

# DOUBLE COBORDISM, FLAG MANIFOLDS AND QUANTUM DOUBLES

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ABSTRACT. Drinfeld's construction of quantum doubles is one of several recent advances in the theory of Hopf algebras (and their actions on rings) which may be attractively presented within the framework of complex cobordism; these developments were pioneered by S P Novikov and the first author. Here we extend their programme by discussing the geometric and homotopy theoretical interpretations of the quantum double of the Landweber-Novikov algebra, as represented by a subalgebra of operations in double complex cobordism. We base our study on certain families of bounded flag manifolds with double complex structure, originally introduced into cobordism theory by the second author. We give background information on double complex cobordism, and discuss the cell structure of the flag manifolds by analogy with the classic Schubert decomposition, allowing us to describe their complex oriented cohomological properties (already implicit in the Schubert calculus of Bressler and Evens). This yields a geometrical realization of the basic algebraic structures of the dual of the Landweber-Novikov algebra, as well as its quantum double. We work in the context of Boardman's eightfold way, which clarifies the relationship between the quantum double and the standard machinery of Hopf algebroids of homology cooperations.

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## 1. INTRODUCTION

In his pioneering work [9], Drinfeld introduced the quantum double construction  $D(H)$  for a Hopf algebra  $H$ . The construction was an immediate source of interest, and Novikov proved in [19] that when  $H$  is cocommutative then  $D(H)$  may be expressed as the smash product (in the sense of [27]) of  $H$  with its dual. Novikov further observed that when  $H$  acts appropriately on any ring  $R$ , then the smash product  $RH$  may be represented as a ring of operators on  $R$ , and he therefore referred to  $RH$  as the *operator double* (or *O-double*), a convention we shall follow here. In consequence, when  $H$  is cocommutative then  $D(H)$  becomes an operator double, given by the adjoint action of  $H$  on its dual. These aspects of Hopf algebra theory are currently under intensive study from a variety of angles, and we refer readers to Montgomery's book [17] for a summary of background material and a detailed survey of the state of the art.

Novikov was actually motivated by an important example from algebraic topology, in which the algebra of cohomology operations in complex cobordism theory may be constructed as an operator double by choosing  $H$  to be the Landweber-Novikov algebra  $S^*$ , and  $R$  the complex cobordism ring  $\Omega_*^U$ . This viewpoint was in turn suggested by the description of  $S^*$ , due to the first author and Shokurov [6], as an algebra of differential operators on a certain algebraic group.

Since the Landweber-Novikov algebra is cocommutative, its quantum double is also an operator double, and the first author has used this property in [8] to prove the remarkable fact that  $D(S^*)$  may be faithfully represented as a ring of operations in an extended version of complex cobordism, known as *double complex cobordism theory*. In this sense, the algebraic and geometric doubling procedures coincide. We shall therefore focus our attention on  $D(S^*)$ , and refer readers to [1], [26] and [28] for comprehensive coverage of basic information in algebraic topology. So far as we are aware, double cobordism theories first appeared in the second author's thesis [23] and in the associated work [25], where the double  $SU$ -cobordism ring was computed.

The above developments are especially appropriate in view of the history of complex cobordism theory. It gained prominence in the context of stable homotopy theory during the late 1960s, but was superseded in the 1970s by Brown-Peterson cohomology because of the computational advantages gained by working with a single prime at a time. Ravenel's book [22] gives an exhaustive account of these events. Work such as [13] has recently led to a resurgence of interest; this has been fuelled by mathematical physics, which was, of course, the driving force behind Drinfeld's original study of quantum groups.

Our principle aim in this work is to give detailed geometrical realizations of the dual and the quantum double of the Landweber-Novikov algebra incorporating the homotopy theory required for a full description of double complex cobordism. The appropriate framework is provided by a family of bounded flag manifolds with double  $U$ -structure. These manifolds were originally constructed by Bott and Samelson in [4] (without reference to flags or  $U$ -structures), but were introduced into complex cobordism theory by the second author in [24]. We therefore allocate considerable space to

a discussion of their algebraic topology, especially with respect to complex oriented cohomology theories (of which complex cobordism is the universal example). Part of this study is implicit in work of Bressler and Evens [5], and we relate our treatment to their generalized Schubert calculus. An exciting program to realize general Bott-Samelson varieties as flag manifolds with combinatorial restrictions is currently under development by Magyar [15], and we look forward to placing our work in his context as soon as possible.

We now describe the contents of each section.

In §2 we give an introduction to double complex cobordism from both the manifold and homotopy theoretic points of view, since the foundations of the construction seem never to have been properly documented. We follow the lead of [8] by writing unreduced bordism functors as  $\Omega_*(\ )$  when emphasizing their geometric origins; if these are of secondary importance, we revert to the notation  $T_*(\ )$  for the corresponding reduced homology theory, where  $T$  is the appropriate Thom spectrum.

Double complex cobordism  $DU^*(\ )$  is the universal example of a cohomology theory  $D^*(\ )$  equipped with two complex orientations, and we discuss this fundamental property in §3, paying particular attention to the consequences for the  $D$ -homology and  $D$ -cohomology of complex projective spaces, Grassmannians, and Thom complexes. These deliberations allow us to introduce the subalgebra  $G_*$  of the double complex cobordism ring  $\Omega_*^{DU}$ , and lay the foundations for our subsequent computations with cohomology operations and flag manifolds.

In §4 we introduce the Landweber-Novikov algebra  $S^*$  as a Hopf subalgebra of the algebra  $A_{MU}^*$  of all complex cobordism operations, and discuss the identification of the dual of each (over  $\mathbb{Z}$  and  $\Omega_*^U$  respectively) with  $G_*$  and  $\Omega_*^{DU}$ . Following Novikov, we describe  $A_{MU}^*$  as the operator double of  $\Omega_*^U$  and  $S^*$ . We then define the algebra  $A_{DU}^*$  of operations in double complex cobordism by analogy, and explain the appearance of a subalgebra isomorphic to the quantum double  $D(S^*)$ . We couch our exposition in terms of Boardman's eightfold way [2], which we believe to be the most comprehensive framework for the multitude of actions and coactions which arise.

We define our family of bounded flag manifolds  $B(Z_{n+1})$  in §5, and study their geometry and topology. We describe a poset of subvarieties  $X_Q$  which serve to desingularize their cell structures, and which are closely related to the Schubert calculus of [5]. We also introduce the basic  $U$ - and double  $U$ -structures on the  $X_Q$  which lie at the heart of our subsequent calculations, and lead to a geometrical realization of  $G_*$ .

We apply this material in §6 by computing the  $E$ -homology and cohomology of the  $X_Q$  for any complex oriented spectrum  $E$ ; the methods readily extend to doubly complex oriented spectra  $D$ , and so enable us to specify the  $DU$ -theory normal characteristic numbers. We interpret the results in terms of our calculus of subvarieties, deducing that  $G_*$  is closed under the action of the operator subalgebra  $S^* \otimes S^*$ . This leads to our description of many of the algebraic structures in  $S^*$  and  $S_*$ , including the commutation law for  $D(S^*)$  considered as an operator double.

We shall use the following notation and conventions in later sections, and without further comment.

We systematically confuse a complex vector bundle  $\rho$  with its classifying map into the appropriate Grassmannian, and write  $\mathbb{C}^m$  for the trivial complex  $m$ -plane bundle over any space  $X$ . We denote the universal complex  $m$ -plane bundle over  $BU(m)$  by  $\xi(m)$ , so that  $\xi(1)$  is the Hopf line bundle over complex projective space  $CP^\infty$ ; we often abbreviate  $\xi(1)$  to  $\xi$ , especially over a finite skeleton  $CP^m$ . If  $\rho$  is a complex  $m$ -plane bundle whose base is a finite CW complex, we let  $\rho^\perp$  denote the complementary  $(p-m)$ -plane bundle in some suitably high dimensional trivial bundle  $\mathbb{C}^p$ .

We write  $\cdot$  for the space consisting of a single point, and  $X_+$  for its disjoint union with an arbitrary space  $X$ .

The Hopf algebras we use are intrinsically geometrical and naturally graded by dimension, as are ground rings such as  $\Omega_*^U$ . Sometimes our algebras are not of finite type, and must therefore be topologized when forming duals and tensor products; this has little practical effect, but is fully explained in [3], for example. Duals are invariably taken in the graded sense and we adapt our notation accordingly. Thus we write  $A_{MU}^*$  for the algebra of complex cobordism operations, and  $A_*^{MU}$  for its continuous dual  $\text{Hom}_{\Omega_*^U}(A_{MU}^*, \Omega_*^U)$ , which in turn forces us to write  $S^*$  for the graded Landweber-Novikov algebra, and  $S_*$  for its dual  $\text{Hom}_{\mathbb{Z}}(S^*, \mathbb{Z})$ ; neither of these notations is entirely standard.

Several of our algebras are polynomial in variables such as  $b_k$  of grading  $2k$ , where  $b_0$  is the identity. An additive basis is therefore given by monomials of the form  $b_1^{\omega_1} b_2^{\omega_2} \dots b_n^{\omega_n}$ , which we denote by  $b^\omega$ , where  $\omega$  is the sequence of nonnegative, eventually zero integers  $(\omega_1, \omega_2, \dots, \omega_n, 0 \dots)$ . The set of all such sequences forms an additive semigroup, and  $b^\psi b^\omega = b^{\psi+\omega}$ . Given any  $\omega$ , we write  $|\omega|$  for  $2 \sum i\omega_i$ , which is the grading of  $b^\omega$ . We distinguish the sequences  $\epsilon(k)$ , which have a single nonzero element 1 and are defined by  $b^{\epsilon(k)} = b_k$  for each integer  $k \geq 1$ . It is often convenient to abbreviate the formal sum  $\sum_{k \geq 0} b_k$  to  $b$ , in which case we write  $(b)_k^n$  for the component of the  $n$ th power of  $b$  in grading  $2k$ ; negative values of  $n$  are permissible.

When dualizing, we choose dual basis elements of the form  $c_\omega$ , defined by  $\langle c_\omega, b^\psi \rangle = \delta_{\omega, \psi}$ ; this notation is designed to be consistent with our convention on gradings, and to emphasize that the elements  $c_\omega$  are not necessarily monomials themselves.

Unless otherwise indicated, tensor products are taken over  $\mathbb{Z}$ .

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## 2. DOUBLE COMPLEX COBORDISM

In this section we outline the theory of double complex cobordism, considering both the manifold and homotopy theoretic viewpoints.

Like all cobordism theories, double complex cobordism is based on a class of manifolds whose stable normal bundle admits a specific structure. Once

this structure is made precise, then the standard procedures described by Stong [26] may be invoked to construct the bordism and cobordism functors on a suitable category of topological spaces, and to describe them from the homotopy theoretic viewpoint in terms of the corresponding Thom spectrum. Nevertheless, in view of the fact that more general indexing sets are required than those considered by Stong, we explain some of the details.

Philosophically, double complex cobordism theory is based on manifolds  $M$  whose stable normal bundle  $\nu^M$  (abbreviated to  $\nu$  whenever the context allows) possesses a specific splitting  $\nu \cong \nu_\ell \oplus \nu_r$  into two complex bundles, which we often label the *left* and *right* components.

