \otimes \omega\circ\theta"] \arrow[d, "\theta\otimes\theta"] & \varPhi_*\langle\mathbb{S}\rangle \arrow[d, "\theta"] \\
\varPhi_*[x] \arrow[r, "\delta"] & \varPhi_*[x] \otimes \varPhi_*[x] \arrow[r, "I \otimes \omega"] & \varPhi_*[x]
\end{tikzcd} \tag{5.8}$$

where ω denotes $\langle \psi(D) \mid \cdot \rangle$. ☐

Clearly, we can take θ to be any one of the maps ρ^ϕ, χ^ϕ or c^ϕ, and combine the results of Proposition 5.7 and Theorem 5.4. By way of example, we record that

$$D^\rho \mathcal{S} = \sum_{W \in \mathcal{S} \setminus \{\emptyset\}} \phi_{|W|-1} \mathcal{S}|\overline{W}, \qquad D^c \mathcal{S} = \sum_{W \in \mathcal{S} \setminus \{\emptyset\}} \mu^\phi_{\Pi(\mathcal{S}|W)}(\widehat{0}, \{W\})\, \mathcal{S}|\overline{W},$$

$$\text{and} \qquad \phi(D^c)\, \mathcal{S} = \sum_{W \in \overline{\mathcal{S}} \setminus \{\emptyset\}} \nu^\phi_{\mathcal{S}|W}(\{W\})\, \mathcal{S}|\overline{W} \qquad \text{(by Theorem 4.12)}$$

in $\Phi_*\langle\mathbb{S}\rangle$. In particular, we have

$$D^\rho\mathcal{N}_n = D^c\mathcal{N}_n = n\,\mathcal{N}_{n-1} \qquad \text{and} \qquad \phi(D^c)\,\mathcal{K}_n = n\,\mathcal{K}_{n-1}\,. \tag{5.9}$$

Let $\varDelta$ be an arbitrary delta operator on $\Phi_*\langle\mathbb{S}\rangle$. Let C_* be a subcoalgebra of $\Phi_*\langle\mathbb{S}\rangle$ which is also a direct summand, and whose complement is $\Phi^*\{\{D_{\mathcal{N}_1}\}\}^\perp$, where $D_{\mathcal{N}_1}$ is interpreted as a functional (see [20]). Then $\varDelta$ acts on C_* non-trivially, and $\Phi^*\{\{\varDelta\}\}$ may be viewed as its dual. In this context, it is consistent to refer to the basis of C_* dual to the pseudobasis $\varDelta^n/n!$ as the associated sequence of $\varDelta$ in C_*; such sequences are obviously binomial. Of special interest are the subcoalgebras $C(A)_*$ spanned by the binomial sequences $(\mathcal{K}_n(A))$ of Proposition 5.6. Note that $(\mathcal{K}_n(A))$ is the associated sequence of $D_{\mathcal{N}_1}$ in $C(A)_*$, but that it is also the associated sequence of other delta operators, as exemplified by 5.9. There is an isomorphism of coalgebras with delta operator between $(C(A)_*, \varDelta)$ and $(\Phi_*[x], \psi(D))$, specified by $\mathcal{K}_n(A) \mapsto x^n$, where $\psi_{n-1} = \langle \varDelta \mid \mathcal{K}_n(A)\rangle$. The determination of the associated sequences of $\varDelta$ therefore reduces to the classical case.

We conclude this section by explaning how the identities (3.7) and (3.8), which hold in $\Phi_*[x]$, can be lifted to $\Phi_*\langle\mathbb{S}\rangle$.

Proposition 5.10. *Let $\varDelta$ be a delta operator on $\Phi_*\langle\mathbb{S}\rangle$ which is a derivation; then the following identities hold in $\Phi_*\langle\mathbb{S}\rangle$:*

$$\phi(\varDelta)\,(\mathcal{S}_1\cdot\mathcal{S}_2) = \sum_{i,j\geq 0} F^\phi_{i,j}\,(\phi(\varDelta)^i/i!\,\mathcal{S}_1)\cdot(\phi(\varDelta)^j/j!\,\mathcal{S}_2) \tag{5.11}$$

$$\phi(\varDelta)\,S(\mathcal{S}) = \sum_{k\geq 1} \iota^\phi_k\, S(\phi(\varDelta)^k/k!\,\mathcal{S})\,. \tag{5.12}$$

Given any map $\theta\colon (\Phi_*\langle\mathbb{S}\rangle,\cdot) \to \Phi_*[x]$ of graded Hopf algebras, D^θ is a derivation, so the above identities hold for it.

