EXTENSIONS OF UMBRAL CALCULUS II: DOUBLE DELTA OPERATORS, LEIBNIZ EXTENSIONS AND HATTORI-STONG THEOREMS

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Abstract. We continue our programme of extending the Roman-Rota umbral calculus to the setting of delta operators over a graded ring E_* with a view to applications in algebraic topology and the theory of formal group laws. We concentrate on the situation where E_* is free of additive torsion, in which context the central issues are number-theoretic questions of divisibility. We study polynomial algebras which admit the action of two delta operators linked by an invertible power series, and make related constructions motivated by the Hattori-Stong theorem of algebraic topology. Our treatment is couched purely in terms of the umbral calculus, but inspires novel topological applications. In particular we obtain a generalised form of the Hattori-Stong theorem.

Contents

1. Introduction

In [20] the basic notions of the Roman-Rota umbral calculus [26] were extended to the setting of delta operators over a commutative graded ring of scalars. In the process fundamental links were established between umbral calculus, the theory of formal group laws and algebraic topology.

In this sequel we extend these links further, introducing the notion of a *double* delta operator, and showing how to pair two delta operators to obtain a double delta operator. Together with the Leibniz property discussed in [20]

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(see also §§6 and 7), this enables us to formulate a generalisation of the Hattori-Stong theorem. Our main result here applies to general torsion-free delta operators. Applying our results to algebraic topology, we obtain a substantial application by determining necessary and sufficient conditions for a complex-oriented cohomology theory to satisfy a Hattori-Stong theorem. This result appears to be new, and confirms that umbral techniques can provide a convenient tool for organising certain types of calculation in the theory of formal group laws.

Throughout this article it is convenient to work over rings which are free of additive torsion; we save the more general case for a future paper, thereby completing a revised version of the programme begun in [20]. Traditionally, umbral calculus has been developed over fields such as the real or complex numbers. The main effect of working over a torsion-free ring is that problems of divisibility arise.

We summarise now the contents of each section, indicating the main results of this paper.

In §2 we recall basic definitions from [20], and introduce our notation, which differs to some extent from that of [20]. The fundamental concepts are that of a *delta operator* $E = (\Delta^e, E_*)$ over a torsion-free ring E_*, and its *penumbral coalgebra* $\Pi(E)_*$. Here Δ^e is a differential operator acting on $E_*[x]$. The E_*-module $\Pi(E)_*$ is generated, by the polynomials $b^e_n(x)$ of the *normalised associated sequence*. The $b^e_n(x)$ satisfy $\Delta^e b^e_n(x) = b^e_{n-1}(x)$ and belong to $E_*[x] \otimes \mathbb{Q}$. Thus $\Pi(E)_*$ is a rational extension of $E_*[x]$. It is important to note that $\Pi(E)_*$ is not in general a sub*algebra* of $E_*[x] \otimes \mathbb{Q}$.

A key example of this set-up arises in algebraic topology from a complex-oriented ring spectrum E for which the coefficient ring E_* of the corresponding generalised homology theory $E_*(\quad)$ is torsion-free. In this case $\Pi(E)_*$ corresponds to $E_*(\mathbb{C}P^\infty)$. We also discuss in §2 the *universal* delta operator, denoted Φ, which does not arise from a spectrum.

The concept of a delta operator is extended in §3 to the notion of a *double* delta operator. This involves a pair of differential operators acting on the ring of polynomials over a ring G_*, but with the crucial extra ingredient of a power series, with coefficients in G_*, relating the two operators. Again there is a universal example, denoted $\Phi \cdot \Phi$. Double delta operators arise in topology from spectra with two complex orientations.

In §4 we show that the concept of a double delta operator is equivalent to that of a Sheffer sequence.

Given two single delta operators over the rings E_* and F_*, it is possible to form a ring $(E \otimes F)_*$, which is a rational extension of $E_* \otimes F_*$, over which the delta operators combine to form a double delta operator. This pairing is defined in §5 by means of the universal example, but in contrast with the universal case, for which $(\Phi \otimes \Phi)_* = (\Phi \cdot \Phi)_*$, it is necessary to quotient out by any additive torsion. This requirement can be avoided by extending the notion of delta operator, both single and double, to apply over rings with torsion. We hope to return to this more general case in a later paper. It is convenient in the present case to give an alternative characterisation of $(E \otimes F)_*$ as the extension of $E_* \otimes F_*$ generated by certain elements defined umbrally, in (5.4), in terms of the associated sequences of the delta operators.

Returning to single delta operators in §6, a *Leibniz* delta operator is one for which the penumbral coalgebra is closed under multiplication of polynomials, and is thus a Hopf algebra. The dual object is then a formal group law. In the case of a non-Leibniz delta operator, extra divisibility can be introduced into the ring E_* to form the *minimal Leibniz extension* $L(E)_*$ over which the delta operator becomes Leibniz. The universal delta operator Φ is not Leibniz, but $L(\Phi)_*$ is isomorphic to the *Lazard ring* over which the universal formal group law is defined. Dually $L(\Phi)$ is the universal Leibniz delta operator. Since $\mathbb{C}P^\infty$ is an H-space, the topological examples of delta operators considered in §2 are always Leibniz. The property of being Leibniz can be expressed in terms of divisibility relations among the coefficients of a delta operator. We formulate and prove a particular case of such relations, a kind of Kummer congruence, in Theorem 6.15.

The Leibniz property can easily be extended to the case of double delta operators. In particular it is shown in §7 that $E \otimes F_*$ is Leibniz if one of the factors E or F is Leibniz. In general $L(E \otimes F)_* = \big(L(E) \otimes F\big)_*$. The universal Leibniz double delta operator is $L(\Phi \cdot \Phi)$.

The pairing operation for topological, and therefore Leibniz, delta operators is considered in §8. We prove that $MU_*(MU)$, the universal ring for strict isomorphisms between formal group laws, is isomorphic to $L(\Phi \cdot \Phi)_*$. For general complex-oriented spectra E and F, we think of $(E \otimes F)_*$ as an algebraic model for the ring $E_*(F)$. We show in Proposition 8.4 that the two are isomorphic if one of E and F satisfies the Landweber exactness conditions [14].

In §9 the pairing construction is considered in the case when one of the factors is the delta operator arising from K-theory (or, in combinatorial terms, from the discrete derivative). It is shown in Corollary 9.2 that $(K \otimes E)_*$ is isomorphic to the ring $\Pi\big(L(E)\big)_*[x^{-1}]$ obtained from the penumbral coalgebra of the Leibniz extension of E by inverting the polynomial variable. This result relates the divisibility involved in the pairing operation, the penumbral coalgebra, and the Leibniz extension.

Corollary 9.2 is central in §10 in which we consider when $L(E)_*$ is rationally closed in $(K \otimes E)_*$. We think of this as an analogue of the Hattori-Stong theorem. The classical Hattori-Stong theorem [11, 30] applies to the universal case $E = \Phi$, for which $L(\Phi)_* = MU_*$ and $(K \otimes \Phi)_* = K_*(MU)$. Using the Kummer congruence of §6, we give in Theorem 10.9 a criterion for the Hattori-Stong theorem in terms of divisibility in $L(E)_*$. The criterion simplifies somewhat when the ring $L(E)_*$ has unique integer factorisation. This case is sufficient to yield a simple proof of the classical Hattori-Stong theorem (Theorem 10.14).

Topological cases of the Hattori-Stong theorem are considered in §11. We show that the theorem holds for the theory E if and only if the first two of Landweber's exactness conditions (up to height one) hold. This gives rise to generalisations of results of G. Laures [16] and L. Smith [27].

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We are particularly grateful to a referee, whose suggestion it was to phrase more of our definitions and constructions in terms of universal examples. This and many other comments have substantially improved the paper.

2. Delta operators and penumbral coalgebras

In this section we give a summary of the background information needed from [20], with embellishments provided by [21] and [23].

Throughout this paper, E_* will be a commutative ring with identity, graded by dimension and free of additive torsion; a homomorphism between two such rings will always respect the product, grading and identity. We abbreviate $E_* \otimes \mathbb{Q}$ to $E\mathbb{Q}_*$. The *binomial coalgebra* $E_*[x]$ over E_* is the free left E_*-module on generators 1, x, $x^2, \ldots$, where x is an indeterminate of dimension 2, invested with the coproduct

$$\psi\colon x^n \longmapsto \sum_{i=0}^{n} \binom{n}{i} x^{n-i} \otimes x^i, \qquad n = 1, 2, \ldots,$$

and augmentation $x^i \mapsto \delta_{i,0}$.

We write D for the linear operator d/dx acting on $E_*[x]$, and then a *delta operator* Δ^e over E_* is a formal differential operator

$$\Delta^e = D + e_1 \frac{D^2}{2!} + \cdots + e_{k-1} \frac{D^k}{k!} + \cdots \tag{2.1}$$

in the divided power series ring $E^*\{\{D\}\}$, where $E^{-2n} = E_{2n}$ for all n. Thus Δ^e acts on $E_*[x]$. We will always assume that e_k lies in E_{2k}. Therefore since D has dimension 2 (being dual to x) so does Δ^e. Observe that

$$\psi \circ \Delta^e = (\Delta^e \otimes 1) \circ \psi = (1 \otimes \Delta^e) \circ \psi \tag{2.2}$$

as functions $E_*[x] \to E_*[x] \otimes_{E_*} E_*[x]$. Dualising the divided powers $(\Delta^e)^k/k!$ gives rise to a new sequence of generators

$$B^e_0(x) = 1, B^e_1(x), B^e_2(x), \ldots$$

for $E_*[x]$ over E_*. These generators satisfy the *binomial* property

$$\psi\colon B^e_n(x) \longmapsto \sum_{i=0}^{n} \binom{n}{i} B^e_{n-i}(x) \otimes B^e_i(x),$$

and are known as the *associated sequence* of Δ^e.

We denote the pair (E_*, Δ^e) by E, and will usually refer to E itself as a *delta operator*; this convention makes explicit the ring on which Δ^e acts.

Together with the coproduct ψ, the usual product of polynomials makes $E_*[x]$ into a Hopf algebra with antipode given by $S(x) = -x$. Since $E^*\{\{D\}\}$

is the graded E_*-linear dual of $E_*[x]$, it too admits a (completed) Hopf algebra structure.

Definition 2.3 (See [20])**.** The *universal delta operator* Φ is defined over the ring $\Phi_* = \mathbb{Z}[\phi_1, \phi_2, \dots]$. The operator is

$$\Delta^\phi = D + \phi_1 \frac{D^2}{2!} + \cdots + \phi_{k-1}\frac{D^k}{k!} + \cdots .$$

It is universal in the sense that any delta operator E is uniquely determined by the homomorphism $\nu_*^e \colon \Phi_* \to E_*$ that sends ϕ_n to e_n. We refer to ν_*^e as the *classifying homomorphism.*

Definition 2.4. A *morphism* $\gamma \colon E \to F$ of delta operators, where $\Delta^e = D + e_1 D^2/2! + \cdots + e_{k-1}D^k/k! + \cdots$ and $\Delta^f = D + f_1 D^2/2! + \cdots + f_{k-1}D^k/k! + \cdots$, is a homomorphism of graded rings $\gamma_* \colon E_* \to F_*$ such that $\gamma_*(e_k) = f_k$ for all $k \geq 1$. Equivalently $\nu_*^f = \gamma_* \circ \nu_*^e$.

This is a more restrictive definition than the one originally given in [20].

Definition 2.5. If $E = (E_*, \Delta^e)$ is a delta operator with E_* torsion-free, let

$$b_n^e(x) = \frac{1}{n!} B_n^e(x) \in E\mathbb{Q}_*[x].$$

The sequence of polynomials

$$b_0^e(x) = 1, b_1^e(x), b_2^e(x), \dots$$

is known as the *normalised associated sequence* of Δ^e. The *penumbral coalgebra* $\Pi(E)_*$ is defined as the free E_*-module generated by the $b_n^e(x)$.

The $b_n^e(x)$ satisfy the *divided power* property

$$\psi \colon b_n^e(x) \longmapsto \sum_{i=0}^{n} b_{n-i}^e(x) \otimes b_i^e(x).$$

Thus $\Pi(E)_*$ is indeed a coalgebra, with $E_*[x] \subseteq \Pi(E)_* \subseteq E\mathbb{Q}_*[x]$.