More precisely, given any positive integers  $m$  and  $n$  let us write  $U(m, n)$  for the product of unitary groups  $U(m) \times U(n)$ , so that the classifying space  $BU(m, n)$  may be canonically identified with  $BU(m) \times BU(n)$ . Thus  $BU(m, n)$  carries the complex  $(m + n)$ -plane bundle  $\xi(m, n)$ , defined as  $\xi(m) \times \xi(n)$  and classified by the Whitney sum map  $BU(m, n) \rightarrow BU(m+n)$ . The standard inclusion of  $U(m)$  in the orthogonal group  $O(2m)$  induces a map of classifying spaces  $f_{m,n}: BU(m, n) \rightarrow BO(2(m+n))$ , and the standard inclusion of  $U(m)$  in  $U(m+1)$  induces a map of classifying spaces  $g_{m,n}: BU(m, n) \rightarrow BU(m+1, n+1)$ . These maps constitute a doubly indexed version of a  $(B, f)$  structure in the sense of Stong, although care is required to ensure that they are sufficiently compatible over both  $m$  and  $n$ . There are also product maps

$$(2.1) \quad BU(m, n) \times BU(p, q) \longrightarrow BU(m+p, n+q)$$

induced by Whitney sum, whose compatibility is more subtle, but confirms that the corresponding  $(B, f)$  cobordism theory is multiplicative; this is our *double complex cobordism* theory, referred to in [23] as  $U \times U$  theory.

We therefore define a *double U-structure* on  $M$  to consist of an equivalence class of lifts of  $\nu$  to  $BU(m, n)$ , for some values of  $m$  and  $n$  which are suitably large compared with the dimension of  $M$ . This class of lifts provides the isomorphism  $\nu \cong \nu_\ell \oplus \nu_r$ , where  $\nu_\ell$  and  $\nu_r$  are classified by the left and right projections onto the respective factors  $BU(m)$  and  $BU(n)$ . If we wish to record a particular choice of  $m$  and  $n$ , we may refer to the resulting  $U(m, n)$ -*structure*. Given such a structure on  $M$ , it is convenient to write  $\chi(M)$  for  $M$  invested with the  $U(n, m)$ -structure induced by the obvious switch map  $BU(m, n) \rightarrow BU(n, m)$ ; we emphasise that  $M$  and  $\chi(M)$  are, in general, distinct. Any manifold with a  $U(m, n)$ -structure has a  $U(m+n)$ -structure, obtained by forgetting the splitting. If  $M$  has a  $U(m, n)$ -structure and  $N$  has a  $U(p, q)$ -structure, then the product  $U(m+p, n+q)$  structure on  $M \times N$  is given by choosing  $\nu_\ell^{M \times N}$  and  $\nu_r^{M \times N}$  to be  $\nu_\ell^M \times \nu_\ell^N$  and  $\nu_r^M \times \nu_r^N$  respectively.

A typical example, of which we shall use analogues in Theorem 6.8 and its applications, is provided by complex projective space  $CP^{n-1}$ , whose stable normal bundle is isomorphic to  $-n\xi$ . If we select  $\nu_\ell$  and  $\nu_r$  to be  $-k\xi$  and  $(k-n)\xi$  respectively, we obtain infinitely many distinct double  $U$ -structures. Also, if  $M$  and  $N$  admit  $U$ -structures, then  $M \times N$  admits the product double  $U$ -structure.

We may choose to impose all our structures on the stable tangent bundle  $\tau^M$  of  $M$ , so long as we observe the usual caution in choosing a canonical trivialization of  $\nu \oplus \tau$ . To avoid this issue and be consistent with the

homotopy theoretic approach, we prefer to use normal structures wherever possible.

The compatibility required of the maps (2.1) is most readily expressed in the language of May's coordinate-free functors (as described, for example, in [10]), which relies on an initial choice of infinite dimensional inner product space  $Z_\infty$ , known as a *universe*. We may here assume that  $Z_\infty$  is complex. This language was originally developed to prove that the multiplicative structure of complex cobordism is highly homotopy coherent [16], and its usage establishes that the same is true for double complex cobordism so long as we consistently embed our double  $U$ -manifolds in finite dimensional subspaces  $V \oplus W$  of the universe  $Z_\infty \oplus Z_\infty$ . We define the classifying space  $B(V, W)$  by appropriately topologizing the set of all subspaces of  $V \oplus W$  which are similarly split. If  $V$  and  $W$  are spanned respectively by (necessarily disjoint)  $m$  and  $n$  element subsets of some predetermined orthonormal basis for  $Z_\infty \oplus Z_\infty$ , we refer to them as *coordinatized*, and write the classifying space as  $BU(m, n)$  to conform with our earlier notation. We then interpret (2.1) as a coordinatized version of the Whitney sum map, on the understanding that the subspaces of dimension  $m$  and  $p$  are orthogonal in  $Z_\infty$ , as are those of dimension  $n$  and  $q$ . The Grassmannian geometry of the universe immediately guarantees the required compatibility.

In our work below, we may safely confine such considerations to occasional remarks, although they are especially pertinent when we define the corresponding Thom spectrum and its multiplicative properties.

The double complex cobordism ring  $\Omega_*^{DU}$  consists of cobordism classes of double  $U$ -manifolds, with the product induced as above; as we shall see, it is exceedingly rich algebraically. The double complex bordism functor  $\Omega_*^{DU}(\ )$  is an unreduced homology theory, defined on an arbitrary topological space  $X$  by means of bordism classes of maps into  $X$  of manifolds with the appropriate structure; it admits a canonical involution (also denoted by  $\chi$ ), induced by switching the factors of the normal bundle. Thus  $\Omega_*^{DU}(X)$  is always a module over  $\Omega_*^{DU}$ , which in this context becomes identified with  $\Omega_*^{DU}(\cdot)$ , and is known as the *coefficient ring* of the theory. Moreover, the product structure ensures that  $\Omega_*^{DU}(X)$  is both a left and a right  $\Omega_*^U$ -module.

Double complex cobordism  $\Omega_{DU}^*(\ )$  is the dual cohomology functor, which we define geometrically using Quillen's techniques [21]. For any double  $U$ -manifold  $X$ , a cobordism class in  $\Omega_{DU}^*(X)$  is represented by an equivalence class of compositions

$$M \xrightarrow{i} E_\ell \oplus E_r \xrightarrow{\pi} X,$$

where  $\pi$  is the projection of a complex vector bundle split into left and right components, and  $i$  is an embedding of a double  $U$ -manifold  $M$  whose normal bundle is split compatibly.

If we ignore the given splitting of each normal bundle (and simultaneously identify  $Z_\infty \oplus Z_\infty$  isometrically with  $Z_\infty$ ), we obtain a forgetful homomorphism  $\pi: \Omega_*^{DU}(X) \rightarrow \Omega_*^U(X)$  for any space  $X$ . Conversely, if we interpret a given  $U(m)$ -structure as either a  $U(m) \times U(0)$ -structure or a  $U(0) \times U(m)$ -structure, we obtain left and right inclusions  $\iota_\ell$  and  $\iota_r: \Omega_*^U(X) \rightarrow \Omega_*^{DU}(X)$ , which are interchanged by  $\chi$ . We note that  $\pi$  is an epimorphism, and that

$\iota_\ell$  and  $\iota_r$  are monomorphisms because both  $\pi \circ \iota_\ell$  and  $\pi \circ \iota_r$  are the identity. We shall be especially interested in the action of these homomorphisms on the coefficient rings. We note further that, for any space  $X$ , the standard product structure in  $\Omega_U^*(X)$  factorizes as

$$(2.2) \quad \begin{aligned} \Omega_U^*(X) \otimes \Omega_U^*(X) &\xrightarrow{\iota_\ell \otimes \iota_r} \Omega_{DU}^*(X) \otimes \Omega_{DU}^*(X) \\ &\longrightarrow \Omega_{DU}^*(X) \xrightarrow{\pi} \Omega_U^*(X), \end{aligned}$$

where the central homomorphism is the product in  $\Omega_{DU}^*(X)$ .

The homotopy theoretic viewpoint of these functors is based on the corresponding Thom spectrum  $DU$ , to which we now turn.

In order to allow spectra which consist of a doubly indexed direct system (rather than the more traditional sequence) of spaces and maps, as well as to ensure that such spectra admit a product which is highly homotopy coherent, it is most elegant to return to the coordinate-free setting. We define the Thom space  $M(V, W)$  by the standard construction on  $B(V, W)$ , and again allow the Grassmannian geometry of the universe to provide the necessary compatibility for both the structure and the product maps. As in (2.1) we give explicit formulae only for coordinatized subspaces.

We write  $MU(m, n)$  for the Thom complex of the bundle  $\xi(m, n)$ , which may of course be canonically identified with  $MU(m) \wedge MU(n)$ . Then the coordinatized structure maps take the form

$$(2.3) \quad S^{2(p+q)} \wedge MU(m, n) \longrightarrow MU(m+p, n+q),$$

given by Thom complexifying the classifying maps of  $(\mathbb{C}^p \times \mathbb{C}^q) \oplus \xi(m, n)$ . Henceforth we take this direct system as our definition of the  $DU$  spectrum, noting that the Thom complexifications of the maps (2.1) provide a product map  $\mu^{DU}$ , which is highly coherent, and equipped with a unit by (2.3) in the case  $m = n = 0$ . It is a left and right module spectrum over  $MU$  by virtue of the systems of maps

$$\begin{aligned} MU(p) \wedge MU(m, n) &\rightarrow MU(m+p, n) \quad \text{and} \\ MU(m, n) \wedge MU(q) &\rightarrow MU(m, n+q), \end{aligned}$$

which are also highly coherent by appeal to the coordinate-free setting.

This setting also enables us to define smash products of spectra [10], and therefore to write  $DU$  as  $MU \wedge MU$ . The involution  $\chi$  is then induced by interchanging factors, and we may represent the bimodule structure by maps  $MU \wedge DU \rightarrow DU$  and  $DU \wedge MU \rightarrow DU$ , induced by applying the  $MU$  product  $\mu^{MU}$  on the left and right copies of  $MU \wedge MU$  respectively.

We define the reduced bordism and cobordism functors on a topological space  $X$  by means of

$$(2.4) \quad \begin{aligned} DU_k(X) &= \varinjlim_{m,n} \pi_{2(n+m)+k}(MU(m, n) \wedge X) \\ \text{and } DU^k(X) &= \varinjlim_{m,n} \{S^{2(n+m)-k} \wedge X, MU(m, n)\}, \end{aligned}$$

where the brackets  $\{ \}$  denote based homotopy classes of maps. The graded groups  $DU_*(X)$  and  $DU^*(X)$  consist of the appropriate direct sums over  $k$ . These definitions exhibit  $DU^*(X)$  as a commutative graded ring, by virtue of the product structure on  $DU$ . The standard complex bordism and

cobordism functors are defined similarly, employing the spectrum  $MU$  in place of  $DU$ .