Proof. Let p denote the multiplication in $\Phi_*\langle\mathbb{S}\rangle$. We have that $\varDelta\circ p = p\circ(\varDelta\otimes I + I\otimes\varDelta)$, whence, by Proposition 3.2,

$$\phi(\varDelta)\circ p = p\circ\phi(\varDelta\otimes I + I\otimes\varDelta) = p\circ F^\phi(\varDelta\otimes I, I\otimes\varDelta)\,.$$

This proves (5.11). On the other hand, we know from (2.4) that $\varDelta\circ S = S\circ\Gamma_{\langle\varDelta\circ S\,|\,\cdot\rangle}$. For all $\mathcal{S}$, $\mathcal{S}_1$ and $\mathcal{S}_2$ in $\mathbb{S}$ with $\mathcal{S}_1,\mathcal{S}_2\neq\{\emptyset\}$, we have that

$$S(\mathcal{S}) = -\mathcal{S} + \text{decomposables in } \Phi_*\langle\mathbb{S}\rangle \qquad (\text{see } (5.1))\,,$$

and

$$\langle\varDelta\mid\mathcal{S}_1\cdot\mathcal{S}_2\rangle = \langle\varDelta\otimes I + I\otimes\varDelta\mid\mathcal{S}_1\otimes\mathcal{S}_2\rangle = 0\,.$$

This implies $\langle\varDelta\circ S|\cdot\rangle = \langle-\varDelta|\cdot\rangle$, whence the linear operator corresponding $\langle\varDelta\circ S|\cdot\rangle$ is $-\varDelta$, and $\varDelta\circ S = -S\circ\varDelta$. Using Proposition 3.2 again, we immediately obtain

$$\phi(\varDelta)\circ S = S\circ\phi(-\varDelta) = S\circ\iota^\phi(\phi(\varDelta))\,,$$

which proves (5.12).

If θ is the map specified above, then, by (2.4), we have that $D^\theta \circ p = p \circ \varGamma_{\langle D^\theta \circ p \,|\, \cdot \rangle}$. For all $\mathcal{S}_1$ and $\mathcal{S}_2$ in $\mathbb{S}$, we have

$$\begin{aligned}\langle D^\theta \circ p \,|\, \mathcal{S}_1 \otimes \mathcal{S}_2 \rangle &= \langle D \mid \theta(\mathcal{S}_1 \cdot \mathcal{S}_2) \rangle = \langle D \mid \theta(\mathcal{S}_1)\, \theta(\mathcal{S}_2) \rangle \\ &= \langle D \otimes I + I \otimes D \mid \theta(\mathcal{S}_1) \otimes \theta(\mathcal{S}_2) \rangle \\ &= \langle D^\theta \otimes I + I \otimes D^\theta \mid \mathcal{S}_1 \otimes \mathcal{S}_2 \rangle \,.\end{aligned}$$

In consequence, the operator corresponding to $\langle D^\theta \circ p \,|\, \cdot \rangle$ is $D^\theta \otimes I + I \otimes D^\theta$, and $D^\theta \circ p = p \circ (D^\theta \otimes I + I \otimes D^\theta)$. □

We may take $\theta = \rho^\phi$ and $\theta = c^\phi$ in Proposition 5.10 to obtain identities concerning the interaction of $\phi(D^\rho)$ and $\phi(D^c)$ with the multiplication and the antipode in $(\varPhi_*\langle \mathbb{S} \rangle, \cdot)$. Analogous identities hold for $\phi(D^\chi)$ in $(\varPhi_*\langle \mathbb{S} \rangle, \odot)$.

6. A Hopf Algebra of Set Systems with Automorphism Group

In this final section, we extend the constructions of §5 by defining Hopf algebra structures on the free H_*-module spanned by the set $\mathbb{SG}$.