In addition we have $\Delta^e b_n^e(x) = b_{n-1}^e(x)$, and $b_n^e(0) = 0$ for $n > 0$, so that

$$\big\langle (\Delta^e)^i \ \big|\ b_j^e(x) \big\rangle = \delta_{i,j}, \tag{2.6}$$

where for any operator Γ and any polynomial $f(x)$ we let

$$\big\langle \Gamma \ \big|\ f(x) \big\rangle = \Gamma f(x)\Big|_{x=0}.$$

Topological Examples 2.7. For any complex-oriented spectrum E, the space ΩS^3 of loops on the 3-sphere has homology and cohomology modules

$$E_*(\Omega S^3) \cong E_*[x] \qquad \text{and} \qquad E^*(\Omega S^3) \cong E^*\{\{D\}\}, \tag{2.8}$$

where $x \in E_2(\Omega S^3)$ is carried by the bottom cell $S^2 \subset \Omega S^3$, and $D \in E^2(\Omega S^3)$ is defined by pullback along the evaluation map $S\Omega S^3 \to S^3$; see [23]. Under the cap product, D acts as d/dx, so that these modules are dual Hopf algebras of the type described above. The coproduct in $E_*(\Omega S^3)$ and the product in $E^*(\Omega S^3)$ arise from the diagonal map, whilst the product in $E_*(\Omega S^3)$ and the coproduct in $E^*(\Omega S^3)$ arise from composition of loops. The antipodes are induced by reversing the loop parameter.

A canonical map $j \colon \Omega S^3 \to \mathbb{C}P^\infty$ into infinite dimensional complex projective space may be defined as a representative for a generator of the group $H^2(\Omega S^3) \cong \mathbb{Z}$. Then the given complex orientation $t^e \in E^2(\mathbb{C}P^\infty)$ pulls back to $j^*t^e \in E^2(\Omega S^3)$, which by virtue of (2.8) may be expressed as

$$D + e_1\frac{D^2}{2!} + \cdots + e_{k-1}\frac{D^k}{k!} + \cdots ,$$

where each e_k lies in E_{2k}. In this way, the spectrum E and its complex orientation t^e give rise to a delta operator $(E_*, \Delta^e) = (E_*, j^*t^e)$. The formula (2.2) expresses the standard interaction between cap product and diagonal.

The resulting sequence of elements $B^e_n(x)$ in $E_*(\Omega S^3)$ are E_*-module generators dual to the divided powers of Δ^e, and satisfy the binomial property.

In this context, $\Pi(E)_*$ is $E_*(\mathbb{C}P^\infty)$ on which Δ^e acts as the Thom isomorphism $\cap\, t^e$, and the inclusion $E_*[x] \subseteq \Pi(E)_*$ is the homomorphism induced by j. The generators $b^e_n(x)$ for $E_*(\mathbb{C}P^\infty)$ are the duals of the powers of t^e.

Topological K-theory gives rise to the delta operator $K = (\mathbb{Z}[u, u^{-1}], \Delta^k)$, where $u \in K_2$ and $\Delta^k = u^{-1}(e^{uD} - 1)$ is the discrete derivative. *Connective* K-theory yields the delta operator $k = (\mathbb{Z}[u], \Delta^k)$, while ordinary cohomology gives rise to the delta operator $H = (\mathbb{Z}, D)$.

It is clear that a map $m\colon E \to F$ of complex-oriented spectra satisfying $m_*(t^e) = t^f$ determines a morphism $m_*\colon (E_*, \Delta^e) \to (F_*, \Delta^f)$ of delta operators.

3. Double delta operators

We now introduce the central idea of this paper.

Definition 3.1. A *double delta operator* consists of a pair of delta operators Δ_1, Δ_2 over a torsion-free, graded commutative ring G_* with identity, together with an operator equation

$$\Delta_2 = \Delta_1 + g_1\Delta_1^2 + \cdots + g_{k-1}\Delta_1^k + \cdots = g(\Delta_1),$$

where g_k lies in G_{2k} for each k. We write G for the double delta operator $(G_*, \Delta_1, \Delta_2, g)$.

We refer to the formal power series $g(y) = \sum_{i\geq 0} g_i y^{i+1} \in G_*[[y]]$, where $g_0 = 1$, as a *strict isomorphism* from Δ_1 to Δ_2, by analogy with the nomenclature of the theory of formal group laws [12]. The compositional inverse (or reverse, or conjugate) power series $\overline{g}(y) = \sum_{i\geq 0} \overline{g}_i y^i$ is, of course, a strict isomorphism from Δ_2 to Δ_1, and has coefficients which are integral combinations of those of $g(y)$; see (3.8) below. There is, therefore, a *dual* double delta operator $\widetilde{G} = \big(G_*, \Delta_2, \Delta_1, \overline{g}\big)$.

Since the set of delta operators over a fixed ring G_* forms a group under composition of divided power series, there is always an expression of the form

$$\Delta_2 = \Delta_1 + g'_1\frac{\Delta_1^2}{2!} + \cdots + g'_{k-1}\frac{\Delta_1^k}{k!} + \cdots ,$$

where $g'_k \in G_{2k}$ for each k; see (5.6). Thus the thrust of Definition 3.1 is that each coefficient g'_{k-1} of this series should be divisible by $k!$ in G_{2k-2}. So if G_* is a field (as in the classical cases $\mathbb{R}$ and $\mathbb{C}$), or at least a $\mathbb{Q}$-algebra, then any two delta operators are strictly isomorphic, and together they define a unique double delta operator.

For each double delta operator there are two associated sequences of polynomials $b^1_n(x)$ and $b^2_n(x)$ in $G\mathbb{Q}_*[x]$. Classic formulæ of the umbral calculus such as

$$b^1_n(x) = x\left(\frac{\Delta_2}{\Delta_1}\right)^n x^{-1} b^2_n(x), \qquad n = 1, 2, \ldots$$

(see [26, Corollary 3.8.2]) explain how to relate them. These formulæ appear at first sight to involve scalars in $G\mathbb{Q}_*$, but the following result shows that the coefficients in fact belong to G_*.

Proposition 3.2. *In $G_*[x]$, the two associated sequences are related as*

$$b^1_k(x) = \sum_{l=1}^{k} \sum \binom{l}{m_1, m_2, \ldots, m_{k-1}} g_1^{m_1} g_2^{m_2} \cdots g_{k-1}^{m_{k-1}} b^2_l(x),$$

where the inner summation is over all sequences $(m_1, m_2, \ldots, m_{k-1})$ of natural numbers such that $m_1 + 2m_2 + \cdots + (k-1)m_{k-1} = k - l$, and $m_1 + m_2 + \cdots + m_{k-1} \le l$, so that the multinomial coefficient

$$\binom{l}{m_1, m_2, \ldots, m_{k-1}} = \frac{l!}{m_1! m_2! \cdots m_{k-1}! (l - m_1 - m_2 - \cdots - m_{k-1})!}$$

is defined.

Thus

$$b^1_k(x) = \sum_{l=1}^{k} \widehat{\mathbf{B}}_{k,l}(1, g_1, g_2, \ldots, g_{k-l}) b^2_l(x),$$

where $\widehat{\mathbf{B}}_{k,l}$ is the partial ordinary Bell polynomial; see [6, Ch. III, [3d]].

Proof. Over $G\mathbb{Q}_*$, we may write $b^1_k(x) = \sum_i \lambda_{k,i} b^2_i(x)$. To evaluate $\lambda_{k,l}$ we apply $\langle \Delta_2^l \mid \ \rangle$ to both sides, substitute $\Delta_2 = g(\Delta_1)$, and use (2.6). □

The action of forgetting one or other of the delta operators, and the isomorphism between them, associates to each double delta operator $G = (G_*, \Delta_1, \Delta_2, g)$ two (single) delta operators denoted by ${}_1G = (G_*, \Delta_1)$ and ${}_2G = (G_*, \Delta_2)$. It follows from Proposition 3.2 that the penumbral coalgebras $\Pi({}_1G)_*$ and $\Pi({}_2G)_*$ are equal; we therefore denote their common value by $\Pi(G)_*$.

The universal delta operator $\Phi = (\Phi_*, \Delta^\phi)$ of Definition 2.3 gives rise to the universal double delta operator.

Definition 3.3. The *universal* double delta operator $\Phi \cdot \Phi$ is defined over the ring $(\Phi \cdot \Phi)_* = \Phi_*[b_1, b_2, \ldots] = \mathbb{Z}[\phi_1, \phi_2, \ldots, b_1, b_2, \ldots]$, where each b_k has dimension $2k$. The delta operator Δ_1 is the extension to $(\Phi \cdot \Phi)_*$ of the universal delta operator Δ^ϕ, the strict isomorphism is $g(y) = \sum_{i \ge 0} b_i y^{i+1}$, and Δ_2 is defined by $\Delta_2 = g(\Delta_1)$.

A double delta operator $G = (G_*, \Delta_1, \Delta_2, g)$, with $\Delta_1 = \sum_{k \ge 0} e_{k-1} D^k / k!$ is uniquely determined by the *classifying homomorphism* $\nu^g_* \colon (\Phi \cdot \Phi)_* \to G_*$ for which $\nu^g_*(\phi_i) = e_i$ and $\nu^g_*(b_i) = g_i$.

The algebra $(\Phi \cdot \Phi)_*$ plays a crucial role in the theory of Sheffer sequences; see §4 and [24].

Definition 3.4. A *morphism* $\mu \colon G \to H$ of double delta operators G and H is a ring homomorphism $\mu_* \colon G_* \to H_*$ satisfying $\nu^h_* = \mu_* \circ \nu^g_*$.

Note that it follows that $\mu_*(g_k) = h_k$ for each coefficient of the respective strict isomorphisms.

Let the classifying homomorphisms for the single delta operators ${}_1(\Phi\cdot\Phi)$ and ${}_2(\Phi\cdot\Phi)$ be denoted by $\lambda_*, \rho_* \colon \Phi_* \to (\Phi\cdot\Phi)_*$. While λ_* is simply the inclusion $\Phi_* \subset \Phi_*[b_1, b_2, \dots] = (\Phi\cdot\Phi)_*$, the homomorphism ρ_* is more complicated, with $\rho_*(\phi_k)$ being the coefficient of $D^{k+1}/(k+1)!$ in $\sum_{i\geq 0} b_i(\Delta^\phi)^{i+1}$. Thus

$$
\begin{aligned}
\rho_*(\phi_k) &= \sum_{i=0}^{k} (i+1)!\mathbf{B}_{k+1,i+1}(1, \phi_1, \phi_2, \dots, \phi_{k-i}) b_i \\
&= \phi_k + \cdots + (k+1)! b_k,
\end{aligned}
\tag{3.5}
$$

where $b_0 = 1$ and $\mathbf{B}_{n,j}(x_1, x_2, \dots, x_{n-j+1})$ is the partial *exponential* Bell polynomial; see [6]. For example

$$
\begin{aligned}
\rho_*(\phi_1) &= \phi_1 + 2b_1, \\
\rho_*(\phi_2) &= \phi_2 + 6\phi_1 b_1 + 6b_2, \\
\rho_*(\phi_3) &= \phi_3 + (8\phi_2 + 6\phi_1^2)b_1 + 36\phi_1 b_2 + 24 b_3.
\end{aligned}
\tag{3.6}
$$

The classifying homomorphism of the dual double delta operator

$$
\widetilde{\Phi\cdot\Phi} = \big((\Phi\cdot\Phi)_*, \Delta_2, \Delta_1, \overline{g}(y)\big)
$$

is the homomorphism $\tau_* \colon (\Phi\cdot\Phi)_* \to (\Phi\cdot\Phi)_*$ for which $\tau_*(\phi_k) = \rho_*(\phi_k)$ and

$$
\tau_*(b_k) = \frac{1}{k+1} \sum_{j=1}^{k} (-1)^j \binom{j+k}{k} \widehat{\mathbf{B}}_{k,j}(b_1, b_2, \dots, b_{k-j+1}),
\tag{3.7}
$$

where $\widehat{\mathbf{B}}_{n,j}$ is the partial *ordinary* Bell polynomial; see [6]. This is, of course, the coefficient of y^{k+1} in the inverse series $\overline{g}(y)$, and so is, despite appearances, an integer polynomial in the b_j. For example

$$
\tau_*(b_1) = -b_1, \quad \tau_*(b_2) = 2b_1^2 - b_2, \quad \tau_*(b_3) = -5b_1^3 + 5b_1 b_2 - b_3.
\tag{3.8}
$$

Clearly τ_* is an involution of the ring $(\Phi\cdot\Phi)_*$ and induces an isomorphism of double delta operators $\tau \colon \Phi\cdot\Phi \to \widetilde{\Phi\cdot\Phi}$.