We abbreviate  $\varinjlim_{m,n} \pi_{2(n+m)+k}(MU(m,n)) \wedge X$  to  $\pi_k(DU \wedge X)$  and  $\varinjlim_{m,n} \{S^{2(n+m)-k} \wedge X, MU(m,n)\}$  to  $[S^{-k}X, DU]$  respectively; this is standard notation for stabilized groups of homotopy classes. Thus the coefficient ring  $DU_*(S^0)$ , written  $DU_*$  for convenience, is simply the homotopy ring  $\pi_*(DU)$  of  $DU$ . In similar vein, given a second homology theory  $E_*(\ )$  we may introduce the  $E$  homology and cohomology groups of  $DU$  by means of

$$(2.5) \quad \begin{aligned} E_k(DU) &= \varinjlim_{m,n} E_{2(n+m)+k}(MU(m,n)) \\ \text{and } E^k(DU) &= \varprojlim_{m,n} E^{2(n+m)+k}(MU(m,n)) \end{aligned}$$

for all integers  $k$ .

Following usual practice, we define the unreduced bordism and cobordism functors of  $X$  to be  $DU_k(X_+)$  and  $DU^k(X_+)$  respectively; the reduced and the unreduced theories differ only by a copy of the coefficient ring, according to the equations

$$DU_k(X_+) = DU_k(X) \oplus DU_k \quad \text{and} \quad DU^k(X_+) = DU^k(X) \oplus DU_{-k},$$

which arise by considering  $X_+$  as the one point union  $X \vee S^0$ . In this context, we often write 1 for the element in  $DU_*(X_+)$  or  $DU^*(X_+)$  which corresponds to the appropriate generator of  $DU_0$ .

The Whitney sum map  $BU(m,n) \rightarrow BU(m+n)$  induces a forgetful map of ring spectra  $\pi: DU \rightarrow MU$ , whilst the respective inclusions of  $BU(m)$  and  $BU(n)$  in  $BU(m+n)$  induce the inclusions  $\iota_\ell$  and  $\iota_r: MU \rightarrow DU$ . All three maps may be extended to the coordinate-free setting, and both  $\iota_\ell$  and  $\iota_r$  yield the identity map after composition with  $\pi$ . Moreover, the  $MU$  product  $\mu^{MU}$  factorizes as

$$\begin{aligned} MU(m) \wedge S^{2n} \wedge S^{2p} \wedge MU(q) &\xrightarrow{\iota_\ell \wedge \iota_r} MU(m,n) \wedge MU(p,q) \\ &\xrightarrow{\mu^{DU}} MU(m+n,p+q) \xrightarrow{\pi} MU(m+n+p+q), \end{aligned}$$

in concert with (2.2). We deduce that  $\iota_\ell$ ,  $\iota_r$  and  $\pi$  all define multiplicative transformations between the appropriate functors, and that  $\iota_\ell$  and  $\iota_r$  are interchanged by  $\chi$ . Given an element  $\theta$  of  $MU^*(X)$  or  $MU_*(X)$ , we shall often write  $\iota_\ell(\theta)$  and  $\iota_r(\theta)$  as  $\theta_\ell$  and  $\theta_r$  respectively.

By way of example, we may combine (2.4) and (2.5) to obtain

$$(2.6) \quad DU_* \cong \varinjlim_m \pi_{2m+*}(MU(m) \wedge MU) = MU_*(MU);$$

this follows at once if we write  $DU$  as  $MU \wedge MU$ . In fact  $MU_*(MU)$  is the Hopf algebroid of cooperations in  $MU$  homology theory, and we shall discuss it in considerable detail in §4 below. Suffice it to say here that its associated homological algebra has been extensively studied in connection with the Adams-Novikov spectral sequence and the stable homotopy groups of spheres. For detailed calculations, however, it has proven more efficient to concentrate on a single prime  $p$  at a time, and work with the  $p$ -local summand  $BP_*(BP)$  given by Brown-Peterson homology [22].

There is a natural isomorphism between the manifold and the homotopy theoretic versions of any bordism functor. This stems from the Pontryagin-Thom construction, which we may summarize in the case of the coefficient ring for double complex cobordism as follows. Consider any manifold  $M^k$  embedded in  $S^{k+2(m+n)}$  with a  $U(m, n)$ -structure, and collapse to  $\infty$  the complement of a normal neighbourhood. Composing with the Thom complexification of the classifying map for  $\nu$  yields a map from  $S^{k+2(m+n)}$  to  $MU(m, n)$ , and thence a homomorphism  $\Omega_k^{DU} \rightarrow \pi_k(DU)$ . This defines the promised isomorphism  $\Omega_*^{DU} \cong DU_*$ , although the verification that it has the necessary algebraic properties requires considerable work, and depends upon Thom's transversality theorems. We note that the isomorphism maps the geometric involution given by interchanging the factors of the normal bundle to the homotopy theoretic involution given by interchanging the factors of the  $DU$  spectrum, and that it may, with further care, be naturally extended to the coordinate-free setting. Henceforth, we shall pass regularly between the manifold and homotopy theoretic viewpoints, assuming the Pontryagin-Thom construction wherever necessary.

### 3. ORIENTATION CLASSES

In this section we explain how  $DU$  is the universal example of a spectrum admitting two distinct complex orientations, and consider the consequences for the double complex bordism and cobordism groups of some well-known spaces in complex geometry. We first recall certain basic definitions and results, which may be found, for example in [1].

We assume throughout that  $E$  is a commutative ring spectrum, whose zeroth homotopy group  $E_0$  is isomorphic to the integers. Then  $E$  is *complex oriented* if the cohomology group  $E^2(CP^\infty)$  contains an element  $x^E$ , known as the *orientation class*, whose restriction to  $E^2(CP^1)$  is a generator when the latter group is identified with  $E_0$ . Under these circumstances, we may deduce that  $E^*(CP^\infty)$  consists of formal power series over  $E_*$  in the variable  $x^E$ , whose powers define dual basis elements  $\beta_k^E$  in  $E_{2k}(CP^\infty)$ . If we continue to write  $\beta_k^E$  for their image under the inclusion of  $BU(1)$  in  $BU(m)$  (for any value of  $m$ , including  $\infty$ ), then  $E_*(BU(m))$  is the free  $E_*$ -module generated by commutative monomials of length at most  $m$  in the elements  $\beta_k^E$ . For  $1 \leq k \leq m$ , the duals of the powers of  $\beta_1^E$  define the *Chern classes*  $c_k^E$  in  $E^{2k}(BU(m))$ , which generate  $E^*(BU(m))$  as a power series algebra over  $E_*$ ; clearly  $c_1^E$  agrees with  $x^E$  over  $CP^\infty$ . When we pass to the direct limit over  $m$ , we obtain

$$(3.1) \quad E_*(BU) \cong E_*[\beta_k^E : k \geq 0] \quad \text{and} \quad E^*(BU) \cong E_*[[c_k^E : k \geq 0]],$$

where the Pontryagin product in homology is induced by Whitney sum. As described in the introduction, we write monomial basis elements in the  $\beta_k^E$  as  $(\beta^E)^\omega$  for any sequence  $\omega$ , and their duals as  $c_\omega^E$ . Thus  $c_{(k)}^E$  and  $c_k^E$  coincide.

Considering  $BU(m-1)$  as a subspace of  $BU(m)$ , we may express the Thom complex  $MU(m)$  of  $\xi(m)$  as the quotient space  $BU(m)/BU(m-1)$ , at least up to homotopy equivalence. Thus  $E_*(MU(m))$  and  $E^*(MU(m))$  may be computed from the cofiber sequence

$$(3.2) \quad BU(m-1) \longrightarrow BU(m) \longrightarrow MU(m)$$

by applying  $E$  homology and cohomology respectively; the resulting two sequences of  $E_*$ -modules are short exact. We may best express the consequences in terms of Thom isomorphisms, for which we first identify the pullback of  $c_m^E$  in  $E^{2m}(MU(m))$  as the *Thom class*  $t^E(m)$  of  $\xi(m)$ , observing that its restriction over the base point is a generator of  $E^{2m}(S^{2m})$  when the latter is identified with  $E_0$ . Indeed, this property reduces to the defining property for  $x^E$  when  $m$  is 1 (thereby identifying  $x^E$  as the Thom class of  $\xi(1)$ ), and the same generator arises for all values of  $m$ . It follows from the definitions that  $t^E(m+n)$  pulls back to the external product  $t^E(m)t^E(n)$  in  $E^{2(m+n)}(MU(m) \wedge MU(n))$  under  $\mu^{MU}$ . The homomorphisms

$$(3.3) \quad \begin{aligned} \phi_*: E_{k+2m}(MU(m)) &\rightarrow E_k(BU(m)_+) \\ \text{and } \phi^*: E^k(BU(m)_+) &\rightarrow E^{k+2m}(MU(m)), \end{aligned}$$

determined by the relative cap and cup products with  $t^E(m)$ , are readily seen to be isomorphisms of  $E_*$ -modules for all integers  $k \geq 0$ ; they are known as the  $E$  theory *Thom isomorphisms* for  $\xi(m)$ . We define elements  $b_k^E$  in  $E_{2(k+m)}(MU(m))$  as  $\phi_*^{-1}(\beta_k)$ , and elements  $s_k^E$  in  $E^{2(k+m)}(MU(m))$  as  $\phi^*(c_k^E)$ ; each of these families extends to a set of generators over  $E_*$  in the appropriate sense.

We may stabilize the Thom isomorphisms by allowing  $m$  to become infinite, in which case (3.1) yields the descriptions

$$(3.4) \quad E_*(MU) \cong E_*[b_k^E : k \geq 0] \quad \text{and} \quad E^*(MU) \cong E_*[[s_k^E : k \geq 0]],$$

where  $b_k^E$  lies in  $E_{2k}(MU)$  and  $s_k^E$  in  $E^{2k}(MU)$ , for all  $k \geq 0$ . We emphasise that the multiplicative structure in homology is induced by  $\mu^{MU}$ , but that in cohomology it exists only as an algebraic consequence of  $\phi^*$ , and is not given by any cup product. We continue to write monomial basis elements in the  $b_k^E$  as  $(b^E)^\omega$ , and their duals as  $s_\omega^E$ , for any sequence  $\omega$ . Again,  $s_{(k)}^E$  and  $s_k^E$  coincide. We write  $t^E$  in  $E^0(MU)$  for the stable Thom class, which corresponds to the element 1 in the description (3.4), and is represented by a multiplicative map of ring spectra.

We have therefore described a procedure for constructing  $t^E$  from our initial choice of  $x^E$ ; in fact this provides a bijection between complex orientation classes in  $E$  and multiplicative maps  $MU \rightarrow E$ .

When  $m$  is 1, the cofiber sequence (3.2) reduces to the homotopy equivalence  $CP^\infty \rightarrow MU(1)$ , and  $\beta_k^E$  is identified with  $b_{k-1}^E$  under the map induced in  $E$  homology. The Thom isomorphisms satisfy  $\phi_*(b_k^E) = \beta_{k-1}^E$  and  $\phi^*((x^E)^{k-1}) = (x^E)^k$  respectively, for all  $k \geq 1$ .