The main difference concerns the hereditary family $\mathfrak{P}$, which now consists of all finite products of intervals in the posets $(\mathcal{K}_{V(\mathcal{S})}/G, \subseteq)$, induced by the action of G on $(\mathcal{K}_{V(\mathcal{S})}, \subseteq)$ for any set system with automorphism group $(\mathcal{S}, G)$. The comultiplication in $H_*\langle \mathbb{SG} \rangle$ is specified by

$$\delta(\mathcal{S}, G) = \sum_{W \in \mathcal{T}} (\mathcal{S}, G)|W \otimes (\mathcal{S}, G)|\overline{W}\,,$$

where $\mathcal{T}$ is a transversal of $\mathcal{K}_{V(\mathcal{S})}/G$. The counit is determined by

$$\varepsilon(\mathcal{S}, G) = \begin{cases} 1 & \text{if } (\mathcal{S}, G) = (\{\emptyset\}, \{1\}) \\ 0 & \text{otherwise}\,. \end{cases}$$

Multiplication is induced by disjoint union, as defined by (4.1), and the unit is the map η specified by $\eta(1) = (\{\emptyset\}, \{1\})$. In order to express the antipode using the general formula in [17], we define the equivalence $\sim$ on $A(\mathcal{K}_{V(\mathcal{S})})$ as follows: $(B_1, \dots, B_k) \sim (B_1', \dots, B_k')$ if and only if B_i and B_i' are in the same orbit of G on $\mathcal{K}_{V(\mathcal{S})}$, for all i. We now have the following formula for the antipode:

$$S(\mathcal{S}, G) = \sum_{(\sigma, \lambda) \in \mathcal{T}} (-1)^{|\sigma|} \prod_{B \in \sigma} (\mathcal{S}, G)|B\,,$$

where $\mathcal{T}$ is an arbitrary transversal of $A(\mathcal{K}_{V(\mathcal{S})})/\sim$. Clearly, the Hopf algebra $H_*\langle \mathbb{SG} \rangle$ is commutative and cocommutative. It is also graded, has finite type, and is isomorphic as an algebra to the polynomial algebra $H_*[\mathbb{SG}_\circ]$. We may define similar Hopf algebra structures by replacing the multiplication $\cdot$ with $\vee$ or $\odot$.

Let $\mathbb{SG}'$ be the subset of $\mathbb{SG}$ consisting of those isomorphism classes for which G has the property that every cycle of an element of G is also in G. Peter

Cameron has determined all permutation groups with this property, and called them *cycle-closed.*

Theorem 6.1 ([5])**.** *A permutation group is cycle-closed if and only if it is the direct product of its transitive constituents, each of which is a symmetric group or a cyclic group of prime order.*

Let $\rho^b, c^b\colon H_*\langle\mathbb{SG}\rangle \to H_*\{x\}$ be the H_*-linear maps specified respectively by $(\mathcal{S}, G) \mapsto \rho^b(\mathcal{S}, G\,; x)$ and $(\mathcal{S}, G) \mapsto c^b(\mathcal{S}, G\,; x)$; note that we can choose the codomains to be $H_*\{x\}$ by virtue of Theorem 4.13. Similarly, let $X\colon \mathbb{Z}\langle\mathbb{SG}\rangle \to \Lambda_{\mathbb{Z}}$ be the $\mathbb{Z}$-linear map specified by $(\mathcal{S}, G) \mapsto X(\mathcal{S}, G\,; x)$. The maps ρ^b, c^b and X are not coalgebra maps, as simple examples show. Nevertheless, this situation is rectified by restricting to the sub-Hopf algebras $H_*\langle\mathbb{SG}'\rangle$ and $\mathbb{Z}\langle\mathbb{SG}'\rangle$ respectively.

Theorem 6.2. *The maps* $\rho^b, c^b\colon (H_*\langle\mathbb{SG}'\rangle, \cdot) \to H_*\{x\}$ *and* $X\colon (\mathbb{Z}\langle\mathbb{SG}'\rangle, \vee) \to \Lambda_{\mathbb{Z}}$ *are surjective maps of graded Hopf algebras.*

Proof. Surjectivity follows from (4.5) and the fact that $X(\mathcal{N}_n, S_n\,; x)$ is the elementary symmetric function e_n. According to Propositions 4.7 and 4.8, it remains only to prove that the given maps preserve comultiplication.