Definition 3.9. Given two delta operators, Δ^e over E_* and Δ^f over F_*, a (Δ^e, Δ^f)*-operator* is a double delta operator G, together with morphisms of single delta operators $\lambda \colon E \to {}_1G$ and $\rho \colon F \to {}_2G$. Thus there are ring homomorphisms $\lambda_* \colon E_* \to G_*$ and $\rho_* \colon F_* \to G_*$ such that

$$
\begin{array}{ccccc}
\Phi_* & \xrightarrow{\lambda_*} & (\Phi\cdot\Phi)_* & \xleftarrow{\rho_*} & \Phi_* \\
{\scriptstyle \nu^e_*}\downarrow & & {\scriptstyle \nu^g_*}\downarrow & & \downarrow{\scriptstyle \nu^f_*} \\
E_* & \xrightarrow{\lambda_*} & G_* & \xleftarrow{\rho_*} & F_*
\end{array}
$$

commutes.

Definition 3.10. A *morphism* from the (Δ^e, Δ^f)-operator (G, λ, ρ) to the $(\Delta^{e'}, \Delta^{f'})$-operator (G', λ', ρ') consists of a triple of homomorphisms which make the diagram

$$\begin{array}{ccccc} E_* & \xrightarrow{\lambda_*} & G_* & \xleftarrow{\rho_*} & F_* \\ \big\downarrow & & \big\downarrow & & \big\downarrow \\ E'_* & \xrightarrow{\lambda'_*} & G'_* & \xleftarrow{\rho'_*} & F'_* \end{array}$$

commute, and which factor through the classifying homomorphisms.

We illustrate these concepts with an example drawn from number theory.

Example 3.11. If p is prime, define the *Artin-Hasse delta operator* as $A = \big(\mathbb{Z}[v], \Delta^a\big)$, with $v \in A_{2p-2}$ and Δ^a determined by the inverse relation

$$D = \Delta^a + v\frac{(\Delta^a)^p}{p} + v^{p+1}\frac{(\Delta^a)^{p^2}}{p^2} + \cdots + v^{\frac{p^i-1}{p-1}}\frac{(\Delta^a)^{p^i}}{p^i} + \cdots . \tag{3.12}$$

It is clear that $a_n = 0$ unless n is a multiple of $p-1$.

We define a double delta operator G as follows. Let $G_* = \mathbb{Z}_{(p)}[u]$, where $u \in G_2$, let Δ_1 be the image of the K-theory operator Δ^k under the inclusion $k_* = \mathbb{Z}[u] \subset G_*$, and let Δ_2 be the image of the Artin-Hasse operator Δ^a under the map $A_* = \mathbb{Z}[v] \subset G_*$ given by $v \mapsto u^{p-1}$. A result of Hasse [10] shows that

$$\Delta_1 = u^{-1}\left(\prod_{p \nmid m}\big(1 - (u\Delta_2)^m\big)^{-\mu(m)/m} - 1\right),$$

where μ is the Möbius function. This is a power series in Δ_2 with coefficients in G_*, showing that $(G_*, \Delta_1, \Delta_2)$ is a double delta operator; by construction it is a (Δ^k, Δ^a)-operator.

Topological Examples 3.13. Examples of double delta operators are provided by cohomology theories with two given complex orientations. If $t_1, t_2 \in E^2(\mathbb{C}P^\infty)$ are two orientations, then since $E^*(\mathbb{C}P^\infty) \cong E_*[[t_2]]$ we can write $t_1 = g(t_2)$ for some power series whose coefficients lie in E_*. Then (E_*, j^*t_1, j^*t_2, g) is a double delta operator, where j is the map discussed in (2.7).

Suppose given two complex-oriented spectra E and F, and hence two delta operators E and F, with the additional property that the ring

$$G_* = (E \wedge F)_* \cong E_*(F) \cong F_*(E)$$

is free of additive torsion. This is the case, for example, when E is the Eilenberg-Mac Lane spectrum H representing integral cohomology, the spectrum K representing complex K-theory, or the Thom spectrum MU representing complex cobordism, and F is either K or MU.

There are two natural inclusions, the *left* and *right units,*

$$l_*\colon E_* \to (E \wedge F)_* \quad \text{and} \quad r_*\colon F_* \to (E \wedge F)_*,$$

which give rise to two complex orientations

$$l_*(t^e) = t_l \quad \text{and} \quad r_*(t^f) = t_r \in (E \wedge F)^2(\mathbb{C}P^\infty).$$

Thus $\big((E \wedge F)_*, j_*(t_l), j_*(t_r)\big)$ is a double delta operator. By the remark at the end of (2.7), it is a (Δ^e, Δ^f)-operator.

If $m_l\colon E \to E'$ and $m_r\colon F \to F'$ are maps of complex-oriented spectra, then the triple $(m_l, m_l \wedge m_r, m_r)$ induces a morphism from the corresponding

(Δ^e, Δ^f)-operator to the corresponding $(\Delta^{e'}, \Delta^{f'})$-operator. As we shall see, not all double delta or (Δ^e, Δ^f)-operators, nor all their morphisms, arise in this fashion.

4. Sheffer sequences

A double delta operator is determined by a divided power series Δ_1 and a power series g over the graded ring G_*, for the second delta operator is given by $\Delta_2 = g(\Delta_1)$, and giving g is equivalent to specifying the unit $g(\Delta_1)/\Delta_1$ in $G_*[[\Delta_1]]$. But the same ingredients give rise to the concept of a Sheffer sequence, which is centrally important in the theory of umbral calculus. We explore briefly the connection here and explain how, just as the universal (or generic) binomial sequence lies in the ring $\Phi_*[x]$, we can define the concept of a *universal Sheffer sequence* which lies in the ring $(\Phi \cdot \Phi)_*[x]$. In fact the whole theory of Sheffer sequences may be recast in the context of double delta operators. We shall merely sketch how this is possible, and leave the interested reader to supply the details.

Definition 4.1. The *(normalised) Sheffer sequence* associated to a double delta operator $G = (G_*, \Delta_1, \Delta_2, g)$ is the sequence of polynomials

$$s_n^G(x) = \frac{\Delta_2}{\Delta_1} b_n^1(x) \in G\mathbb{Q}_*[x],$$

where $b_n^1(x)$ is the normalised associated sequence of the delta operator ${}_1G$.

This corresponds in the terminology of [20, Definition 4.11] to the Sheffer sequence for the pair $(\overline{g}, {}_1G)$. In [24] Sheffer sequences are studied mainly in terms of the *unnormalised* polynomials $S_n^G(x) = n! s_n^G(x)$.

Recall that for a double delta operator, we can write

$$\Delta_2 = g(\Delta_1) = \Delta_1(1 + g_1\Delta_1 + g_2\Delta_1^2 + \cdots),$$

where $\Delta_1 b_n^1(x) = b_{n-1}^1(x)$, so that

$$s_n^G(x) = \sum_{j=0}^{n} g_j b_{n-j}^1(x), \tag{4.2}$$

which shows that $s_n^G(x)$ belongs to the penumbral coalgebra $\Pi(G)_*$.

Definition 4.3. A (normalised) *Sheffer system* consists of a delta operator $E = (E_*, \Delta^e)$ and a sequence of polynomials $s_n(x) \in E\mathbb{Q}_*[x]$, where $s_n(x)$ has degree n and $s_0(x) = 1$, such that $\Delta^e s_n(x) = s_{n-1}(x)$, and $s_n(0) \in E_{2n}$.

Since, by (4.2), $s_n^G(0) = g_n \in E_{2n}$, a double delta operator gives rise to a Sheffer system, but clearly the process is reversible. The Sheffer system $\big(E, s_n(x)\big)$ determines the double delta operator $(E_*, \Delta^e, \Delta^s)$, where

$$\Delta^s = \sum_{n\geq 0} s_n(0)(\Delta^e)^{n+1}.$$

It is clear that the appropriate concept of *morphism* of Sheffer systems $\big(E, s_n(x)\big) \to \big(F, t_n(x)\big)$ is a morphism $\gamma\colon E \to F$ of delta operators such that $\gamma_*\big(s_n(x)\big) = t_n(x)$. Equivalently, it is a morphism of double delta operators $(E_*, \Delta^e, \Delta^s) \to (F_*, \Delta^f, \Delta^t)$.

The Sheffer system corresponding to the double delta operator $\Phi \cdot \Phi$ is the *universal Sheffer system.* The polynomial $s_n^{\Phi\cdot\Phi}(x)$ is written as $s_n^{\psi,\phi}(x)$ in [24], where its universal properties are elaborated. The variable ψ_k of [24] corresponds to b_k in Definition 3.3 so that the ring $\Psi \otimes \Phi$ of that paper is isomorphic to our $(\Phi \cdot \Phi)_*$.

We end this section by drawing attention to Roman and Rota's formula for delta operators (see [26, Theorem 2.3.8]).

Theorem 4.4. *Suppose* $\big(E, s_n(x)\big)$ *is a Sheffer system, and* Δ *is a delta operator defined over* E_**. Then*

$$\Delta s_n(x) = \sum_{k=0}^{n} \big\langle \Delta \bigm| b_k^e(x) \big\rangle s_{n-k}(x).$$

5. A pairing of delta operators

Recall the homomorphisms $\lambda_*, \rho_* \colon \Phi_* \to (\Phi \cdot \Phi)_*$ which classify the delta operators ${}_1(\Phi \cdot \Phi)$ and ${}_2(\Phi \cdot \Phi)$. They endow $(\Phi \cdot \Phi)_*$ with the structure of a bimodule over Φ_*.

Definition 5.1. If E and F are delta operators, define the ring $(E \cdot F)_*$ as

$$(E \cdot F)_* = E_* \otimes_{\Phi_*} (\Phi \cdot \Phi)_* \otimes_{\Phi_*} F_*.$$

Here the left-hand tensor product is defined with respect to the Φ_*-module structures determined by the homomorphisms $\nu_*^e \colon \Phi_* \to E_*$ and $\lambda_* \colon \Phi_* \to (\Phi \cdot \Phi)_*$, and the right-hand tensor product is defined with respect to the structures determined by ρ_* and ν_*^f. Note that $(E \cdot F)_*$ is indeed a ring, since each of the Φ_*-modules involved is in fact an algebra over Φ_*.

The homomorphism $(\Phi \cdot \Phi)_* \to (E \cdot F)_*$ given by $z \mapsto 1 \otimes z \otimes 1$ will determine a double delta operator over $(E \cdot F)_*$ as long as $(E \cdot F)_*$ is torsion-free. However this not always the case. For example in $(H \cdot H)_*$ the element $1 \otimes b_1 \otimes 1$ is 2-torsion, for, by (3.6),

$$\begin{aligned} 1 \otimes 2b_1 \otimes 1 &= 1 \otimes \big(\rho_*\big(\phi_1\big) - \lambda_*(\phi_1)\big) \otimes 1 \\ &= 1 \otimes 1 \otimes \nu_*^h(\phi_1) - \nu_*^h(\phi_1) \otimes 1 \otimes 1 \end{aligned}$$

is zero since $\nu_*^h(\phi_1) = h_1 = 0$.

In order to construct a double delta operator it is therefore necessary to quotient out by the torsion ideal.

Definition 5.2. If E and F are delta operators, define the ring $(E \otimes F)_*$ as the quotient of $(E \cdot F)_*$ by the ideal of elements $\alpha \in (E \cdot F)_*$ such that $n\alpha = 0$ for some non-zero integer n. Thus $(E \otimes F)_*$ is a torsion-free ring, and the homomorphism $(\Phi \cdot \Phi)_* \to (E \cdot F)_* \to (E \otimes F)_*$ determines a double delta operator $E \otimes F$. Moreover the obvious homomorphisms from E_* and F_* to $(E \otimes F)_*$ show that $E \otimes F$ is a (Δ^e, Δ^f)-operator.