Any complex  $m$ -plane bundle  $\rho$  over a space  $X$  has a Thom class  $t^E(\rho)$  in  $E^{2m}(M(\rho))$ , obtained by pulling back the universal example  $t_m^E$  along the classifying map for  $\rho$ . We may use this Thom class exactly as in (3.3) to define Thom isomorphisms

$$\phi_*: E_{k+2m}(M(\rho)) \rightarrow E_k(X_+) \quad \text{and} \quad \phi^*: E^k(X_+) \rightarrow E^{k+2m}(M(\rho)).$$

If  $\rho$  is a virtual bundle its Thom space is stable, and so long as we insist that its bottom cell has dimension zero, we acquire Thom isomorphisms in the format of (3.4).

We remark that  $MU$  is itself complex oriented if we choose  $x^{MU}$  to be represented by the homotopy equivalence  $CP^\infty \rightarrow MU(1)$ ; we shall abbreviate this class to  $x$ . The resulting Thom class  $t$  is represented by the identity map on  $MU$ . In fact  $MU$  is the universal example, since any Thom class  $t^E$  induces a homomorphism  $MU_* \rightarrow E_*$  in homotopy, which extends to the unique homomorphism  $MU^*(CP^\infty) \rightarrow E^*(CP^\infty)$  satisfying  $x^{MU} \mapsto x^E$ . In view of these properties, we shall dispense with the superscript  $MU$  in the universal case wherever possible. So far as Quillen's geometrical interpretation of cobordism is concerned, a Thom class  $t(\rho)$  in  $\Omega_U^{2m}(M(\rho))$  is represented by the inclusion of the zero section  $M \subset M(\rho)$ , whenever  $\rho$  lies over a  $U$ -manifold  $M$ .

We combine (2.6) with the Thom isomorphism  $MU_*(MU) \cong MU_*(BU_+)$  to obtain a left  $MU_*$ -isomorphism  $h: DU_* \cong MU_*(BU_+)$ , which has an important geometrical interpretation.

**Proposition 3.5.** *Suppose that an element of  $\Omega_*^{DU}$  is represented by a manifold  $M^k$  with double  $U$ -structure  $\nu_\ell \oplus \nu_r$ ; then its image under  $h$  is represented by the singular  $U$ -manifold  $\nu_r: M^k \rightarrow BU$ .*

*Proof.* By definition, the image we seek is represented by the composition

$$\begin{aligned} S^{k+2(m+n)} &\rightarrow M(\nu) \rightarrow MU(m) \wedge MU(n) \\ &\xrightarrow{1 \wedge \delta} MU(p) \wedge MU(m) \wedge BU(m)_+ \xrightarrow{\mu \wedge 1} MU(p+m) \wedge BU(m)_+, \end{aligned}$$

where the first map is obtained by applying the Pontryagin-Thom construction to an appropriate embedding  $M^k \subset S^{k+2(m+n)}$ , and the second classifies the double  $U$ -structure on  $M^k$ . We may identify the final three maps as the Thom complexification of the composition

$$\begin{aligned} M^k &\xrightarrow{\nu_\ell \oplus \nu_r} BU(m) \times BU(n) \xrightarrow{1 \times \delta} BU(m) \times BU(n) \times BU(n) \\ &\xrightarrow{\oplus \times 1} BU(m+n) \times BU(n), \end{aligned}$$

which simplifies to  $\nu \oplus \nu_r$ , as sought.  $\square$

**Corollary 3.6.** *Suppose that an element of  $\Omega_*^U(BU_+)$  is represented by a singular  $U$ -manifold  $f: M^k \rightarrow BU(q)$  for suitably large  $q$ ; then its inverse image under  $h$  is represented by the double  $U$ -structure  $(\nu \oplus f^\perp) \oplus f$  on  $M$ .*

We remark that our proof of Proposition 3.5 shows that  $h$  is actually multiplicative, so long as we invest  $MU_*(BU_+)$  with the Pontryagin product which arises from the Whitney sum map on  $BU$ . Moreover,  $h$  conjugates the involution  $\chi$  so as to act on  $MU_*(BU_+)$ , where it interchanges the map  $f$  of Corollary 3.6 with  $\nu \oplus f^\perp$ .

Returning to our complex oriented spectrum  $E$ , we record the fundamental relationship with the theory of formal groups, as introduced by Novikov [18] and developed by Quillen in his celebrated work [20]. It depends on the fact that the Künneth isomorphism identifies  $E^*(CP^\infty \times CP^\infty)$  with the ring of formal power series  $E_*[[x^E, y^E]]$ , where  $x^E$  and  $y^E$  denote the pullbacks of  $x^E$  from projection onto the first and second factors  $CP^\infty$  respectively. Since the first Chern class  $c_1^E(\xi \otimes \xi)$  of the external tensor product lies in

$E^2(CP^\infty \times CP^\infty)$ , we obtain a formal power series expansion

$$(3.7) \quad c_1^E(\xi \otimes \xi) = x^E + y^E + \sum_{i,j \geq 1} a_{i,j}^E (x^E)^i (y^E)^j,$$

where each  $a_{i,j}^E$  lies in the coefficient group  $E_{2(i+j-1)}$ . This formal power series, which we usually denote by  $F^E(X, Y)$ , is the *formal group law* for  $E$ . It is associative, commutative, and 1-dimensional, and admits an inverse induced by complex conjugation on  $CP^\infty$ . The universal example is provided by the spectrum  $MU$ , in which case the coefficients  $MU_*$  form the Lazard ring  $L$ ; it is well-known that  $L$  is generated (as a ring, but with redundancy) by the elements  $a_{i,j}$ . Other examples are given by the integral Eilenberg-MacLane spectrum  $H$ , and the complex  $K$ -theory spectrum  $K$ , for which canonical orientations may be chosen such that

$$F^H(X + Y) = X + Y \quad \text{and} \quad F^K(X + Y) = X + Y + uXY$$

are the *additive* and *multiplicative* formal group laws respectively (where  $u$  is the Bott periodicity element generating the coefficient group  $\pi_2(K)$ ).

So long as  $E_*$  is free of additive torsion, we may construct a formal power series  $\exp^E(X)$  which gives a strict isomorphism between the additive formal group law and  $F^E$ ; this is defined over  $E\mathbb{Q}_* = E_* \otimes \mathbb{Q}$ , and was shown by the first author [7] to be the image of  $x^E$  under the Chern-Dold character (or rationalization map). It is known as the *exponential series* for  $F^E(X, Y)$ , and its defining property may be expressed as

$$\exp^E(X + Y) = F^E(\exp^E(X), \exp^E(Y)).$$

Its substitutional inverse  $\log^E(X)$  is the *logarithm* for  $F^E(X, Y)$ .

We refer readers to Hazewinkel's encyclopaedic book [11] for further details on formal group laws.

It is certainly true that  $DU$  is complex oriented, because we have already defined two multiplicative maps  $\iota_\ell$  and  $\iota_r: MU \rightarrow DU$  which therefore serve as Thom classes  $t_\ell$  and  $t_r$ . The corresponding orientation classes are  $\iota_\ell(x)$  and  $\iota_r(x)$  in  $DU^2(CP^\infty)$ , which we shall denote, as promised, by  $x_\ell$  and  $x_r$  respectively. Thus

$$DU^*(CP_+^\infty) \cong DU_*[[x_\ell]] \cong DU_*[[x_r]],$$

and there are inverse formal power series

$$x_r = \sum_{k \geq 0} g_k x_\ell^{k+1} \quad \text{and} \quad x_\ell = \sum_{k \geq 0} \bar{g}_k x_r^{k+1},$$

written  $g(x_\ell)$  and  $\bar{g}(x_r)$  respectively. Clearly the elements  $g_k$  and  $\bar{g}_k$  lie in  $DU_{2k}$  for all  $k$ , and are interchanged by the involution  $\chi$ . Furthermore,  $g_0 = \bar{g}_0 = 1$  because  $x_\ell$  and  $x_r$  restrict to the same generator of  $DU^2(S^2)$ . The geometrical significance of the  $g_k$  is illustrated by the following property.

**Proposition 3.8.** *Under the isomorphism  $h$ , we have that*

$$h(g_k) = \beta_k$$

*in  $MU_{2k}(BU_+)$ , for all  $k \geq 0$ .*

*Proof.* We may express  $g_k$  as the Kronecker product  $\langle x_r, \beta_{k+1, \ell} \rangle$  in  $DU_{2k}$ , which is represented by the composition

$$S^{2(p+k+1)} \xrightarrow{\beta_{k+1}} MU(p) \wedge CP^\infty \xrightarrow{1 \wedge x} MU(p) \wedge MU(1),$$

for suitably large  $p$ . This stabilizes to  $b_k$  in  $MU_{2k}(MU)$ , and hence to  $\beta_k$  in  $MU_{2k}(BU_+)$ , as required.  $\square$

**Corollary 3.9.** *The subalgebra  $G_*$  of  $DU_*$  generated by the elements  $g_k$  is polynomial over  $\mathbb{Z}$ .*

*Proof.* This result follows from the multiplicativity of  $h$  and the independence of monomials in the  $\beta_k$  over  $\mathbb{Z}$ .  $\square$

We also infer from Proposition 3.8 that  $g_k$  may be realized geometrically by choosing a singular  $U$ -manifold representing  $\beta_k$  in  $MU_{2k}(CP^\infty)$ , and amending its double  $U$ -structure according to Corollary 3.6. Interchanging  $\nu_\ell$  and  $\nu_r$  yields  $\bar{g}_k$ . We make extensive use of the algebra  $G_*$  in later sections.

We refer to any spectrum  $D$  equipped with two complex orientations which restrict to the same element of  $D_0$  as *doubly complex oriented*. Obviously  $E \wedge F$  is such a spectrum whenever  $E$  and  $F$  are complex oriented, but not all examples take this form. Nonetheless, we shall write the two orientation classes as  $x_\ell^D$  and  $x_r^D$  respectively, so that they are related by inverse formal power series

$$(3.10) \quad x_r^D = \sum_{k \geq 0} g_k^D (x_\ell^D)^{k+1} \quad \text{and} \quad x_\ell^D = \sum_{k \geq 0} \bar{g}_k^D (x_r^D)^{k+1}$$

in  $D^2(CP^\infty)$ .

By mimicking the programme laid out above for  $E$ , we may construct *left* and *right* sets of  $D_*$ -generators for  $D_*(CP^\infty)$ ,  $D_*(BU(m))$ ,  $D_*(MU(m))$ , and for their cohomological counterparts. Thus, for example, there are left and right Chern classes  $c_{k, \ell}^D$  and  $c_{k, r}^D$  in  $D^{2k}(BU(m))$  for  $k \leq m$ , and left and right Thom classes  $t_\ell^D(m)$  and  $t_r^D(m)$  in  $D^{2m}(MU(m))$ ; the latter give rise to left and right Thom isomorphisms associated to an arbitrary complex bundle  $\rho$ . There are also left and right formal group laws  $F_\ell^D(X, Y)$  and  $F_r^D(X, Y)$ .

In the light of (3.10), we would expect any left and right constructions of this form to be interrelated by the elements  $g_k^D$ . This is indeed the case, and is further testimony to their importance.