Let $(\mathcal{S}, G)$ be a set system with automorphism group for which $\mathcal{S}$ has d vertices, and let $\mathcal{T}$ be a transversal of the orbits of G on $\mathcal{K}_{V(\mathcal{S})}$. Then

$$((\rho^b \otimes \rho^b) \circ \delta)\,(\mathcal{S}, G) = \sum_{W \in \mathcal{T}} \frac{\rho^\phi(\mathcal{S}|W\,; x)}{|G|W|} \otimes \frac{\rho^\phi(\mathcal{S}|\overline{W}\,; x)}{|G|\overline{W}|}\,.$$

Using Theorem 5.4 and the standard fact that $|G(W)| = |G|/|G_W|$, we obtain

$$(\delta \circ \rho^b)\,(\mathcal{S}, G) = \sum_{W \subseteq V(\mathcal{S})} \frac{\rho^\phi(\mathcal{S}|W\,; x) \otimes \rho^\phi(\mathcal{S}|\overline{W}\,; x)}{|G|} = \sum_{W \in \mathcal{T}} \frac{\rho^\phi(\mathcal{S}|W\,; x) \otimes \rho^\phi(\mathcal{S}|\overline{W}\,; x)}{|G_W|}\,.$$

Since G is cycle-closed, we have $G_W \cong G|W \times G|\overline{W}$, whence $(\rho^b \otimes \rho^b) \circ \delta = \delta \circ \rho^b$. The result for c^b follows similarly.

In order to prove that X is a coalgebra map, it suffices to show that

$$X(\mathcal{S}, G\,; x_{[2d]}) = \sum_{W \in \mathcal{T}} X(\mathcal{S}|W, G|W\,; x_{[d]})\, X(\mathcal{S}|\overline{W}, G|\overline{W}\,; x_{d+[d]})\,. \tag{6.3}$$

Let us denote $\Xi_{[d]}(\mathcal{S}|W) \times \Xi_{d+[d]}(\mathcal{S}|\overline{W})$ by Ξ_W, for simplicity. Recall the bijection from $\Xi_{[2d]}(\mathcal{S})$ to $\bigcup_{W \subseteq V(\mathcal{S})} \Xi_W$ constructed in the proof of Theorem 5.5. The group G acts on the second set via this bijection, and we have a restricted action of G_W on Ξ_W. There is a second bijection, from $\bigcup_{W \in \mathcal{T}} G_W \backslash\backslash \Xi_W$ to $G \backslash\backslash \bigcup_{W \subseteq V(\mathcal{S})} \Xi_W$, given by $G_W(f', f'') \mapsto G(f', f'')$. Since G is cycle-closed we have $G_W \cong G|W \times G|\overline{W}$, and hence a third bijection, from $G_W \backslash\backslash \Xi_W$ to $(G|W \backslash\backslash \Xi_{[d]}(\mathcal{S}|W)) \times (G|\overline{W} \backslash\backslash \Xi_{d+[d]}(\mathcal{S}|\overline{W}))$. These three bijections together yield a fourth, from $G \backslash\backslash \Xi_{[2d]}(\mathcal{S})$ to $\bigcup_{W \in \mathcal{T}} (G|W \backslash\backslash \Xi_{[d]}(\mathcal{S}|W)) \times (G|\overline{W} \backslash\backslash \Xi_{d+[d]}(\mathcal{S}|\overline{W}))$, with the property that if

$G(f) \mapsto ((G|W)(f'), (G|\overline{W})(f''))$, then $x^f = x^{f'} x^{f''}$ (see the proof of Theorem 5.5 for the significance of the notation). This proves (6.3). □

Now let $\theta\colon H_*\langle \mathbb{SG}' \rangle \to H_*\{x\}$ be a surjective map of graded coalgebras, such as ρ^b or c^b. As in §5, we may employ θ^* to associate a delta operator $a(D^\theta)$ on $H_*\langle \mathbb{SG}' \rangle$ with the delta operator $a(D)$ on $H_*\{x\}$, and check that the map $\theta\colon (H_*\langle \mathbb{SG}' \rangle, a(D_\theta)) \to (H_*\{x\}, a(D))$ becomes a map of coalgebras with delta operator. This result then yields a suitably strengthened version of Theorem 6.2.