Since $(\Phi \cdot \Phi)_*$ is torsion-free, $\Phi \otimes \Phi$ and $\Phi \cdot \Phi$ are equal. We will continue, however, to use the notation $\Phi \cdot \Phi$ for this universal case.

Clearly whenever there are morphisms $\gamma \colon E \to E'$ and $\delta \colon F \to F'$ of delta operators, then there is a unique morphism $\gamma \otimes \delta$ from the (Δ^e, Δ^f)-operator $E \otimes F$ to the $(\Delta^{e'}, \Delta^{f'})$-operator $E' \otimes F'$.

Proposition 5.3. *The double delta operator $E \otimes F$ is the universal (Δ^e, Δ^f)-operator in the sense that given any (Δ^e, Δ^f)-operator G there is a morphism $E \otimes F \to G$ of (Δ^e, Δ^f)-operators.*

Proof. The homomorphism $(E \cdot F)_* \to G_*$ sending $e \otimes z \otimes f$ to $\lambda_*(e)\nu_*^g(z)\nu_*(f)$ induces a homomorphism $(E \otimes F)_* \to G_*$ since G_* is torsion-free. □

Since the equations (3.5) can be solved rationally for the b_k in terms of the ϕ_j and $\rho_*(\phi_j)$, we have $(\Phi \cdot \Phi)_* \otimes \mathbb{Q} = \Phi_* \otimes \Phi_* \otimes \mathbb{Q}$ as (Φ_*, Φ_*)-bimodules, and so there are proper inclusions

$$\Phi_* \otimes \Phi_* \subset (\Phi \cdot \Phi)_* \subset \Phi_* \otimes \Phi_* \otimes \mathbb{Q}.$$

Applying $E_* \otimes_{\Phi_*} \_ \otimes_{\Phi_*} F_*$ to this chain of inclusions, we obtain homomorphisms

$$E_* \otimes F_* \to (E \cdot F)_* \to E_* \otimes F_* \otimes \mathbb{Q},$$

the last of which factors through $(E \otimes F)_*$. Since E_* and F_* are torsion-free, so is $E_* \otimes F_*$ [7, 9]. Thus there are inclusions

$$E_* \otimes F_* \subseteq (E \otimes F)_* \subseteq E_* \otimes F_* \otimes \mathbb{Q}.$$

It is therefore clear that $(E \otimes F)_*$ can be characterised as the extension of $E_* \otimes F_*$ in $E_* \otimes F_* \otimes \mathbb{Q}$ generated by the elements g_k which are the images of the elements $1 \otimes b_k \otimes 1 \in (E \cdot F)_*$. We identify these elements in $E_* \otimes F_* \otimes \mathbb{Q}$ as follows.

The delta operators Δ^e and Δ^f extend in the obvious fashion over the ring $E_* \otimes F_* \otimes \mathbb{Q}$ and give rise to a double delta operator over that ring. Then

$$\Delta^f = \sum_{k \geq 1} g_{k-1} (\Delta^e)^k,$$

with, by (2.6),

$$g_{k-1} = \langle \Delta^f \mid b_k^e(x) \rangle = \Delta^f b_k^e(x) \Big|_{x=0};$$

see also [26]. We may conveniently compute $\langle \Delta^f \mid b_k^e(x) \rangle$ by the umbral substitution

$$b_k^e(f) \qquad f^k \equiv f_{k-1}. \tag{5.4}$$

For example, since, by [20],

$$\begin{aligned} b_1^e(x) &= x, \\ b_2^e(x) &= \tfrac{1}{2}(x^2 - e_1 x), \\ b_3^e(x) &= \tfrac{1}{6}\big(x^3 - 3e_1 x^2 + (3e_1^2 - e_2)x\big), \end{aligned} \tag{5.5}$$

it follows that

$$g_0 = 1, \quad g_1 = \tfrac{1}{2}(f_1 - e_1), \quad g_2 = \tfrac{1}{6}(f_2 - 3e_1 f_1 + 3e_1^2 - e_2). \tag{5.6}$$

Here and below we write e_k for $e_k \otimes 1$ and f_k for $1 \otimes f_k$ in contexts where this does not cause confusion.

We have thus proved the following result, which we will use to obtain the structure of rings of the form $(E \otimes F)_*$.

Proposition 5.7. *The ring $(E \otimes F)_*$ is isomorphic to the extension of $E_* \otimes F_*$ in $E_* \otimes F_* \otimes \mathbb{Q}$ generated either by the elements $b_k^e(f)$ or by the elements $b_k^f(e)$, for $k > 1$.* □

To illustrate this construction, we describe how it works in some specific cases. Recall the definitions of the delta operators H, k and K given in (2.7).

Proposition 5.8. *The ring $(H \otimes E)_* \subset E\mathbb{Q}_*$ is generated over E_* by the elements $e_n/(n+1)!$ for $n \geq 1$.*

Proof. The coefficients of the power series expressing Δ^e in terms of $D = \Delta^h$, as in (2.1), are $e_n/(n+1)!$. □

Corollary 5.9. *The ring $(H \otimes k)_*$ is the subring of $\mathbb{Q}[u]$ generated by the elements u^{p-1}/p, where p is prime.*

Proof. By Proposition 5.8, $(H \otimes k)_*$ is generated by the elements $u^n/(n+1)!$. It thus contains the subring generated by the elements u^{p-1}/p, but it is easy to see that the two subrings are equal, having the elements $u^n/m(n)$ as an additive basis, where $m(n)$ is the function of [1]. □

Corollary 5.10. *The ring $(H \otimes K)_*$ is $\mathbb{Q}[u, u^{-1}] = K\mathbb{Q}_*$.* □

The double delta operator $E \otimes K$ for general E is considered in §9.

Let us briefly consider the relationship between $E \otimes F$ and $F \otimes E$. It is clear that the involution $\tau_* \colon (\Phi \cdot \Phi)_* \to (\Phi \cdot \Phi)_*$ of (3.7) interchanges the left and right Φ_*-module structures on $(\Phi \cdot \Phi)_*$ and hence induces an isomorphism $\tau_* \colon (E \cdot F)_* \to (F \cdot E)_*$. This in turn factors to give an isomorphism $\tau_* \colon (E \otimes F)_* \to (F \otimes E)_*$, which is the restriction of the switch map $\tau_* \colon E_* \otimes F_* \otimes \mathbb{Q} \to F_* \otimes E_* \otimes \mathbb{Q}$. We thus have an isomorphism of double delta operators $\tau \colon E \otimes F \to \widetilde{F \otimes E}$.

6. Leibniz delta operators and Leibniz extensions

We now consider the Leibniz property of a delta operator. Since all our delta operators are torsion-free, we are able to take a slightly different approach from that of [20]. Rather than define a delta operator (E_*, Δ^e) as Leibniz when there exists a formula expressing how the operator Δ^e acts on a product (hence the name), we concentrate, dually, on the closure of the penumbral coalgebra under multiplication.

Definition 6.1. A torsion-free delta operator E is *Leibniz* if the penumbral coalgebra $\Pi(E)_*$ is a subring of $E\mathbb{Q}_*[x]$.

Since the polynomials $b^e_n(x) \in \Pi(E)_*$ form a basis for $E\mathbb{Q}_*[x]$ as an $E\mathbb{Q}_*$-module, there are always elements $e(i,j;m) \in E\mathbb{Q}_{2(i+j-m)}$ such that

$$b^e_i(x) b^e_j(x) = \sum_{m=1}^{i+j} e(i,j;m) b^e_m(x), \tag{6.2}$$

for each $i, j \geq 1$. The delta operator E is Leibniz precisely when all the $e(i,j;m)$ belong to E_*. In this case the multiplicative structure enjoyed by the coalgebra $\Pi(E)_*$ makes it into a Hopf algebra. The antipode is the ring homomorphism given by $x \mapsto -x$, and is thus determined on the basis of associated polynomials by

$$b^e_n(x) \longmapsto b^e_n(-x) = \sum_i^n (-1)^i \widehat{\mathbf{B}}_{n,i}\big(b^e_1(x), \ldots, b^e_{n-i+1}(x)\big),$$

where $\widehat{\mathbf{B}}_{n,i}$ is the ordinary Bell polynomial [6]. This integral formula clearly involves multiplication of the $b_j^e(x)$. Note that for a non-Leibniz delta operator E, the polynomials $b_n^e(-x)$ do not, in general, belong to $\Pi(E)_*$. For example,

$$b_4^\phi(-x) = \tfrac{1}{12}(15\phi_1^3 - 4\phi_1\phi_2 + \phi_3)b_1^\phi(x) + 3\phi_1^2 b_2^\phi(x) + 3\phi_1 b_3^\phi(x) + b_4^\phi(x) \notin \Pi(\Phi)_*.$$

For a Leibniz delta operator, the dual of $\Pi(E)_*$, the ring $E_*[[\Delta^e]]$, is a (completed) Hopf algebra whose coproduct is given by

$$\psi(\Delta^e) = \Delta^e \otimes 1 + 1 \otimes \Delta^e + \sum_{i,j>0} e(i,j;1)(\Delta^e)^i \otimes (\Delta^e)^j. \tag{6.3}$$

But a Hopf algebra structure on a power series ring is precisely a formal group. The series (6.3) is a formal group law for which (2.1) is the exponential series; see [12]. There is thus a close relation between the theory of (torsion-free) Leibniz delta operators and that of formal group laws.

Associativity in $\Pi(E)_*$, or coassociativity in $E_*[[\Delta^e]]$, imposes a large number of relations on the $e(i,j;m)$. A preliminary simplification can be made by concentrating on the elements $e(i,j;1)$, which will be abbreviated to $e(i,j)$. They may be specified by the rational equations

$$e(i,j) = (b_i^e b_j^e)(e) \qquad e^k \equiv e_{k-1}, \tag{6.4}$$

which are to be interpreted by first multiplying together the polynomials $b_i^e(x)$ and $b_j^e(x)$, and then making the usual umbral substitution $x^k \equiv e_{k-1}$ for all $k \leq i+j$. The order of performing these operations is important!

A simple computation, appealing to (5.5), reveals the first few examples to be

$$\begin{aligned} e(1,1) &= e_1, \\ e(1,2) = e(2,1) &= \tfrac{1}{2}(e_2 - e_1^2), \\ e(1,3) = e(3,1) &= \tfrac{1}{6}(e_3 - 4e_1e_2 + 3e_1^3), \\ e(2,2) &= \tfrac{1}{4}(e_3 - 2e_1e_2 + e_1^3). \end{aligned} \tag{6.5}$$

For a general delta operator these equations take place in $E\mathbb{Q}_*$. When the delta operator is Leibniz they imply that divisibility relations must hold in E_*; see Theorem 6.15 below.

Lemma 6.6 (See [2, Part II, §3])**.** *For each i and j, and for each $1 < m \leq i+j$, the element $e(i,j;m)$ is an integer polynomial in the elements $e(i',j')$ with $i'+j' < i+j$.*

Proof. We use induction on m, noting that the statement is empty for $m = 1$. Working in the Hopf algebra $E\mathbb{Q}_*[x]$, we apply the diagonal to both sides of (6.2), and equate coefficients of $b_{m-r}^e(x) \otimes b_r^e(x)$ (with $0 < r < m$) to obtain

$$e(i,j;m) = \sum e(i-s, j-t; m-r)e(s,t;r),$$

where the summation is over $0 \leq s \leq i$ and $0 \leq t \leq j$, with $0 < s+t < i+j$. The inductive step now follows. □

It is now clear how to extend the ring E_* in order that a torsion-free delta operator becomes Leibniz.

Definition 6.7. For a torsion-free delta operator E, the *minimal Leibniz extension of* E is the delta operator $L(E) = \big(L(E)_*, \Delta^e\big)$, where $L(E)_*$ is the subring of $E\mathbb{Q}_*$ generated over E_* by the elements $e(i,j)$.