**Lemma 3.11.** *We have that*

(1) *in  $D^0(MU)$ ,*

$$t_r^D = t_\ell^D + \sum_{\omega} (g^D)^\omega s_{\omega, \ell}^D;$$

(2) *the formal power series  $g^D(X)$  provides a strict isomorphism between the formal group laws  $F_\ell^D(X, Y)$  and  $F_r^D(X, Y)$ ;*

(3) *in  $D_{2n}(CP^\infty)$ ,*

$$\beta_{n, \ell}^D = \sum_{k=0}^n (g^D)_{n-k}^k \beta_{k, r}^D$$

*for all  $n \geq 0$ .*

*Proof.* For (1), we dualize (3.10) and pass to  $D_*(BU)$ , then dualize back again to  $D^*(BU)$  and apply the Thom isomorphism.

For (2), we apply (3.10) twice to obtain

$$F_\ell^D(g^D(x_r^D), g^D(y_r^D)) = F_\ell^D(x_\ell^D, y_\ell^D) \quad \text{and} \quad g^D(F_r^D(x_r^D, y_r^D)) = c_{1,\ell}^D(\xi \otimes \xi)$$

in  $D^2(CP^\infty \times CP^\infty)$ . Thus  $F_\ell^D(g^D(x_r^D), g^D(y_r^D)) = g^D(F_r^D(x_r^D, y_r^D))$ , from (3.7).

For (3), we write  $(x_r^D)^k$  as  $(x_\ell^D)^k(\sum_{i \geq 0} g_i(x_\ell^D)^i)^k$ , and obtain

$$\langle (x_r^D)^k, \beta_{n,\ell}^D \rangle = (g^D)_{n-k}^k$$

as required.  $\square$

When  $DU$  is equipped with the orientation classes  $x_\ell$  and  $x_r$  it becomes the universal example of a doubly complex oriented spectrum, since the exterior product  $t_\ell^D t_r^D$  is represented by a multiplicative spectrum map  $t^D: DU \rightarrow D$  whose induced transformation  $DU^2(CP^\infty) \rightarrow D^2(CP^\infty)$  maps  $x_\ell$  and  $x_r$  to  $x_\ell^D$  and  $x_r^D$  respectively. It therefore often suffices to consider the case  $DU$  (as we might in Lemma 3.11, for example). We shall continue to omit the superscript  $DU$  whenever the context makes clear that we are dealing with the universal case. We note from the definitions that the homomorphism of coefficient rings  $DU_* \rightarrow D_*$  induced by  $t^D$  satisfies

$$(3.12) \quad g_k \mapsto g_k^D \quad \text{and} \quad \bar{g}_k \mapsto \bar{g}_k^D$$

for all  $k \geq 0$ . Thus  $DU_*$  is universal for rings equipped with two formal group laws which are linked by a strict isomorphism.

Whenever a complex vector bundle is given with a prescribed splitting  $\rho \cong \rho_\ell \oplus \rho_r$ , then  $t^D(\rho)$  acts as a canonical Thom class  $t_\ell^D(\rho_\ell)t_r^D(\rho_r)$ , and so defines a Thom isomorphism which respects the splitting. In the universal case,  $t(\rho)$  is represented geometrically by the inclusion of the zero section  $M \subseteq M(\rho_\ell \oplus \rho_r)$  whenever  $\rho$  lies over a double  $U$ -manifold  $M$ .

As an example, it is instructive to consider the case when  $D$  is  $MU$ , doubly oriented by setting  $x_\ell^D = x_r^D = x$ . The associated Thom class is the forgetful map  $\pi: DU \rightarrow MU$ , since  $\pi(x_\ell) = \pi(x_r) = x$ ; we therefore deduce from (3.12) that

$$\pi(g_k) = \pi(\bar{g}_k) = 0$$

for all  $k > 0$ .

We briefly consider the  $D_*$ -modules  $D_*(BU(m, n))$  and  $D_*(MU(m, n))$ , together with their cohomological counterparts. These may all be described by application of the Künneth formula. For example  $D^*(BU(m, n))$  is a power series algebra, generated by any one of the four possible sets of Chern classes

$$(3.13) \quad \{c_{j,\ell}^D \otimes 1, 1 \otimes c_{k,r}^D\}, \quad \{c_{j,\ell}^D \otimes 1, 1 \otimes c_{k,\ell}^D\}, \\ \{c_{j,r}^D \otimes 1, 1 \otimes c_{k,\ell}^D\}, \quad \text{or} \quad \{c_{j,r}^D \otimes 1, 1 \otimes c_{k,r}^D\},$$

where  $1 \leq j \leq m$  and  $1 \leq k \leq n$ . The first of these are by far the most natural, and we shall choose them whenever possible. The stable versions, in which we take limits over one or both of  $m$  and  $n$ , are obtained by the obvious relaxation on the range of  $j$  and  $k$ . We shall use notation such as  $(\beta^D)^\psi \otimes (\beta^D)^\omega$  (omitting the subscripts  $\ell$  and  $r$ ) to indicate our first-choice

basis monomials in  $D_*(BU(m, n))$ , and  $(b^D)^\psi \otimes (b^D)^\omega$  for their images in  $D_*(MU(m, n))$  under the Thom isomorphism induced by  $t^D(m, n)$ ; we write  $c_\psi^D \otimes c_\omega^D$  and  $s_\psi^D \otimes s_\omega^D$  respectively for the corresponding dual basis elements in  $D^*(BU(m, n))$  and  $D^*(MU(m, n))$ . As before, it often suffices to consider the universal example  $DU$ .

#### 4. OPERATIONS AND COOPERATIONS

In this section we consider the operations and cooperations associated with  $MU$  and  $DU$ . We work initially with our arbitrary spectra  $E$  and  $D$ , specializing to  $MU$  and  $DU$  as required; we study both the geometric and the homotopy theoretic aspects, using Boardman's eightfold way [2] (and its update [3]) as a convenient algebraic framework. All comments concerning the singly oriented  $E$  apply equally well to  $D$  unless otherwise stated.

For any integer  $n$ , the cohomology group  $E^n(E)$  consists of homotopy classes of spectrum maps  $E \rightarrow S^n \wedge E$ , and therefore encodes  $E$ -theory cohomology operations of degree  $n$ . Thus  $E^*(E)$  is a noncommutative, graded  $E_*$ -algebra with respect to composition of maps, and realizes the algebra  $A_E^*$  of stable  $E$ -cohomology operations. It is important to observe that  $E^*(E)$  is actually a bimodule over the coefficients  $E_*$ , which act naturally on the left (as used implicitly above), but also on the argument and therefore on the right. The same remarks apply to  $E_*(E)$ , on which the product map  $\mu^E$  induces a commutative  $E_*$ -algebra structure; the two module structures are then defined respectively by the left and right inclusions  $\eta_\ell$  and  $\eta_r$  of the coefficients in  $E_*(E) \cong \pi_*(E \wedge E)$ . We shall normally maintain the convention of assuming the left action without comment, and using the right action only when explicitly stated. We refer to  $E_*(E)$  as the algebra  $A_E^*$  of stable  $E$ -homology *cooperations*, for reasons which will become clear below.

In fact  $E_*(X)$  is free and of finite type for all spaces and spectra  $X$  that we consider here. This simplifies the topologizing of  $E^*(X)$ , which involves little more than accommodating the appearance of formal power series in certain computations. Moreover, it ensures that  $\mu^E$  induces a cocommutative coproduct  $\delta^E: E^*(E) \rightarrow E^*(E) \widehat{\otimes}_{E_*} E^*(E)$ , where the tensor product is completed whenever  $E^*(E)$  fails to be of finite type.

We consider the  $E_*$ -algebra map  $t_*^E: E_*(MU) \rightarrow E_*(E)$  induced by the Thom class  $t^E$ , and define monomials  $e^\omega$  as  $t_*^E(b^E)^\omega$ . When  $E$  is singly oriented we assume that these monomials form a basis for  $E_*(E)$ , which is therefore isomorphic to the polynomial algebra  $E_*[e_k^E : k \geq 0]$ ; thus  $E^*(E)$  is given by  $\text{Hom}_{E_*}(E_*(E), E_*)$ , and admits the dual topological basis  $e_\omega$ . It follows that the composition product dualizes to a noncocommutative coproduct  $\delta_E: E_*(E) \rightarrow E_*(E) \otimes_{E_*} E_*(E)$  (where  $\otimes_{E_*}$  is taken over the right action on the left factor), with counit given by projection onto the coefficients. Together with the left and right units  $\eta_\ell$  and  $\eta_r$ , and the antipode  $\chi_E$  induced by interchanging the factors in  $E_*(E)$ , this coproduct turns  $E_*(E)$  into a cogroupoid object in the category of  $E_*$ -algebras. Such an object generalizes the notion of Hopf algebra, and is known as a *Hopf algebroid*; for a detailed discussion, see [22].

We write  $s$  for a generic operation in  $A_E^*$ . By virtue of our discussion above we may interpret  $s$  as a spectrum map  $E \rightarrow S^n \wedge E$ , or as an  $E_*$ -homomorphism  $E_*(E) \rightarrow E_*$ .

When  $E$  is replaced by  $D$ , we have monomials  $e_\ell^\psi$  and  $e_r^\omega$  in  $D_*(D)$ , induced by  $t_\ell^D$  and  $t_r^D$  respectively; the products  $e_\ell^\psi e_r^\omega$  are induced by  $t_\ell^D t_r^D$ . In the cases of interest these products form a basis for  $D_*(D)$ , which is therefore isomorphic to the polynomial algebra  $D_*[e_{j,\ell}^D, e_{k,r}^D : j, k \geq 0]$ .

Interpreting the elements of  $E^*(E)$  as selfmaps of  $E$ , we first define the *standard action*

$$(4.1) \quad E^*(E) \otimes_{E_*} E^*(X) \longrightarrow E^*(X)$$

of  $E_*$ -modules (where  $\otimes_{E_*}$  is taken over the right action on  $E^*(E)$ ) for any space or spectrum  $X$ . We write this action functionally; when  $X$  is  $E$  it reduces to the composition product in  $E^*(E)$ . The Cartan formula asserts that the product map in  $E^*(X)$  is a homomorphism of left  $E^*(E)$ -modules with respect to the action, and was restated by Milnor in the form

$$(4.2) \quad s(yz) = \sum s'(y)s''(z) \quad \text{where} \quad \delta^E(s) = \sum s' \otimes s''$$

for all classes  $y$  and  $z$  in  $E^*(X)$ . Following Novikov [19], we refer to any such module with property (4.2) as a *Milnor module*.

When  $X$  is a point (or the sphere spectrum) then (4.1) describes the action of  $E^*(E)$  on the coefficient ring  $E_*$ , and we immediately deduce that

$$(4.3) \quad s(x) = \langle s, \eta_r(x) \rangle$$

for all  $x$  in  $E_*$ .

Given our freeness assumptions we may dualize and conjugate (4.1) in seven further ways, whose interrelationships are discussed by Boardman with great erudition. We require four of these (together with a fifth and sixth which are different), so we select from [2] and [3] without further comment, relying on the straightforward nature of the algebra. We ignore all issues concerning signs because our spaces and spectra have no cells in odd dimensions.