As an analogue of (5.9), we have

$$D^\rho(\mathcal{N}_n, S_n) = D^c(\mathcal{N}_n, S_n) = (\mathcal{N}_{n-1}, S_{n-1}), \qquad \text{and} \qquad b(D^c)\,(\mathcal{K}_n, S_n) = (\mathcal{K}_{n-1}, S_{n-1}).$$

With reference to the binomial sequences $(\mathcal{K}_n(A))$ defined in §5, we remark that the sequence $(\mathcal{K}_n(A), S_n)$ is a divided power sequence, in the sense of (2.10).

We also obtain an analogue of Proposition 5.10, whose proof is similar.

Proposition 6.4. *Let Δ be a delta operator on $H_*\langle \mathbb{SG} \rangle$ which is a derivation; then the following identities hold in $H_*\langle \mathbb{SG} \rangle$:*

$$b(\Delta)\,((\mathcal{S}_1, G_1)\cdot(\mathcal{S}_2, G_2)) = \sum_{i,j\ge 0} f^b_{i,j}\,(b(\Delta)^i\,(\mathcal{S}_1, G_1))\cdot(b(\Delta)^j\,(\mathcal{S}_2, G_2))$$

$$\textit{and} \qquad b(\Delta)\,S(\mathcal{S}, G) = \sum_{k\ge 1} i^b_k\, S(b(\Delta)^k\,(\mathcal{S}, G))$$

Given any map $\theta\colon (H_*\langle \mathbb{SG}' \rangle, \cdot) \to H_*\{x\}$ of graded Hopf algebras, D^θ is a derivation on $H_*\langle \mathbb{SG}' \rangle$, so the above identities hold for it (in $H_*\langle \mathbb{SG}' \rangle$).

All our results for the map c^b may be reformulated by complementation for the map $\chi^b\colon (H_*\langle \mathbb{SG}' \rangle, \odot) \to H_*\{x\}$, specified by $(\mathcal{S}, G) \mapsto \chi^b(\mathcal{S}, G\,; x)$.

In conclusion, we address the problem of finding a model for the covariant bialgebra $L_*\langle \beta^b_i(x) \rangle$ of the universal formal group law (see §3). Our answer is incomplete insofar as we avoid the thorny issue of modelling the scalars L_*; we are currently still investigating this important question.

Let $\mathbb{SG}''$ denote the subset of $\mathbb{SG}'$ consisting of those isomorphism classes $(\mathcal{S}, G)$ for which there is a partition $\pi \in \Pi(\mathcal{S})$ such that $\mathcal{K}_\pi \subseteq \mathcal{S} \subseteq \mathcal{K}_\pi \cup \mathrm{Bool}(\pi)$, and G is a subgroup of $\bigoplus_{B\in\pi} S_B$. Note that G depends only on the partition π, and not on the set system $\mathcal{S}$. It is easy to see that the free L_*-module $L_*\langle \mathbb{SG}'' \rangle$ spanned by $\mathbb{SG}''$ is a sub-Hopf algebra of $L_*\langle \mathbb{SG}' \rangle$.

Theorem 6.5. *The map $c^b\colon L_*\langle \mathbb{SG}'' \rangle \to L_*\langle \beta^b_i(x) \rangle$ is a surjective map of graded Hopf algebras.*

Proof. We establish that $c^b(\mathbb{SG}'') \subseteq L_*\langle \beta^b_i(x) \rangle$, from which the result follows by Theorem 5.4 and (4.5). Consider $(\mathcal{S}, G) \in \mathbb{SG}''$, and let π be the associated partition. We will prove that $c^b(\mathcal{S}, G\,; x) \in L_*\langle \beta^b_i(x) \rangle$ by induction with respect to $|\mathcal{S} \setminus \mathcal{K}_\pi|$; the induction starts successfully at 0, by the definition of $L_*\langle \beta^b_i(x) \rangle$. If $|\mathcal{S} \setminus \mathcal{K}_\pi| > 0$, choose a set $U \in \mathcal{S} \setminus \mathcal{K}_\pi$ minimal with respect to inclusion, and recall the deletion/contraction formula (4.10). By the choice of G, we have