In general the formal group law (6.3) will be defined over the ring $L(E)_*$. Inverting the series (2.1), we may write

$$D = \Delta^e + c_1 \frac{(\Delta^e)^2}{2} + \cdots + c_{k-1} \frac{(\Delta^e)^k}{k} + \cdots \tag{6.8}$$

(note the absence of factorials), where the coefficients c_n belong to $L(E)_*$ but not, in general, to E_*; see, for example, [8, IV, §1, Proposition 1]. This is the logarithm series of the formal group law.

In the case of the universal delta operator Φ, the generators ϕ_k can be expressed as integer polynomials in the c_n, with

$$\phi_n \equiv -(n-1)! c_n \quad \text{mod decomposables.}$$

Thus $\Phi_* \subset \mathbb{Z}[c_1, c_2, \ldots] \subset L(\Phi)_*$, and $L(\Phi)_*$ is generated by the $\phi(i,j)$, which are the coefficients of the formal group law

$$\sum_{n \geq 1} \frac{\phi_{n-1}}{n!} \left(\sum_{k \geq 1} c_{k-1} \frac{X^k + Y^k}{k} \right)^n \in \Phi\mathbb{Q}_*[[X, Y]].$$

But this is precisely Lazard's universal formal group law; see, for example, [12]. We review the method of Milnor [18] for constructing polynomial generators for the Lazard ring $L_* = L(\Phi)_*$. This throws some light on the extension $\Phi_* \subset L_*$. Let

$$h_n = \begin{cases} p, & \text{if } n+1 \text{ is a power of the prime } p, \\ 1, & \text{if } n+1 \text{ is not a prime power,} \end{cases} \tag{6.9}$$

then h_n is the highest common factor of the integers $\binom{n+1}{i}$, for $1 \leq i \leq n$. There are, therefore, integers λ_i^n such that

$$\sum_{i=1}^{n} \lambda_i^n \binom{n+1}{i} = h_n.$$

Now let

$$u_n = \sum_{i=1}^{n} \lambda_i^n \phi(i, n-i+1) \in L_{2n}.$$

Lazard's theorem asserts that $L_* = \mathbb{Z}[u_1, u_2, \ldots]$. For example, one choice of the λ_i^n leads to

$$\begin{aligned} u_1 &= \phi_1, \\ u_2 &= \tfrac{1}{2}(\phi_2 - \phi_1^2), \\ u_3 &= \tfrac{1}{12}(\phi_3 + 2\phi_1\phi_2 - 3\phi_1^3). \end{aligned} \tag{6.10}$$

By (6.4),

$$\phi(i, n-i+1) = \frac{1}{i!(n-i+1)!}\big(\phi_n + z_i(\phi)\big),$$

where $z_i(\phi)$ is an integer polynomial in the ϕ_r, for $r < n$. Writing Λ_n for the integer $(n+1)!/h_n$,

$$\Lambda_n u_n = \sum_{i=1}^{n} \lambda_i^n \frac{1}{h_n} \binom{n+1}{i} \big(\phi_n + z_i(\phi)\big)$$

which shows that $\Lambda_n u_n \in \Phi_{2n}$ and that

$$\Lambda_n u_n \equiv \phi_n \quad \text{mod decomposables} \tag{6.11}$$

in Φ_*. Thus letting $f_n = \Lambda_n u_n$, the f_n are alternative polynomial generators for Φ_*, with the inclusion

$$\Phi_* = \mathbb{Z}[f_1, f_2, \dots] \subset L_* = \mathbb{Z}[u_1, u_2, \dots]$$

given by $f_n \mapsto \Lambda_n u_n$.

For a general delta operator E the universal morphism $\nu \colon \Phi \to E$ induces $L(\nu) \colon L(\Phi) \to L(E)$. Let $L(\nu)_*(u_n) = v_n \in E_{2n}$, then $L(E)_*$ is generated over E_* by the elements

$$v_n = \sum_{i=1}^{n} \lambda_i^n e(i, n-i+1).$$

In general there are relations among the v_n.

Example 6.12. Consider the delta operator $\Phi/\phi_1 = \big(\mathbb{Z}[\phi_2, \phi_3, \dots], D + \phi_2 D^3/3! + \cdots\big)$. Here

$$L(\Phi/\phi_1)_* = \mathbb{Z}[u_1, u_2, u_3, \dots]/(u_1) = \mathbb{Z}[u_2, u_3, \dots],$$

since $\phi_1 = u_1$. On the other hand,

$$L(\Phi/\phi_2)_* = \mathbb{Z}[u_1, u_2, u_3, \dots]/(2u_2 + u_1^2),$$

since $\phi_2 = 2u_2 + u_1^2$ in L_*.

In [20, §8] the *universal Leibniz extension* ${}^L E$ was introduced. This differs in general from $L(E)_*$; in particular, and contrary to what was asserted in [20], ${}^L E_*$ may have torsion when E_* is torsion-free. However, as was shown in [20, Theorem 9.14], the universal case ${}^L \Phi_*$ is torsion-free and is isomorphic to L_*. In fact ${}^L E_*$ may be defined as $E_* \otimes_{\Phi_*} L_*$. For example, the calculations above show that ${}^L H_*$ is isomorphic to

$$\frac{\mathbb{Z}[u_2, u_3, \dots, u_n, \dots]}{(2u_2, 12u_3, \dots, \Lambda_n u_n, \dots)},$$

which has torsion of all orders. In the cases of Example 6.12, the two extensions coincide. In general, $L(E)_*$ is isomorphic to the quotient of ${}^L E_*$ by its torsion ideal. Thus a delta operator is Leibniz if and only if the classifying homomorphism $\Phi_* \to E_*$ factors through $\Phi_* \subset L_*$.

Topological Examples 6.13. Delta operators arising from topology are always Leibniz, for the map $j \colon \Omega S^3 \to \mathbb{C}P^\infty$ discussed in (2.7) is, up to homotopy, a map of H-spaces, so that $\Pi(E)_*$ is already a Hopf algebra.

Since the universal delta operator is not Leibniz, it does not stem from a spectrum. Yet its Leibniz extension $L_* = L(\Phi)_*$ is isomorphic to the complex bordism ring MU_* [19], and so corresponds to the case $E = MU$, the universal complex-oriented spectrum. It is this relationship which makes it possible to investigate Leibniz extensions by methods adapted from algebraic topology.

In general the extension $E_* \subseteq L(E)_*$ may be very complicated. We conclude this section by giving a specific example, and by proving a result which shows that there are general divisibility relations which must hold among the e_n in $L(E)_*$.

Example 6.14. We refer to the quadratic delta operator $R = \big(\mathbb{Z}[u], D + uD^2/2\big)$ as the *Bessel operator*, since the associated sequence is made up of graded Bessel polynomials; see [26]. This delta operator is not Leibniz; for example, since $r_1 = u$, with $r_i = 0$ for $i \geq 2$, by (5.5),

$$b_1^r(x) = x, \quad b_2^r(x) = \tfrac{1}{2}(x^2 - ux), \quad b_3^r(x) = \tfrac{1}{6}(x^3 - 3ux^2 + 3u^2x),$$

so that

$$b_1^r(x)b_2^r(x) = -\tfrac{1}{2}u^2 b_1^r(x) + 2ub_2^r(x) + 3b_3^r(x).$$

Thus $u^2/2 = -r(1,2) \in L(R)_*$. The associated formal group law is

$$X + Y + u^{-1}\big(\sqrt{1+2uX} - 1\big)\big(\sqrt{1+2uY} - 1\big).$$

Hence

$$r(i,j) = (-1)^{i+j} C_{i-1} C_{j-1} \frac{u^{i+j-1}}{2^{i+j-2}},$$

where C_n is the Catalan number

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

By analysing the 2-divisibility of such numbers it is not hard to show that $L(R)_*$ is multiplicatively generated by the elements $u^j/2^{j-1}$, where j is of the form $2^m + 2^n - 1$. Note also that the relations (6.10) show that $u^3 + 4u_3$ is 3-torsion in the ring ${}^L R_*$ which cannot, therefore, be isomorphic to $L(R)_*$.

The following Kummer congruence for the coefficients of a delta operator is related to Theorem 2 of [4]; see also [28, 29]. We will use its corollary in the proof of Proposition 10.8.

Theorem 6.15. *If p is prime, then*

$$e_{n+p-1} \equiv e_n e_{p-1} \mod p$$

in the ring $L(E)_*$.

Simple cases of these congruences arise from the formulas in (6.5). For example, since $e_2 - e_1^2 = 2e(1,2)$ and $e_3 - 4e_1e_2 + 3e_1^3 = 6e(1,3)$ in $L(E)_*$, we have $e_2 \equiv e_1^2 \mod 2$, $e_3 \equiv e_1^3 \equiv e_1e_2 \mod 2$, and $e_3 \equiv e_1e_2 \mod 3$.

Proof. We will in fact show that the congruences hold modulo p in the subring $\mathbb{Z}[c_1, c_2, \dots]$ of $L(E)_*$ which is generated by the elements c_n defined by (6.8).

Lagrange inversion applied to the equations (2.1) and (6.8) yields

$$e_n = \sum (-1)^\kappa \frac{(n+\kappa)!}{2^{k_1}3^{k_2}\cdots(s+1)^{k_s} k_1! k_2! \cdots k_s!} c_1^{k_1} c_2^{k_2} \cdots c_s^{k_s},$$

where the sum is over all sequences $(k_1, k_2, \ldots, k_s)$ such that $k_1 + 2k_2 + \cdots + sk_s = n$, and $\kappa = k_1 + k_2 + \cdots + k_s$. Writing $e(c_1^{k_1} c_2^{k_2} \cdots c_s^{k_s})$ for the coefficient of $c_1^{k_1} c_2^{k_2} \cdots c_s^{k_s}$ in e_n, this formula can be rearranged to give

$$e(c_1^{k_1} c_2^{k_2} \cdots c_s^{k_s}) = (-1)^\kappa \binom{n+\kappa}{2k_1, 3k_2, \ldots, (s+1)k_s} \prod_{t=1}^{s} \frac{\big((t+1)k_t\big)!}{(t+1)^{k_t} k_t!}.$$

We will show that

$$e(c_1^{k_1} c_2^{k_2} \cdots c_s^{k_s} c_{p-1}) \equiv e(c_1^{k_1} c_2^{k_2} \cdots c_s^{k_s}) \mod p \tag{6.16}$$

in $\mathbb{Z}[c_1, c_2, \ldots]$. This will be sufficient, since we will also show that

$$e_{p-1} \equiv c_{p-1} \mod p. \tag{6.17}$$

The following result, generalising Wilson's theorem (the case $m = p$ and $a = 1$), is trivial; see §1 of [5] for the proof of stronger results.

Lemma 6.18. *If p is prime and $m, a \geq 1$, then*

$$\frac{(ma)!}{m^a a!} \equiv \begin{cases} 1 \mod p, & \text{if } m = 1, \\ (-1)^a \mod p, & \text{if } m = p, \\ 0 \mod p, & \text{if } m \nmid p \text{ and } ma > p. \end{cases}$$ □

It follows that $e(c_1^{k_1} c_2^{k_2} \cdots c_s^{k_s}) \equiv 0 \mod p$ unless $(t+1)k_t < p$ for all $t \neq p-1$. Hence the congruence (6.16) holds for all monomials for which $(t+1)k_t > p$ for some $t \neq p$, in particular for all monomials divisible by c_t for some $t \geq p$.

Consider now the multinomial coefficient

$$\binom{n+\kappa}{2k_1, 3k_2, \ldots, pk_{p-1}}.$$

If each term $(t+1)k_t$ is less than p for $t < p-1$, then this multinomial coefficient will be divisible by p if $n + \kappa - pk_{p-1} \geq p$. Thus the congruence (6.16) holds (again because both sides are zero modulo p) for all monomials $c_1^{k_1} c_2^{k_2} \cdots c_{p-1}^{k_{p-1}}$ for which

$$2k_1 + 3k_2 + \cdots + (p-1)k_{p-2} \geq p.$$

Together with Wilson's theorem, this argument also shows that congruence (6.17) holds.