The first of these duals (and our second structure) is the *Adams coaction*

$$(4.4) \quad \psi: E_*(X) \longrightarrow E_*(E) \otimes_{E_*} E_*(X)$$

of  $E_*$ -modules (where  $\otimes_{E_*}$  is taken over the right action on  $E_*(E)$ ) for any space or spectrum  $X$ . This is defined by dualizing the standard action over  $E_*$ , and when  $X$  is  $E$  then  $\psi$  reduces to the coproduct  $\delta_E$ . For each operation  $s$ , the duality may be expressed by

$$(4.5) \quad \langle s(y), a \rangle = \sum_{\omega} \langle s, e^{\omega} \langle y, a^{\omega} \rangle \rangle,$$

where  $a$  lies in  $E_*(X)$  with  $a^{\omega}$  defined by  $\psi(a) = \sum_{\omega} e^{\omega} \otimes a^{\omega}$ , and  $y$  lies in  $E^*(X)$ .

If we assume that  $X$  is a spectrum (or stable complex), we may interpret  $\pi_*(X \wedge E)$  as  $X_*(E)$ , and consider the isomorphism  $c: E_*(X) \cong X_*(E)$  of *conjugation*. Our third structure is the *right coaction*

$$(4.6) \quad \psi: X_*(E) \longrightarrow X_*(E) \otimes_{E_*} E_*(E)$$

of right  $E_*$ -modules (where  $\otimes_{E_*}$  is taken over the right action of the scalars on  $X_*(E)$ ); it is evaluated as  $\psi(ca) = \sum_{\omega} ca^{\omega} \otimes \chi_E(e^{\omega})$ , by conjugating (4.4). When  $X$  is  $E$ , then  $c$  reduces to  $\chi_E$  and  $\psi$  becomes  $\delta_E$ , as before.

Fourthly, the standard action partially dualizes over  $E_*$  to give the *Milnor coaction*

$$(4.7) \quad \rho: E^*(X) \longrightarrow E^*(X) \widehat{\otimes}_{E_*} E_*(E)$$

of  $E_*$ -modules. As Milnor famously observed, the Cartan formulae are encapsulated in the fact that  $\rho$  makes  $E^*(X)$  a *Hopf comodule* over  $E_*(E)$ , by virtue of being an algebra map. For each operation  $s$  and each  $x$  in  $E^*(X)$ , the partial duality may be described by

$$(4.8) \quad s(y) = \sum_{\omega} \langle s, e^{\omega} \rangle y_{\omega},$$

where  $y_{\omega}$  is defined by  $\rho(y) = \sum_{\omega} y_{\omega} \otimes e^{\omega}$ ; thus  $y_{\omega} = e_{\omega}(y)$ . In view of the completion required of the tensor product in (4.7), we describe  $\rho$  more accurately as a *formal coaction*. The Chern-Dold character is most naturally expressed by a simple generalization of (4.7).

A fifth possibility is provided by the *left action*

$$E^*(E) \otimes_{E_*} E_*(X) \longrightarrow E_*(X)$$

of  $E_*$ -modules (where  $\otimes_{E_*}$  is taken over the right action of the scalars on  $X_*(E)$ ), which is defined by means of spectrum maps in similar fashion to the standard action (4.1). It is evaluated by partially dualizing the Adams coaction, for which we write

$$(4.9) \quad s_{\ell}a = \sum_{\omega} \langle s, \chi_E(e^{\omega}) \rangle a^{\omega},$$

with notation as above. The left action satisfies

$$(4.10) \quad \langle y, s_{\ell}a \rangle = \langle s, c(y_*a) \rangle,$$

where  $y_*: E_*(X) \rightarrow E_*(E)$  is the homomorphism induced by  $y$ , with  $y$  and  $a$  as above.

For our sixth and seventh structures we again assume that  $X$  is stable, so the selfmaps of  $E$  induce a left action

$$(4.11) \quad E^*(E) \otimes_{E_*} X_*(E) \longrightarrow X_*(E)$$

(where  $\otimes_{E_*}$  is taken over the right action of the scalars on both factors), and a right action

$$X^*(E) \otimes_{E_*} E^*(E) \longrightarrow X^*(E)$$

(where  $\otimes_{E_*}$  is taken over the right action on  $X^*(E)$ ). Neither of these seems to be discussed explicitly by Boardman, although (4.11) appears regularly in the literature. It is evaluated by partially dualizing (4.6), for which we write

$$(4.12) \quad s_r d = \sum_{\omega} d^{\omega} \langle s, e^{\omega} \rangle,$$

where  $d$  lies in  $X_*(E)$  with  $\psi(d) = \sum_{\omega} d^{\omega} \otimes e^{\omega}$ . The two actions are related according to

$$(4.13) \quad \langle w, s_r d \rangle = \langle (w)s, d \rangle,$$

where  $w$  lies in  $X^*(E)$ ; this should be compared with (4.10), and justifies the interpretation of (4.11) as a right action (on the left!).

When  $X$  is  $E$ , then (4.10) may be rewritten as

$$(4.14) \quad \langle y, s_\ell a \rangle = \langle s, \chi_E(y_r a) \rangle,$$

and (4.13) reduces to the right action of  $E^*(E)$  on its dual  $E_*(E)$ . We may evaluate the coproduct  $\delta_E(d)$  as

$$(4.15) \quad \sum_{\omega} e_{\omega,r} d \otimes e^{\omega} = \sum_{\omega} \chi_E(e^{\omega}) \otimes e_{\omega,\ell} d,$$

given any  $d$  in  $E_*(E)$ .

It is important to record how the elements  $e_k$  and the orientation class  $x^E$  are intertwined by certain of these actions and coactions.

**Proposition 4.16.** *We have that*

$$\rho(x^E) = \sum_{k \geq 0} (x^E)^{k+1} \otimes e_k$$

in  $E^*(CP^\infty) \widehat{\otimes}_{E_*} E_*(E)$ .

*Proof.* The induced homomorphism  $x_*^E: E_{*+2}(CP^\infty) \rightarrow E_*(E)$  acts such that  $x_*^E(\beta_{k+1}^E) = e_k$ , by definition of the elements  $e^\omega$ . Dualizing, we obtain

$$(4.17) \quad s(x^E) = \sum_{k \geq 0} \langle s, e_k \rangle (x^E)^{k+1},$$

and the formula follows from (4.8).  $\square$

By a simple extension of (4.17), the entire algebra  $A_*^E$  may be faithfully represented by its action on the bottom cell of the smash product of infinitely many copies of  $CP^\infty$ .

**Corollary 4.18.** *The coproduct and antipode of the Hopf algebroid  $E_*(E)$  are given by*

$$\delta_E(e_n) = \sum_{k \geq 0} (e)_{n-k}^{k+1} \otimes e_k \quad \text{and} \quad \chi_E(e_n) = (e)_n^{-(n+1)}$$

respectively.

*Proof.* Since  $\rho$  is a coaction, we have that  $\rho \otimes 1(\rho(x^E)) = 1 \otimes \delta^E(\rho(x^E))$  as maps  $E^*(CP^\infty) \rightarrow E^*(CP^\infty) \widehat{\otimes}_{E_*} E_*(E) \otimes_{E_*} E_*(E)$ , and the formula for  $\delta^E$  follows. The properties of the antipode, coupled with Lagrange inversion, yield  $\chi_E$  immediately.  $\square$

We remark in passing that Boardman uses the formulae for  $\rho(x^E)$  to define the elements  $e_k$ .

When  $E$  is replaced by  $D$ , we find that

$$(4.19) \quad \rho(x_\ell^D) = \sum_{k \geq 0} (x_\ell^D)^{k+1} \otimes e_{k,\ell} \otimes 1 \quad \text{and} \quad \rho(x_r^D) = \sum_{k \geq 0} (x_r^D)^{k+1} \otimes 1 \otimes e_{k,r}$$

in  $D^*(CP^\infty) \widehat{\otimes}_{D_*} D_*(D)$ . These two expressions are linked by the series  $x_r^D = g(x_\ell^D)$  of (3.10), and were exploited by the first author in [8]. In

similar vein,  $\delta_D$  and  $\chi_D$  on  $e_{n,\ell}^D$  and  $e_{n,r}^D$  are given by the left and right forms of Corollary 4.18.

Armed with these results, we now consider the universal example  $MU$ .

As described in §3, the operations  $e_\omega$  are written  $s_\omega$  in  $A_{MU}^*$ , and are known as the Landweber-Novikov operations; dually, the elements  $e^\omega$  reduce to  $b^\omega$  in  $A_*^{MU}$ . We consider the integral spans  $S^*$  and  $S_*$  of the  $s_\omega$  and  $b^\omega$  respectively, so that  $A_{MU}^* \cong \Omega_*^U \otimes S^*$  and  $A_*^{MU} \cong \Omega_*^U \otimes S_*$  as  $\Omega_*^U$ -modules, where  $S_*$  is the polynomial algebra  $\mathbb{Z}[b_k : k \geq 0]$ . The  $\Omega_*^U$ -duality between  $A_{MU}^*$  and  $A_*^{MU}$  therefore restricts to an integral duality between  $S^*$  and  $S_*$ , for which no topological considerations are necessary because  $S^*$  has finite type.

The formulae of Proposition 4.16 show that  $S_*$  is closed with respect to the coproduct and antipode of  $A_*^{MU}$ , whilst the left unit and the counit survive with respect to  $\mathbb{Z}$ . Therefore  $S_*$  is a Hopf subalgebra of the Hopf algebroid. Duality then ensures that  $S^*$  is also a Hopf algebra, with respect to composition of operations and the Cartan formula

$$(4.20) \quad \delta(s_\omega) = \sum_{\omega' + \omega'' = \omega} s_{\omega'} \otimes s_{\omega''},$$

which is dual to the product of monomials. Of course  $S^*$  is the Landweber-Novikov algebra. Alternatively, and following the original constructions, we may use the action of  $S^*$  on  $\Omega_U^*(\wedge_\infty CP^\infty)$  to prove directly that  $S^*$  is a Hopf algebra. Many of our actions and coactions restrict to  $S^*$  and  $S_*$  and will be important below. We emphasize that  $A_{MU}^*$  has no  $\Omega_*^U$ -linear antipode, and that the antipode in  $S^*$  is induced from the antipode in  $S_*$  by  $\mathbb{Z}$ -duality.

Choosing  $E$  and  $X$  to be  $MU$  in (4.13) and (4.14) provides the left and right action of  $A_{MU}^*$  on its dual. Explicitly, if  $s$  and  $y$  lie in  $A_{MU}^*$  and  $b$  in  $A_*^{MU}$ , then

$$(4.21) \quad \langle y, s_\ell b \rangle = \langle s, \chi(y_r b) \rangle \quad \text{and} \quad \langle y, s_r b \rangle = \langle y s, b \rangle.$$

Alternatively, by appealing to (4.9) and (4.12) we may write

$$(4.22) \quad s_\ell b = \sum \langle s, \chi(b') \rangle b'' \quad \text{and} \quad s_r b = \sum \langle s, b'' \rangle b',$$

where  $\delta(b) = \sum b' \otimes b''$ . By restriction we obtain identical formulae for the left and right actions of  $S^*$  on  $A_*^{MU}$  and on  $S_*$ . In the latter case,  $\mathbb{Z}$ -duality allows us to rewrite the left action as

$$\langle y, s_\ell b \rangle = \langle \chi(s) y, b \rangle,$$

thereby (at last) according it equivalent status to the right action.