$G \cong G|U \times G|\overline{U}$, whence $(\mathcal{S} \setminus U, G) \in \mathbb{SG}''$. Also, $G|\overline{U}$ can be viewed as an automorphism group of $\mathcal{S}//U$ in the obvious way, whence $(\mathcal{S}//U, G|\overline{U})$ lies in $\mathbb{SG}''$. Using these facts, formula (4.10) can be rewritten as

$$c^b(\mathcal{S}, G\,; x) = c^b(\mathcal{S} \setminus U, G\,; x) + \frac{\mu^{\phi}_{\Pi(\mathcal{S}|U)}(\widehat{0}, \{U\})}{|G|U|}\, c^b(\mathcal{S}//U, G|\overline{U}\,; x)\,.$$

Given the choice of U, we have that $\mu^{\phi}_{\Pi(\mathcal{S}|U)}(\widehat{0}, \{U\})/|G|U| = -\nu^{\phi}_{\mathcal{K}_{\pi|U}}(\{U\})/|G|U|$; but this is an integer multiple of $F^{\phi}_{n_1,\dots,n_r}/(n_1! \dots n_r!)$ in L_* by (4.15) and (3.9), where $\pi|U = \{B_1, \dots, B_r\}$ and $n_i = |B_i|$. The induction is then complete. ☐

References

[1] J. F. Adams. *Stable Homotopy and Generalised Homology.* Chicago Univ. Press, Chicago, 1972.

[2] M. Aigner. *Combinatorial Theory.* Springer-Verlag, Berlin–New York, 1978.

[3] Bourbaki. *Algèbre.* Hermann, Paris, 1970.

[4] M. Butler. *Formal Power Series and Polynomial Invariants of Graphs.* PhD thesis, Manchester Univ., 1992.

[5] P. Cameron. Private communication, October 1994.

[6] L. Carlitz. Some properties of Hurwitz series. *Duke Math. J.*, 16:285–295, 1949.

[7] L. Comtet. *Advanced Combinatorics.* Reidel, Dordrecht, 1974.

[8] L. Geissinger. Hopf algebras of symmetric functions and class functions. In *Combinatoire et Représentation du Group Symétrique*, volume 579 of *Lecture Notes in Math.*, pages 168–181. Springer-Verlag, Berlin–Heidelberg–New York, 1977.

[9] M. Hazewinkel. *Formal Groups and Applications.* Academic Press, New York, 1978.

[10] C. Lenart and N. Ray. Polynomial invariants of partition systems and posets. Preprint, University of Manchester, 1995.

[11] W. Nichols and M. Sweedler. Hopf algebras and combinatorics. In R. Morris, editor, *Umbral calculus and Hopf algebras*, volume 6 of *Contemp. Math.*, pages 49–84. Amer. Math. Soc., Providence, RI, 1980.

[12] D. C. Ravenel. *Complex Cobordism and Stable Homotopy Groups of Spheres.* Academic Press, New York, 1986.

[13] N. Ray. Extensions of umbral calculus; penumbral coalgebras and generalised Bernoulli numbers. *Adv. Math.*, 61:49–100, 1986.

[14] N. Ray. Tutte algebras of graphs and formal group theory. *Proc. London Math. Soc.*, 65:23–45, 1992.

[15] N. Ray, W. Schmitt, and C. Wright. Colouring simplicial complexes. *Ars Combin.*, 29A:161–169, 1990.

[16] W. Schmitt. Antipodes and incidence coalgebras. *J. Combin. Theory Ser. A*, 46:264–290, 1987.

[17] W. R. Schmitt. Incidence Hopf algebras. *J. Pure Appl. Algebra*, 96:299, 1994.

[18] R. P. Stanley. *Enumerative Combinatorics.* Wadsworth & Brooks/Cole, Monterey, CA, 1986.

[19] R. P. Stanley. A symmetric function generalization of the chromatic polynomial of a graph. *Adv. Math.*, 111:166, 1995.

[20] M. E. Sweedler. *Hopf Algebras.* W. A. Benjamin, New York, 1969.

[21] D. Wagner. The partition polynomial of a finite set system. *J. Combin. Theory Ser. A*, 56:138–159, 1991.

Mathematics Department, University of Manchester, Manchester M13 9PL, England
E-mail address: `lenart@ma.man.ac.uk`

Mathematics Department, University of Manchester, Manchester M13 9PL, England
E-mail address: `nige@ma.man.ac.uk`