For the remaining cases, Lemma 6.18 shows that

$$e(c_1^{k_1} c_2^{k_2} \cdots c_{p-1}^{k_{p-1}+1}) \equiv e(c_1^{k_1} c_2^{k_2} \cdots c_{p-1}^{k_{p-1}}) \frac{(n+\kappa+p)(n+\kappa+p-1)\cdots(n+\kappa+1)}{(pk_{p-1}+p)(pk_{p-1}+p-1)\cdots(pk_{p-1}+1)}$$

modulo p. Now we know that $pk_{p-1} < n + \kappa < pk_{p-1} + p$, so that the factor $pk_{p-1} + p$ can be cancelled in the fraction, leaving a numerator and denominator which are, by Wilson's theorem, both congruent to -1. This shows that

$$e(c_1^{k_1} c_2^{k_2} \cdots c_{p-1}^{k_{p-1}+1}) \equiv e(c_1^{k_1} c_2^{k_2} \cdots c_{p-1}^{k_{p-1}}) \mod p$$

and completes the proof. □

Corollary 6.19. *For any delta operator E,*

$$e_{p^j-1} \equiv e_{p-1}^{1+p+\cdots+p^{j-1}} \mod p$$

in $L(E)_*$. □

7. Leibniz double delta operators

In this section we extend the Leibniz concept to double delta operators and discuss how the Leibniz properties of the double delta operator $E \otimes F$ are influenced by the corresponding properties of its constituent components E and F.

Given a double delta operator G, equation (6.2) yields two sets of elements $g_1(i,j) = g_1(i,j;1)$ and $g_2(i,j) = g_2(i,j;1)$ in $G\mathbb{Q}_*$ corresponding to the two delta operators ${}_1G$ and ${}_2G$.

Lemma 7.1. *Given a double delta operator G, the delta operator ${}_1G$ is Leibniz if and only if ${}_2G$ is Leibniz.*

Proof. As remarked in §3, the two penumbral coalgebras $\Pi({}_1G)_*$ and $\Pi({}_2G)_*$ are equal. Hence if one is closed under multiplication so is the other. □

Definition 7.2. A double delta operator G is *Leibniz* if ${}_1G$ and ${}_2G$ are Leibniz delta operators. If G fails to be Leibniz, the Leibniz extension

$$G_* \subseteq L(G)_* \subseteq G\mathbb{Q}_*$$

is defined by adjoining either the elements $g_1(i,j)$ or the elements $g_2(i,j)$; Lemma 7.1 shows that both choices yield the same result.

The concept of a Leibniz double delta operator is precisely equivalent to a pair of formal group laws over a torsion-free ring together with a strict isomorphism between them.

Proposition 7.3. *Given two delta operators E and F,*

$$L(E \otimes F) = L(E) \otimes F = E \otimes L(F) = L(E) \otimes L(F).$$

In particular, if E is Leibniz, then $E \otimes F = E \otimes L(F)$, and if F is Leibniz, then $E \otimes F = L(E) \otimes F$; in both cases $E \otimes F$ is Leibniz.

Proof. It suffices to note that, if $G = E \otimes F$, then $g_1(i,j) = e(i,j)$ and $g_2(i,j) = f(i,j)$. □

We can now construct the universal Leibniz double delta operator.

Proposition 7.4.

$$L(\Phi \cdot \Phi)_* = \big(L(\Phi) \otimes L(\Phi)\big)_* = L_*[b_1, b_2, \dots].$$

Proof. By Proposition 7.3, $L(\Phi \cdot \Phi)_* = \big(L(\Phi) \otimes L(\Phi)\big)_* = \big(L(\Phi) \otimes \Phi\big)_*$. Now

$$\big(L(\Phi) \cdot \Phi\big)_* = L_* \otimes_{\Phi_*} (\Phi \cdot \Phi)_* \otimes_{\Phi_*} \Phi_* = L_*[b_1, b_2, \dots],$$

which is torsion-free and hence equal to $\big(L(\Phi) \otimes \Phi\big)_*$. □

In parallel with the case of single delta operators, a double delta operator G is Leibniz if and only if the classifying map $(\Phi \cdot \Phi)_* \to G_*$ factors through $(\Phi \cdot \Phi)_* \subset L(\Phi \cdot \Phi)_*$. Moreover $L(G)_*$ may be identified with the quotient of $G_* \otimes_{(\Phi \cdot \Phi)_*} L(\Phi \cdot \Phi)_*$ by its torsion ideal. We may thus characterise $L(E \otimes F)$.

Proposition 7.5. *For torsion-free delta operators E and F,*

$$L(E \otimes F)_* = \frac{E_* \otimes_{\Phi_*} L(\Phi \cdot \Phi)_* \otimes_{\Phi_*} F_*}{\text{Torsion}}.$$

Hence if E and F are Leibniz,

$$(E \otimes F)_* = \frac{E_* \otimes_{L_*} L(\Phi \cdot \Phi)_* \otimes_{L_*} F_*}{\text{Torsion}}.$$

□

Even in the Leibniz case the two tensor products may not be isomorphic before taking the torsion quotient. For example, $H_* \otimes_{\Phi_*} L(\Phi \cdot \Phi)_* \otimes_{\Phi_*} H_*$ is isomorphic to

$$\frac{\mathbb{Z}[b_1, b_2, \dots, b_n, \dots]}{(2b_1, 6b_2, \dots, (n+1)!b_n, \dots)},$$

while $H_* \otimes_{L_*} L(\Phi \cdot \Phi)_* \otimes_{L_*} H_*$ is isomorphic to

$$\frac{\mathbb{Z}[b_1, b_2, \dots, b_n, \dots]}{(2b_1, 3b_2, \dots, h_n b_n, \dots)},$$

where h_n is as defined in (6.9).

8. Pairing topological delta operators

As for single delta operators in §6, the double delta operators arising from complex-oriented ring spectra, in the way described in (3.13), are always Leibniz. We discuss now their relationship with the pairing of §5. Recall from (2.7) that if E and F are complex-oriented ring spectra with torsion-free coefficient rings E_* and F_*, they each give rise to a delta operator, also denoted by E and F. Thus $(E \otimes F)_*$ denotes the domain of the double delta operator $E \otimes F$; see Definition 5.2. We can think of this ring as an algebraic model for the ring $(E \wedge F)_* \cong E_*(F) \cong F_*(E)$. Under certain conditions the two are isomorphic.

Since, as discussed in (3.13), there is a (Δ^e, Δ^f)-operator over the ring $E_*(F)$ if it is torsion-free, Proposition 5.3 provides, in this case, a homomorphism

$$\mu_{E,F} \colon (E \otimes F)_* \to E_*(F).$$

In particular this applies to $MU_*(MU)$, which is torsion-free. In the general case the orientations of E_* and F_* and the classifying maps of the double delta operators combine to give a commutative diagram

$$\begin{array}{ccccc} L(\Phi \cdot \Phi)_* & \xrightarrow{\nu_*} & (MU \otimes MU)_* & \xrightarrow{\mu_{MU,MU}} & MU_*(MU) \\ & & \downarrow & & \downarrow \\ & & (E \otimes F)_* & \xrightarrow[\mu_{E,F}]{} & E_*(F) \end{array}$$

Theorem 8.1. *The homomorphisms $\nu_* \colon L(\Phi \cdot \Phi)_* \to (MU \otimes MU)_*$ and $\mu_{MU,MU} \colon (MU \otimes MU)_* \to MU_*(MU)$ are isomorphisms.*

Proof. Since $MU_* \cong L_*$, the composition of these two maps is can be identified, using Proposition 7.4, with the MU_*-module homomorphism

$$MU_*[b_1, b_2, \dots] \to MU_*(MU)$$

which sends the b_k to the coefficients of the series expressing one of the delta operators over $MU_*(MU)$ in terms of the other. But it is shown in [2, Part II], for example, that $MU_*(MU)$ is polynomially generated by these coefficients.

Now it follows by Proposition 7.5 that $(MU \otimes MU)_*$ is isomorphic to $MU_* \otimes_{MU_*} MU_*(MU) \otimes_{MU_*} MU_* = MU_*(MU)$. □

The observation that $MU_*(MU)$ is the universal ring for strict isomorphisms of formal group laws is due to Landweber [13].

Recall that the complex-oriented spectrum E is said to be *Landweber exact* if the homology theory $E_*(\quad)$ can be defined for all spaces X as

$$E_*(X) = E_* \otimes_{MU_*} MU_*(X).$$

Criteria on E_* for this to hold were set down by Landweber in [14]; they are discussed at the beginning of §11. An elementary consequence of these criteria is that E_* must be torsion-free. Examples of such spectra include complex K-theory, the elliptic spectrum Ell, the Johnson-Wilson spectra $E(n)$, the Brown-Peterson spectra BP and the complex bordism spectrum MU itself.

Lemma 8.2. *If E is Landweber exact, and F is a complex-oriented spectrum, then $E_*(F)$ is a flat F_*-module.*

Proof. The argument is essentially the same as that of [17, Remark 3.7]. If F is any complex-oriented ring spectrum, then (see [2, Part II])

$$MU_*(MU) \otimes_{MU_*} F_* \cong MU_*(F). \tag{8.3}$$

Hence

$$E_*(F) \cong E_* \otimes_{MU_*} MU_*(MU) \otimes_{MU_*} F_*.$$

Thus the functor $E_*(F) \otimes_{F_*} \_$ can be written as the composition of the functors $MU_*(MU) \otimes_{MU_*} \_$ and $E_* \otimes_{MU_*} \_$. The first of these is exact because $MU_*(MU)$ is a flat MU_*-module, and it is a functor into the category of $MU_*(MU)$-comodules. But the second is exact on this category; this is ensured by the Landweber exactness conditions [14]. □

Proposition 8.4. *If E is Landweber exact, and F is a complex-oriented spectrum with F_* torsion-free, then*

$$\mu_{E,F} \colon (E \otimes F)_* \to E_*(F)$$

is an isomorphism.

Proof. Proposition 7.5, Theorem 8.1 and (8.3) show that $(E \otimes F)_*$ is isomorphic to $E_* \otimes_{MU_*} MU_*(F)$ modulo torsion. But, since E is Landweber exact, $E_* \otimes_{MU_*} MU_*(F) \cong E_*(F)$, and, since F is torsion-free, there is an exact sequence of F_*-modules

$$0 \to F_* \to F_* \otimes \mathbb{Q}.$$

Applying $E_*(F) \otimes_{F_*} \_$ there is, by Lemma 8.2, an exact sequence

$$0 \to E_*(F) \to E_*(F) \otimes \mathbb{Q},$$

so that $E_*(F)$ is torsion-free. □

9. Stably penumbral polynomials and K-theory

Recall from (2.7) the Leibniz delta operator K given by $K_* = \mathbb{Z}[u, u^{-1}]$ and $\Delta^k = u^{-1}(e^{uD} - 1)$. The coefficients of Δ^k are thus given by $k_n = u^n$. It has $B^k_n(x) = x(x-u)\cdots\big(x-(n-1)u\big)$ as its associated sequence, and therefore the normalised version may be written as $b^k_n(x) = u^n\binom{x/u}{n}$. Hence K is Leibniz by virtue of the *Vandermonde convolution* identity

$$\binom{x/u}{i}\binom{x/u}{j} = \sum_{m=1}^{i+j}\binom{m}{i}\binom{i}{m-j}\binom{x/u}{m},$$

for $i, j \geq 1$; see, for example, [25].

We will show that in a "stable" sense the divisibility introduced into a delta operator by pairing with K can be identified with the divisibility involved in forming the Leibniz extension and the penumbral coalgebra.

If F is a Leibniz delta operator, then the penumbral coalgebra $\Pi(F)_*$ of Definition 2.5 is closed under multiplication by x, and we may construct the localised F_*-algebra $\Pi(F)_*[x^{-1}]$ as the limit of the directed system of modules

$$\Pi(F)_* \xrightarrow{x} \Pi(F)_* \xrightarrow{x} \Pi(F)_* \xrightarrow{x} \cdots,$$

where the maps are multiplication by x.

Theorem 9.1. *If F is a Leibniz delta operator, with F_* torsion-free, then $\Pi(F)_*[x^{-1}]$ and $(K \otimes F)_*$ are isomorphic as F_*-algebras.*

Proof. By Proposition 5.7, the F_*-algebra $(K\otimes F)_*$ is generated as an algebra over $K_* \otimes F_* = F_*[u, u^{-1}]$ by the umbral elements $b^f_n(k)$. The nature of the coefficients $k_n = u^n$ means that $b^f_n(k)$, as defined in (5.4), is just the polynomial $u^{-1}b^f_n(u)$. Hence $(K \otimes F)_*$ is multiplicatively generated over $F_*[u, u^{-1}]$ by the polynomials $b^f_n(u)$.