The *adjoint actions* of  $S^*$  on  $A_*^{MU}$  and  $S_*$  are similarly defined by

$$(4.23) \quad \langle y, \text{ad}(s)(b) \rangle = \sum \langle \chi(s') y s'', b \rangle \quad \text{and} \\ \text{ad}(s)(b) = \sum \langle \chi(s'), b' \rangle \langle s'', b''' \rangle b'',$$

which give rise to the adjoint Milnor module structure on  $A_*^{MU}$  and  $S_*$ .

By way of example, we recall from Corollary 4.18 that the diagonal for  $S_*$  is given by  $\delta(b_n) = \sum_{k \geq 0} (b)_{n-k}^{k+1} \otimes b_k$ ; (4.15) therefore yields

$$(4.24) \quad \begin{aligned} s_{\epsilon(k),\ell} b_n &= (k - n - 1) b_{n-k}, & s_{\epsilon(k),r} b_n &= (b)_{n-k}^{k+1} \\ \text{and} \quad \text{ad}(s_{\epsilon(k)}) b_n &= (k - n - 1) b_{n-k} + (b)_{n-k}^{k+1} \end{aligned}$$

for all  $0 \leq k \leq n$ .

The Thom isomorphisms (3.4) ensure that the action of  $S^*$  on the coefficients  $\Omega_*^U$  has certain special geometrical attributes. By restricting the arguments of Proposition 3.5, a representative for the action of the right unit

$$(4.25) \quad \eta_r : \Omega_*^U \longrightarrow A_*^{MU} \cong \Omega_*^U(BU_+)$$

on the cobordism class of a  $U$ -manifold  $M^k$  is given by the singular  $U$ -manifold  $\nu : M^k \rightarrow BU$ . Since the operation  $s_\omega$  corresponds to the Chern class  $c_\omega$  under  $A_{MU}^* \cong \Omega_U^*(BU_+)$ , we deduce from (4.3) that

$$s_\omega(M^k) = \langle c_\omega(\nu), \sigma \rangle,$$

where  $\sigma$  in  $\Omega_k^U(M^k)$  is the canonical orientation class represented by the identity map on  $M^k$ . In other words,  $s_\omega(M^k)$  in  $\Omega_{k-2|\omega|}^U$  is represented by the domain of the Poincaré dual of the the normal Chern class  $c_\omega(\nu)$ .

Following Novikov we use this action to illuminate the product structure in  $A_{MU}^*$ , within which we have already identified the subalgebras  $\Omega_*^U$  and  $S^*$ . It therefore suffices to describe the *commutation rule* for expressing products of the form  $sx$ , where  $s$  and  $x$  lie in  $S^*$  and  $\Omega_*^U$  respectively. Recalling (4.1) and (4.2) we obtain

$$(4.26) \quad sx = \sum s'(x)s'',$$

and write  $A_{MU}^* = \Omega_*^U S^*$  for the resulting algebra. We may construct an algebra in this fashion from any Milnor module structure, and we refer to it as the associated *operator double*. It is an important special case of Sweedler's smash product [27]. The coproduct in  $A_{MU}^*$  is obtained from (4.20) by  $\Omega_*^U$ -linearity.

We may describe the coproduct for  $A_*^{MU}$  in terms of the polynomial algebra  $\Omega_*^U \otimes S_*$ , simply by using  $\Omega_*^U$ -linearity to extend the coproduct for  $S_*$ . Dually, the form of (4.26) is governed by the Adams coaction, as expressed in (4.5).

There are other ways of interpreting the algebra  $S_*$ . For example, the projection  $t_*^H : \Omega_*^U(MU) \rightarrow H_*(MU)$  restricts to an isomorphism on  $S_*$ , with  $t_*^H(b^\omega) = (b^H)^\omega$ . This isomorphism often appears implicitly in the literature, although the induced coproduct and antipode in  $H_*(MU)$  are purely algebraic. A second interpretation, for which we recall the canonical isomorphism  $\Omega_*^{DU} \cong A_*^{MU}$  of (2.6), is crucial.

**Proposition 4.27.** *The subalgebra  $G_*$  of  $\Omega_*^{DU}$  is identified with the dual of the Landweber-Novikov algebra  $S_*$  in  $A_*^{MU}$  under the canonical isomorphism.*

*Proof.* By appealing to Proposition 3.8, it suffices to show that the Thom isomorphism  $A_*^{MU} \cong \Omega_*^U(BU_+)$  satisfies  $b_k \mapsto \beta_k$ . This follows by definition,

and ensures (by multiplicativity) that monomials  $g^\omega$  and  $b^\omega$  are identified for all  $\omega$ .  $\square$

We now consider operations in double complex cobordism theory, and their interaction with the Landweber-Novikov algebra. For this purpose we apply our general theory in the case when  $E$  is  $DU$ , from which we immediately obtain the algebra of  $DU$ -operations  $A_{DU}^*$ , and of  $DU$ -cooperations  $A_*^{DU}$ . These act and coact according to the eightfold way.

Identifying  $D$  with  $MU \wedge MU$ , we note that an element  $s$  of  $A_{MU}^*$  yields operations  $s \otimes 1$  and  $1 \otimes s$  in  $A_{DU}^*$  by action on the appropriate factor; this description is consistent with our choice of  $\Omega_*^{DU}$ -basis elements  $s_\psi \otimes s_\omega$  from amongst the four possibilities in (3.13). We refer to the operations  $s_\psi \otimes 1$  and  $1 \otimes s_\omega$  as left and right Landweber-Novikov operations, and observe that they commute by construction. Thus  $A_{DU}^*$  contains the subalgebra  $S^* \otimes S^*$ , and  $A_*^{DU}$  contains the subalgebra  $S_* \otimes S_* \cong \mathbb{Z}[b_j \otimes 1, 1 \otimes b_k : j, k \geq 0]$ . Occasionally we write  $S^* \otimes 1$  and  $S_* \otimes 1$  as  $S_\ell^*$  and  $S_{*,\ell}$  respectively, with similar conventions on the right. As before, the  $\Omega_*^{DU}$ -duality between  $A_{DU}^*$  and  $A_*^{DU}$  restricts to an integral duality between  $S^* \otimes S^*$  and  $S_* \otimes S_*$ . Since  $S_* \otimes S_*$  is a Hopf subalgebra of the Hopf algebroid  $A_*^{DU}$ , this duality ensures that  $S^* \otimes S^*$  is also a Hopf algebra with respect to composition of operations. The coproduct is dual to the product of monomials, being given by left and right Cartan formulae of the form (4.20), and the antipode is induced from  $S_* \otimes S_*$  by  $\mathbb{Z}$ -duality. These structures are, of course, identical with those obtained by forming the  $\otimes$  square of each of the Hopf algebras  $S^*$  and  $S_*$ .

The coproduct  $\delta: S^* \rightarrow S^* \otimes S^*$  provides a third (and extremely important) *diagonal embedding* of  $S^*$  in  $A_{DU}^*$ .

Writing  $E$  as  $DU$  and  $X$  as a point (or the sphere spectrum) in (4.1) yields the action of  $A_{DU}^*$  on the coefficient ring  $\Omega_*^{DU}$ ; the action of  $S^* \otimes S^*$  follows by restriction. Both of these give rise to a Milnor module structure, so that we may express  $A_{DU}^*$  as the operator double  $\Omega_*^{DU}(S^* \otimes S^*)$ . The actions are closely related to those of (4.21) under the canonical isomorphism  $\Omega_*^{DU} \cong A_*^{MU}$ .

**Proposition 4.28.** *The canonical isomorphism identifies the actions of  $S_\ell^*$  and  $S_r^*$  on  $\Omega_*^{DU}$  with the left and right actions of  $S^*$  on  $A_*^{MU}$  respectively.*

*Proof.* This follows from the definitions by identifying the respective actions in terms of maps of spectra; thus, on an element  $x$  in  $\pi_k(MU \wedge MU)$ , the action of a left operation  $s_\ell$  and the left action of  $s$  are both represented by

$$S^k \xrightarrow{x} MU \wedge MU \xrightarrow{s \wedge 1} MU \wedge MU.$$

For the right actions, we replace  $s \wedge 1$  with  $1 \wedge s$ .  $\square$

**Corollary 4.29.** *The subalgebra  $G_*$  of  $\Omega_*^{DU}$  is closed under the action of the subalgebra  $S^* \otimes S^*$  of  $A_{DU}^*$ .*

*Proof.* We apply Proposition 4.27, and the result is immediate.  $\square$

We shall say more about Corollary 4.29 in §6, where we give an interpretation in terms of the geometry of flag manifolds.

We may combine Proposition 4.28 and Corollary 4.29 to ensure that the diagonal action of  $S^*$  on  $\Omega_*^{DU}$  and  $G_*$  is identified with the adjoint action

on  $S_*$ . This result was established in [8] by appeal to the coaction  $\rho$  as in (4.19), and we now restate its consequences; the proofs above are valid for any spectrum of the form  $E \wedge F$ .

Since  $S^*$  is cocommutative, we utilize Novikov's construction [19] (as also described in [17]) of  $D(H^*)$  as the operator double  $S_*S^*$  with respect to the adjoint action (4.23) of  $S^*$  on its dual. Our realization of the quantum double  $D(S^*)$  follows.

**Theorem 4.30** ([8]). *The algebra of operations  $A_{DU}^*$  contains a subalgebra isomorphic to  $D(S^*)$ .*

*Proof.* As explained above, we may express  $A_{DU}^*$  as the operator double  $\Omega_*^{DU}(S^* \otimes S^*)$  with respect to the standard action of  $S^* \otimes S^*$  on  $\Omega_*^{DU}$ . The action of the diagonal subalgebra  $S^*$  restricts to  $G_*$ , and is identified with the adjoint action on  $S_*$  by Corollary 4.29. Since the operator double  $S^*G_*$  is a subalgebra of  $A_{DU}^*$ , the result follows.  $\square$

It follows directly from the definitions, coupled with the formulae (4.24), that the commutation law in  $D(S^*)$  obeys

$$(4.31) \quad s_{\epsilon(k)}b_n = (k - n - 1)b_{n-k} + (b)_{n-k}^{k+1} + b_n s_{\epsilon(k)}$$

for all  $0 \leq k \leq n$ .

By analogy with (4.25) we describe the action of  $S^* \otimes S^*$  on  $\Omega_*^{DU}$  geometrically. A representative for the action of the right unit

$$\eta_r: \Omega_*^{DU} \longrightarrow A_*^{DU} \cong \Omega_*^{DU}(BU \times BU_+)$$

on the cobordism class of a double  $U$ -manifold  $M^k$  is given by the singular double  $U$ -manifold  $\nu_\ell \times \nu_r: M^k \rightarrow BU \times BU$ . Since the operation  $s_\psi \otimes s_\omega$  corresponds to the Chern class  $c_\psi \otimes c_\omega$  under  $A_{DU}^* \cong \Omega_{DU}^*(BU \times BU_+)$ , we conclude from (4.3) that

$$(4.32) \quad s_\psi \otimes s_\omega(M^k) = \langle c_{\psi,\ell}(\nu_\ell)c_{\omega,r}(\nu_r), \sigma \rangle,$$

where  $\sigma$  in  $\Omega_k^{DU}(M^k)$  is the canonical orientation class represented by the identity map on  $M^k$ .