Recalling that the polynomials $b^f_n(x)$, for $n \geq 0$ form a basis for $\Pi(F)_*$, define the F_*-module homomorphism $\alpha\colon \Pi(F)_* \to (K \otimes F)_*$ by setting $\alpha\big(b^f_n(x)\big) = b^f_n(u)$. Clearly α is a monomorphism of rings, with $\alpha(x) = u$. Hence the diagram

$$\begin{array}{ccc} \Pi(F)_* & \xrightarrow{\ x\ } & \Pi(F)_* \\ \alpha\downarrow & & \downarrow\alpha \\ (K\otimes F)_* & \xrightarrow{\ u\ } & (K\otimes F)_* \end{array}$$

commutes. Since multiplication by u is an isomorphism on $(K \otimes F)_*$, there is an induced map of F_*-algebras $\widehat{\alpha}\colon \Pi(F)_*[x^{-1}] \to (K\otimes F)_*$. In fact, if $z \in \Pi(F)_*[x^{-1}]$ and $i \geq 0$ is such that $x^i z \in \Pi(F)_*$, we have $\widehat{\alpha}(z) = u^{-i}\alpha(x^i z)$. It is clear that $\widehat{\alpha}$ is a monomorphism.

But, since F is Leibniz, $\Pi(F)_*$ is closed under multiplication, thus any product of the polynomials $b^f_n(x)$ can be written as an F_*-linear combination of the $b^f_n(x)$. Hence $(K \otimes F)_*$ is *additively* generated over $F_*[u, u^{-1}]$ by the polynomials $b^f_n(u)$, and this implies that $\widehat{\alpha}$ is an epimorphism. □

In the case of a general torsion-free delta operator which is not necessarily Leibniz, we need to consider the minimal Leibniz extension.

Corollary 9.2. *If E is a torsion-free delta operator, then $\Pi\big(L(E)\big)_*[x^{-1}]$ and $(K \otimes E)_*$ are isomorphic as $L(E)_*$-algebras.*

Proof. By Proposition 7.3, $(K \otimes E)_* = \big(K \otimes L(E)\big)_*$. It is this identification which gives $(K \otimes E)_*$ the structure of an $L(E)_*$-algebra. The result is then the special case of Theorem 9.1 in which $F = L(E)$. □

10. Hattori-Stong theorems

The classical Hattori-Stong theorem [11, 30], for which we will give a proof as Theorem 10.14, asserts, in Hattori's formulation, that the MU_*-module map $MU_* \to K_*(MU)$ induced by the right unit $MU \to K \wedge MU$ is the inclusion of a direct summand [2, Part II, §14]. Note that MU_* is a direct summand as a subgroup of the abelian group $K_*(MU)$, not a summand as an MU_*-module.

Proposition 10.1. *No splitting map $K_*(MU) \to MU_*$ is an MU_*-module map.*

Proof. Suppose given a factorisation

$$MU_* \to K_*(MU) \to MU_*$$

of the identity on MU_* by MU_*-module maps. Now applying $\_ \otimes_{MU_*} H_*$ to this sequence, there is a factorisation of the identity

$$H_* \to K_*(MU) \otimes_{MU_*} H_* = K_*(H) \to H_*.$$

However, $H_* = \mathbb{Z}$, concentrated in degree zero, while $K_*(H)$ is a rational vector space as shown by Propositions 5.10 and 8.4. □

We are unaware of any splitting $K_*(MU) \to MU_*$ having been written down explicitly.

What Hattori actually shows is that if $\alpha \in MU_*$ is divisible by an integer m in $K_*(MU)$, then α is already divisible by m in MU_*. In other words MU_* is a *pure* subgroup of $K_*(MU)$; see [9, Ch. IV]. The connective K-theory group $k_*(MU) = (k \otimes MU)_*$ lies between MU_* and $K_*(MU)$. Since $k_*(MU)$ is finitely generated in each degree, it follows that MU_* is a summand of $k_*(MU)$, and, since $k_*(MU)$ is a summand of $K_*(MU)$, so is MU_*. Such finiteness arguments may not be available for a general delta operator, so we will phrase our generalisations in terms of the concept of purity.

We can interpret MU_* as $L_* = L(\Phi)_*$ and $K_*(MU)$ as $(K \otimes MU)_* = \big(K \otimes L(\Phi)\big)_* = (K \otimes \Phi)_*$. For any torsion-free delta operator E, Corollary 9.2 shows that $E_* \subseteq L(E)_* \subset (K \otimes E)_*$ so the Hattori-Stong theorem motivates us to ask when $L(E)_*$ is a pure subgroup of $(K \otimes E)_*$. In order to discuss this question we consider the smallest pure subgroup containing $L(E)_*$.

Definition 10.2. Let $\Sigma(E)_* = E\mathbb{Q}_* \cap (K \otimes E)_*$ denote the rational closure of E_* in $(K \otimes E)_*$.

Thus $\Sigma(E)_*$ consists of all $\alpha \in (K \otimes E)_*$ for which there exists a non-zero integer m such that $m\alpha \in E_*$. Since $E_* \subseteq L(E)_* \subseteq E\mathbb{Q}_*$, it is equivalent to ask that there is an integer m such that $m\alpha \in L(E)_*$. It is also clear that $\Sigma(E)_*$ is a subring of $(K \otimes E)_*$, and that $L(E)_*$ is a subring of $\Sigma(E)_*$.

Definition 10.3. We say that the *Hattori-Stong theorem* holds for the torsion-free delta operator E if $\Sigma(E)_* = L(E)_*$, so that $L(E)_*$ is a pure subgroup of $(K \otimes E)_*$.

We will study the ring $\Sigma(E)_*$ by using Corollary 9.2 to identify $(K \otimes E)_*$ with the ring $\Pi\big(L(E)\big)_*[x^{-1}]$.

Writing

$$x^n = \sum_{r=1}^{n} \sigma(n,r) b_r^e(x), \tag{10.4}$$

in $\Pi\big(L(E)\big)_*$, the coefficients $\sigma(n,r) \in L(E)_*$ may be computed as

$$\sigma(n,r) = \big\langle (\Delta^e)^r \bigm| x^n \big\rangle.$$

In the notation of [22], $\sigma(n,r) = r! S^E(n,r)$, where $S^E(n,r)$ is an E-theory Stirling number of the second kind. The leading coefficient $\sigma(n,n)$ is equal to $n!$, while $\sigma(n,1) = e_{n-1}$, the coefficient of $D^n/n!$ in Δ^e; see (2.1). In fact Proposition 3.2, applied to the double delta operator $H \otimes E$, gives

$$\sigma(n,r) = n! \sum \binom{r}{m_1, m_2, \ldots, m_k} \left(\frac{e_1}{2!}\right)^{m_1} \left(\frac{e_2}{3!}\right)^{m_2} \cdots \left(\frac{e_k}{(k+1)!}\right)^{m_k},$$

where the summation is over all sequences $(m_1, m_2, \ldots, m_k)$ such that $m_1 + 2m_2 + \cdots + km_k = n - r$, and $m_1 + m_2 + \cdots + m_k \leq r$. If $m_1 + m_2 + \cdots + m_k = r - s$, then the coefficient of the monomial $e_1^{m_1} e_2^{m_2} \cdots e_k^{m_k}$ in $\sigma(n,r)$ is equal to

$$n(n-1)\cdots(n-s+1) \binom{n-s}{\underbrace{2,\ldots,2}_{m_1}, \underbrace{3,\ldots,3}_{m_2}, \ldots, \underbrace{k+1,\ldots,k+1}_{m_k}} \binom{r}{m_1, m_2, \ldots, m_k, s}. \tag{10.5}$$

This formula shows that $\sigma(n,r) \in E_*$, as expected since $x^n \in E_*[x] \subseteq \Pi(E)_* \subseteq \Pi\big(L(E)\big)_*$.

Lemma 10.6. *If p is prime and $j \geq 1$, the coefficient $\sigma(p^j, r)$ is divisible by p in E_* for $r > 1$.*

Proof. It is clear that if $s > 0$ in (10.5), then p divides the coefficient of the corresponding monomial in $\sigma(p^j, r)$. While if $s = 0$, the first multinomial coefficient is

$$\binom{p^j}{\underbrace{2,\ldots,2}_{m_1}, \underbrace{3,\ldots,3}_{m_2}, \ldots, \underbrace{k+1,\ldots,k+1}_{m_k}}$$

which is divisible by p in every case except that corresponding to $\sigma(p^j, 1)$. ☐

Lemma 10.7. *If p is prime and $j \geq 1$, then $x^{p^j} \equiv e_{p-1}^{1+p+\cdots+p^{j-1}} x \mod p$ in $\Pi\big(L(E)\big)_*$.*

Proof. By Lemma 10.6 and (10.4), $x^{p^j} \equiv \sigma(p^j, 1) b_1^e(x)$, but $b_1^e(x) = x$ and $\sigma(p^j, 1) = e_{p^j-1}$, so that the result follows from Corollary 6.19. ☐

Proposition 10.8. *Let p be a prime and $l \in L(E)_*$, then p divides l in $\Sigma(E)_*$ if and only if p divides le_{p-1}^n in $L(E)_*$ for some non-negative integer n.*

Proof. It is clear by Definition 10.2 and Corollary 9.2 that $p \mid l$ in $\Sigma(E)_*$ if and only if $p \mid lx^{p^j}$ in $\Pi\big(L(E)\big)_*$ for some positive integer j. By Lemma 10.7 this is equivalent to p dividing $le_{p-1}^{1+p+\cdots+p^{j-1}}x$ in $\Pi\big(L(E)\big)_*$ for some j.

However, since x is an element of the $L(E)_*$-module basis for $\Pi\big(L(E)\big)_*$ provided by the normalised associated sequence, $p \mid l'x$ in $\Pi\big(L(E)\big)_*$, where $l' \in L(E)_*$, if and only if $p \mid l'$ in $L(E)_*$. □

The Hattori-Stong theorem (that $L(E)_*$ is equal to $\Sigma(E)_*$) amounts to saying that if a prime p divides $l \in L(E)_*$ in $\Sigma(E)_*$, then p already divides l in $L(E)_*$. Hence Proposition 10.8 gives a criterion for when the Hattori-Stong theorem applies.

Theorem 10.9. *The Hattori-Stong theorem holds for the delta operator E if and only if, for all primes p and all $l \in L(E)_*$, whenever p divides le_{p-1} in $L(E)_*$, then p divides l in $L(E)_*$.*

Proof. If $\Sigma(E)_* \neq L(E)_*$, then there must be a prime p and an element $l \in L(E)_*$ such that p divides l in $\Sigma(E)_*$ but p does not divide l in $L(E)_*$. By Proposition 10.8, p divides le_{p-1}^n for some n. Applying the condition of the statement n times shows that p must divide l in $L(E)_*$, which is a contradiction.

Conversely, if $\Sigma(E)_* = L(E)_*$ and p divides le_{p-1} in $L(E)_*$, then Proposition 10.8 with $n = 1$ shows that p divides l in $L(E)_*$. □

Applying Proposition 10.8 in the case $l = 1$ will tell us which primes are invertible in $\Sigma(E)_*$.

Theorem 10.10. *The prime p is invertible in the ring $\Sigma(E)_*$ if and only if p divides e_{p-1}^n in $L(E)_*$ for some non-negative integer n.* □

Theorem 10.10 raises the question of which primes are invertible in $L(E)_*$. Definition 6.7 shows that $L(E)_*$ is multiplicatively generated over E_* by elements of positive degree. It follows that if $E_n = 0$ for $n < 0$, then no new relations can be introduced in degree 0, so that a prime is invertible in $L(E)_*$ if and only if it is invertible in E_*. On the other hand, if we invert the two-dimensional generator of the Bessel delta operator R to give $R[u^{-1}] = \big(\mathbb{Z}[u, u^{-1}], D + uD^2/2\big)$ we have $L\big(R[u^{-1}]\big)_* = \mathbb{Z}[1/2][u, u^{-1}]$ since $u^2/2 \in L(R)_4$; see Example 6.14. Hence 2 is invertible in $L\big(R[u^{-1}]\big)_*$, but not in $R[u^{-1}]_*$.

If the prime divisibility structure of the ring $L(E)_*$ is reasonably simple, we can make some simplification of Theorems 10.9 and 10.10.

Definition 10.11. A ring R has *unique integer factorisation* if, for all $r, s \in R$ and prime integers p, whenever p divides rs, then either p divides r or p divides s.

Proposition 10.12. *Assume that $L(E)_*$ has unique integer factorisation, then the Hattori-Stong theorem for E holds if and only if, for all primes p, either p is invertible in $L(E)_*$ or p does not divide e_{p-1} in $L(E)_*$.*

Proof. If $L(E)_*$ has unique integer factorisation, the statement "$p \mid le_{p-1}$ implies $p \mid l$" is equivalent to the statement "$p \mid e_{p-1}$ implies $p \mid l$ for all l". □

It is useful to phrase what is essentially the same result in a different way.

Proposition 10.13. *If $L(E)_*$ has unique integer factorisation, then $\Sigma(E)_*$ is the localisation of $L(E)_*$ in which those primes p which divide e_{p-1} are inverted.* □

We conclude this section by considering a number of examples. Firstly we can give a simple proof of Hattori and Stong's original result.

Theorem 10.14 (The classical Hattori-Stong theorem)**.**
$\Sigma(\Phi)_* = L(\Phi)_*$.

Proof. Since $L(\Phi)_*$ is a polynomial ring over $\mathbb{Z}$, it has unique integer factorisation and no primes are invertible in $L(\Phi)_*$. Hence, by Proposition 10.12, we need only show that $p \nmid \phi_{p-1}$ in $L(\Phi)_*$. There are many ways of doing this. We could remark that the congruence (6.11) shows that ϕ_{p-1} is congruent modulo decomposables to $(p-1)!u_{p-1}$, where u_{p-1} is one of the polynomial generators of $L(\Phi)_*$. Alternatively, the morphism of delta operators $\Phi \to K$ given by the universality of Φ maps ϕ_n to $u^n \in K_*$. Since u^{p-1} is indivisible in $K_* = L(K)_*$, the prime p cannot divide ϕ_{p-1} in $L(\Phi)_*$. □

The second of these arguments can be abstracted as follows.

Proposition 10.15. *Suppose that the Hattori-Stong theorem holds for the delta operator F, and we are given a morphism $E \to F$ of delta operators. If $L(E)_*$ has unique integer factorisation, and the same primes are invertible in each of the rings $L(E)_*$ and $L(F)_*$, then the Hattori-Stong theorem holds for E.*

Proof. We apply Proposition 10.12. Supposing that the prime p divides e_{p-1} in $L(E)_*$, we deduce that p divides f_{p-1} in $L(F)_*$, so that the Hattori-Stong theorem for F implies that p is invertible in $L(F)_*$ and hence in $L(E)_*$. □

Proposition 10.16. *For the Artin-Hasse delta operator A of Example* 3.11*,* $\Sigma(A)_* = A_* \otimes \mathbb{Z}_{(p)}$

Proof. We apply Proposition 10.13. The delta operator A is Leibniz; see [12, §3.2]. The series (3.12) giving D in terms of Δ^a is at the same time a divided power series with coefficients in A_* and a power series with coefficients in $A_*[1/p]$. It follows that the same is true for the inverse series for Δ^a in terms of D. Hence a_{q-1} is divisible by $q!/p^{\nu_p(q!)}$. So for a prime $q \neq p$, the coefficient a_{q-1} is divisible by q. But $a_{p-1} = -(p-1)!v$ which is not divisible by p. □

We consider now two examples of delta operators where

$$E_* \subset L(E)_* \subset \Sigma(E)_* \subset E\mathbb{Q}_*$$

are all proper inclusions.

Recall from Example 6.12 that for the delta operator Φ/ϕ_1 we have $L(\Phi/\phi_1)_* = \mathbb{Z}[u_2, u_3, \dots]$, which has unique integer factorisation. Since $\phi_1 = 0$ in $L(\Phi/\phi_1)_*$, the prime 2 is invertible in $\Sigma(\Phi/\phi_1)_*$. On the other hand, for an odd prime p, we saw in (6.11) that $(p-1)!u_{p-1} \equiv \phi_{p-1}$ modulo decomposables in L_*, so that p does not divide ϕ_{p-1} in $L(\Phi/\phi_1)_*$, hence p is not invertible in $\Sigma(\Phi/\phi_1)_*$. Thus by Proposition 10.13 $\Sigma(\Phi/\phi_1)_* = \mathbb{Z}[\frac{1}{2}][u_2, u_3, \dots]$.

Our second example is the delta operator Φ/ϕ_2 for which $L(\Phi/\phi_2)_* = \mathbb{Z}[u_1, u_2, u_3, \dots]/(2u_2 + u_1^2)$ (see Example 6.12) does not have unique integer factorisation. However unique integer factorisation fails only for the prime 2, and since $\phi_1^2 = -2u_2$ in $L(\Phi/\phi_2)_*$, Theorem 10.10 shows that 2 is invertible in $\Sigma(\Phi/\phi_2)_*$. Similarly 3 is invertible, since $\phi_2 = 0$ in $L(\Phi/\phi_2)_*$. For all other primes we may use the argument which applied to the previous example to conclude that $\Sigma(\Phi/\phi_2)_* = \mathbb{Z}[\frac{1}{6}][u_1, u_3, \dots]$.

11. Topological Hattori-Stong theorems

Finally we examine topological Hattori-Stong theorems. Suppose that E is a complex-oriented spectrum with E_* torsion-free. The corresponding delta operator E will already be Leibniz, and, since the K-theory spectrum is Landweber exact, Proposition 8.4 shows that $(K \otimes E)_*$ is isomorphic to $K_*(E)$. Thus the Hattori-Stong theorem for E-theory asserts that E_* is isomorphic to its rational closure $\Sigma(E)_*$ under the right unit $E_* \to K_*(E)$.

We recall the conditions for a spectrum to be Landweber exact [14]. Fixing a prime p, for $n \geq 0$ let $u_n \in E_{2p^n-2}$ be the coefficient of t^{p^n} in the p-series $[p]_E(t)$ of the E-theory formal group law. Clearly $u_0 = p$. The exactness conditions at the prime p are that $p, u_1, u_2, \dots$ is a *regular sequence* in the ring E_*. That is, for all $n \geq 0$, multiplication by u_n on the quotient $E_*/(p, u_1, \dots, u_{n-1})$ should be injective. This is required to hold for all primes p.

The first of these conditions says that multiplication by p is injective on E_*, and thus E_* is torsion-free, which is a blanket assumption for all the spectra we consider.

The next condition, at height one, says that multiplication by u_1 on $E_*/(p)$ is injective. That is to say, if p divides $u_1 e$ in E_*, then p divides e. Now $u_1 \equiv c_{p-1}$ modulo p, where c_{p-1} is the coefficient of t^p/p in the log series of the formal group law (see [15, Lemma 2.1]), and we saw in (6.17) that $c_{p-1} \equiv e_{p-1} \mod p$ in E_*. So, given that $L(E)_* = E_*$, this height-one condition is equivalent to the criterion of Theorem 10.9 for the Hattori-Stong theorem to hold. We shall say that E_* (or more generally an E_*-module M_*) satisfies the height-one Landweber exactness condition for all primes if E_* (or M_*) is torsion-free and for each prime p the sequence p, e_{p-1} is regular. We have thus proved

Theorem 11.1. *If E is a complex-oriented ring spectrum with E_* torsion-free, then E_* is a pure subgroup of $K_*(E)$ if and only if E satisfies the height-one Landweber exactness condition for all primes.* □

Corollary 11.2. *If E is a complex-oriented ring spectrum which is Landweber exact, then E_* is a pure subgroup of $K_*(E)$.* □

The following generalisation closely parallels a result of Laures [16, Theorem 1.6] which applies to the case of elliptic cohomology.

Theorem 11.3. *Let E be a complex-oriented ring spectrum and X a space or spectrum such that $E_*(X)$ satisfies the height-one Landweber exactness condition for all primes, then $E_*(X)$ is a pure subgroup of $K_*(E \wedge X)$.*

Proof. Since $E_*(\mathbb{C}P^\infty) = \Pi(E)_*$ is a free E_*-module, there is a Künneth isomorphism $E_*(X \wedge \mathbb{C}P^\infty) \cong E_*(X) \otimes_{E_*} \Pi(E)_*$. Similarly, since Lemma 8.2 shows that $K_*(E) = E_*(K)$ is a flat E_*-module, $K_*(E \wedge X) = E_*(X \wedge K) \cong E_*(X) \otimes_{E_*} K_*(E)$. Moreover the isomorphism of Theorem 9.1 is compatible with these isomorphisms so that $K_*(E \wedge X) \cong E_*(X) \otimes_{E_*} \Pi(E)_*[x^{-1}]$. Now apply the arguments used in the proofs of Proposition 10.8 and Theorem 10.9. □

It is striking that for these results only the first two of Landweber's criteria are needed. It is tempting to suspect that using the higher conditions one might prove that E_* is a pure subgroup of $F_*(E)$, where *both* E and F are Landweber exact theories. In the absence of any analogue of Theorem 9.1, or indeed of any space to play the role that $\mathbb{C}P^\infty$ plays for K-theory, it is difficult to see how to generalise our proofs.

Of course, if $E_*(X)$ satisfies the height-one exactness condition, then so does E_*; the converse is true if $E_*(X)$ is free over E_*. This follows in turn if X is a finite complex and $H_*(X)$ is a free $\mathbb{Z}$-module. Smith [27] states the classical Hattori-Stong theorem, for $E = MU$, in this last form.

Suppose that E satisfies the height-one exactness condition, and M_* is a flat E_*-module, then tensoring the exact sequences

$$0 \to E_* \xrightarrow{p} E_* \qquad \text{and} \qquad 0 \to E_*/(p) \xrightarrow{e_{p-1}} E_*/(p)$$

with M_*, it follows that M_* satisfies the height-one exactness condition. In particular, if E is Landweber exact, then Lemma 8.2 implies that $E_*(E)$ satisfies the height-one exactness condition. Hence $E_*(E)$ is a pure subgroup of $K_*(E \wedge E)$, More generally a similar result will hold for $E_*(E \wedge E \wedge \cdots \wedge E)$. These remarks follow closely the case of elliptic cohomology considered in Theorem 2.10 of [16].

Though Theorem 11.1 gives the complete picture, in some cases the following results provide a simple way to verify that the Hattori-Stong theorem holds.

Proposition 11.4. *Suppose that the complex orientation $MU \to E$ extends via a map $E \to K$ to an orientation of K, and E_* has unique integer factorisation, then the Hattori-Stong theorem holds for E-theory.*

Proof. The K-theory orientation which factors through E may not be the standard orientation. But together the two orientations determine a double delta operator over K_*. For the standard orientation, for each prime p, the sequence p, u^{p-1} is certainly regular in $K_* = \mathbb{Z}[u, u^{-1}]$, so the Hattori-Stong theorem holds. It follows that the Hattori-Stong theorem holds for the delta operator determined by the other orientation. Since no primes are invertible in K_*, none can be in E_*, so Proposition 10.15 applies. □

For p-local spectra there is a corresponding result. Let G denote a ring spectrum which is a summand of p-local K-theory.

Proposition 11.5. *Suppose that E is a complex-oriented ring spectrum for which E_* is a $\mathbb{Z}_{(p)}$-module. If the complex orientation $MU \to E$ extends via a map $E \to G$ to an orientation of G, and E_* has unique integer factorisation, then the Hattori-Stong theorem holds for E-theory.*

Proof. A suitable orientation for G gives rise to the delta operator $A \otimes \mathbb{Z}_{(p)}$, where A is the Artin-Hasse operator of Example 3.11. Proposition 10.16 shows that the Hattori-Stong theorem holds for G-theory. The remainder of the proof follows that of Proposition 11.4. □

It is clearly possible to state results which are intermediate between Propositions 11.4 and 11.5, for example for theories in which 2 is invertible and which map into $KO[\frac{1}{2}]$.

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