We deduce from (4.32) that the left, right, and diagonal actions of  $s_\omega$  on  $M^k$  give  $\langle c_{\omega,\ell}(\nu_\ell), \sigma \rangle$ ,  $\langle c_{\omega,r}(\nu_r), \sigma \rangle$ , and

$$\sum_{\omega'+\omega''=\omega} \langle c_{\omega',\ell}(\nu_\ell)c_{\omega'',r}(\nu_r), \sigma \rangle$$

respectively, in  $\Omega_{k-2|\omega|}^{DU}$ . In conjunction with Proposition 4.28, these formulae yield a geometrical realization of the three actions of  $S^*$  on  $A_*^{MU}$ ; we take the double  $U$ -cobordism class of  $M^k$ , form the Poincaré duals of  $c_{\omega,\ell}(\nu_\ell)$ ,  $c_{\omega,r}(\nu_r)$ , and  $\sum_{\omega'+\omega''=\omega} c_{\omega',\ell}(\nu_\ell)c_{\omega'',r}(\nu_r)$  respectively, and record the double  $U$ -cobordism class of the domain. We shall implement this procedure in terms of bounded flag manifolds in §6.

## 5. BOUNDED FLAG MANIFOLDS

In this section we introduce our family of bounded flag manifolds, and discuss their topology in terms of a cellular calculus which is intimately related to the Schubert calculus for classic flag manifolds. Our description

is couched in terms of nonsingular subvarieties, anticipating applications to cobordism in the next section. We also invest the bounded flag manifolds with certain canonical  $U$ - and double  $U$ -structures, and so relate them to our earlier constructions in  $\Omega_*^{DU}$ . Much of our notation differs considerably from that introduced by the second author in [24].

We shall follow combinatorial convention by writing  $[n]$  for the set of natural numbers  $\{1, 2, \dots, n\}$ , equipped with the standard linear ordering  $<$ . Every interval in the poset  $[n]$  has the form  $[a, b]$  for some  $1 \leq a \leq b \leq n$ , and consists of all  $m$  satisfying  $a \leq m \leq b$ ; our convention therefore dictates that we abbreviate  $[1, b]$  to  $[b]$ . It is occasionally convenient to interpret  $[0]$  as the empty set, and  $[\infty]$  as the natural numbers. We work in the context of the Boolean algebra  $\mathcal{B}(n)$  of finite subsets of  $[n]$ , ordered by inclusion. We decompose each such subset  $Q \subseteq [n]$  into maximal subintervals  $I(1) \cup \dots \cup I(s)$ , where  $I(j) = [a(j), b(j)]$  for  $1 \leq j \leq s$ , and assign to  $Q$  the monomial  $b^\omega$ , where  $\omega_i$  records the number of intervals  $I(j)$  of cardinality  $i$  for each  $1 \leq i \leq n$ ; we refer to  $\omega$  as the *type* of  $Q$ , noting that it is independent of the choice of  $n$ . We display the elements of  $Q$  in increasing order as  $\{q_i : 1 \leq i \leq d\}$ , and abbreviate the complement  $[n] \setminus Q$  to  $Q'$ . We also write  $I(j)^+$  for the subinterval  $[a(j), b(j) + 1]$  of  $[n + 1]$ , and  $Q^\wedge$  for  $Q \cup \{n + 1\}$ . It is occasionally convenient to set  $b(0)$  to 0 and  $a(s + 1)$  to  $n + 1$ .

We begin by recalling standard constructions of complex flag manifolds and some of their simple properties, for which a helpful reference is [12]. We work in an ambient complex inner product space  $Z_{n+1}$ , which we assume to be invested with a preferred orthonormal basis  $z_1, \dots, z_{n+1}$ , and we write  $Z_E$  for the subspace spanned by the vectors  $\{z_e : e \in E\}$ , where  $E \subseteq [n + 1]$ . We abbreviate  $Z_{[a,b]}$  to  $Z_{a,b}$  (and  $Z_{[b]}$  to  $Z_b$ ) for each  $1 \leq a < b \leq n + 1$ , and write  $CP(Z_E)$  for the projective space of lines in  $Z_E$ . We let  $V - U$  denote the orthogonal complement of  $U$  in  $V$  for any subspaces  $U < V$  of  $Z_{n+1}$ , and we regularly abuse notation by writing 0 for the subspace which consists only of the zero vector. A complete flag  $V$  in  $Z_{n+1}$  is a sequence of proper subspaces

$$0 = V_0 < V_1 < \dots < V_i < \dots < V_n < V_{n+1} = Z_{n+1},$$

of which the *standard* flag  $Z_0 < \dots < Z_i < \dots < Z_{n+1}$  is a specific example. The flag manifold  $F(Z_{n+1})$  is the set of all flags in  $Z_{n+1}$ , topologized appropriately. Since the unitary group  $U(n + 1)$  acts transitively on  $F(Z_{n+1})$  in such a way that the stabilizer of the standard flag is the maximal torus  $T$ , we may identify  $F(Z_{n+1})$  with the coset space  $U(n + 1)/T$ ; in this guise,  $F(Z_{n+1})$  acquires the quotient topology.

In fact the flag manifold has many extra properties. It is a nonsingular complex projective algebraic variety of dimension  $\binom{n+1}{2}$ , and is therefore a closed complex manifold. It has a CW-structure whose cells are even dimensional and may be described in terms of the Bruhat decomposition, which indexes them by elements  $\alpha$  of the symmetric group  $\Sigma_{n+1}$  (the Weyl group of  $U(n + 1)$ ), and partially orders them by the decomposition of  $\alpha$  into products of transpositions. The closure of each Bruhat cell  $e_\alpha$  is an algebraic subvariety, generally singular, known as the *Schubert variety*  $X_\alpha$ . Whether

considered as cells or subvarieties, these subspaces define a basis for the integral homology  $H_*(F(Z_{n+1}))$ , and therefore also for the integral cohomology  $H^*(F(Z_{n+1}))$  which is simply the integral dual. The manipulation of cup and cap products and Poincaré duality with respect to these bases is known as the *Schubert calculus* for  $F(Z_{n+1})$ , and has long been a source of delight to geometers and combinatorialists.

An alternative description of  $H^*(F(Z_{n+1}))$  is provided by Borel's computations with the Serre spectral sequence. The canonical torus bundle over  $U(n+1)/T$  is classified by a map  $U(n+1)/T \rightarrow BT$ , which induces the *characteristic homomorphism*  $H^*(BT) \rightarrow H^*(U(n+1)/T)$  in integral cohomology. Noting that  $H^*(BT)$  is a polynomial algebra over  $\mathbb{Z}$  on 2-dimensional classes  $x_i$ , where  $1 \leq i \leq n+1$ , Borel identifies  $H^*(F(Z_{n+1}))$  with the ring of *coinvariants* under the action of the Weyl group, defined as the quotient  $\mathbb{Z}[x_1, \dots, x_{n+1}]/J$ , where  $J$  is the ideal generated by all symmetric polynomials. With respect to this identification,  $x_i$  is the first Chern class of the line bundle over  $F(Z_{n+1})$  obtained by associating  $V_i - V_{i-1}$  to each flag  $V$ .

The interaction between the Schubert and Borel descriptions of the cohomology of  $F(Z_{n+1})$  is a fascinating area of combinatorial algebra and has led to a burgeoning literature on the subject of *Schubert polynomials*, beautifully surveyed in MacDonald's book [14]. The entire study may be generalized to quotients such as  $G/B$ , where  $G$  is a semisimple algebraic group over a field  $k$ , and  $B$  is an arbitrary Borel subgroup.

We call a flag  $U$  in  $Z_{n+1}$  *bounded* if each  $i$ -dimensional component  $U_i$  contains the first  $i-1$  basis vectors  $z_1, \dots, z_{i-1}$ , or equivalently, if  $Z_{i-1} < U_i$  for every  $1 \leq i \leq n+1$ . We define the *bounded flag manifold*  $B(Z_{n+1})$  to be the set of all bounded flags in  $Z_{n+1}$ , topologized as a subspace of  $F(Z_{n+1})$ ; its complex manifold structure arises by choosing a neighbourhood of  $U$  to consist of all bounded flags  $T$  satisfying  $T_i \cap U_i^\perp = 0$ . It is simple to check that, as  $i$  decreases, this condition restricts each proper subspace  $T_i$  to a single degree of freedom and defines a chart of dimension  $n$ . Clearly  $B(Z_2)$  is isomorphic to the projective line  $CP(Z_2)$  with the standard complex structure, whilst  $B(Z_1)$  consists solely of the trivial flag. We occasionally abbreviate  $B(Z_{n+1})$  to  $B_n$ , in recognition of its dimension.

We shall devote the remainder of this section to the topology of bounded flag manifolds and a discussion of their  $U$ - and double  $U$ -structures.

There is a map  $p_h: B(Z_{n+1}) \rightarrow B(Z_{h+1, n+1})$  for each  $1 \leq h \leq n$ , defined by factoring out  $Z_h$ . Thus  $p_h(U)$  is given by

$$0 < U_{h+1} - Z_h < \dots < U_i - Z_h < \dots < U_n - Z_h < Z_{h+1, n+1}$$

for each bounded flag  $U$  in  $Z_{n+1}$ . Since  $Z_{i-1} < U_i$  for all  $1 \leq i \leq n+1$ , we deduce that  $Z_{h+1, i-1} < U_i - Z_h$  for all  $i > h+1$ , ensuring that  $p_h(U)$  is indeed bounded. We may readily check that  $p_h$  is the projection of a fiber bundle, with fiber  $B(Z_{h+1})$ . In particular,  $p_1$  has fiber the projective line  $CP(Z_2)$ , and so after  $n-1$  applications we may exhibit  $B(Z_{n+1})$  as an iterated bundle

$$(5.1) \quad B(Z_{n+1}) \rightarrow \dots \rightarrow B(Z_{h, n+1}) \rightarrow \dots \rightarrow B(Z_{n, n+1})$$

over  $B(Z_{n,n+1})$ , where the fiber of each map is isomorphic to  $CP^1$ . This construction was used by the second author in [24].

We define maps  $q_h$  and  $r_h: B(Z_{n+1}) \rightarrow CP(Z_{h,n+1})$  by letting  $q_h(U)$  and  $r_h(U)$  be the respective lines  $U_h - Z_{h-1}$  and  $U_{h+1} - U_h$ , for each  $1 \leq h \leq n$ . We remark that  $q_h = q_1 \cdot p_{h-1}$  and  $r_h = r_1 \cdot p_{h-1}$  for all  $h$ , and that the appropriate  $q_h$  and  $r_h$  may be assembled into maps  $q_Q$  and  $r_Q: B(Z_{n+1}) \rightarrow \times_Q CP(Z_{h,n+1})$ , where  $h$  varies over an arbitrary subset  $Q$  of  $[n]$ . In particular,  $q_{[n]}$  and  $r_{[n]}$  are embeddings, which associate to each flag  $U$  the  $n$ -tuple  $(U_1, \dots, U_h - Z_{h-1}, \dots, U_n - Z_{n-1})$  and the  $n$ -tuple  $(U_2 - U_1, \dots, U_{h+1} - U_h, \dots, Z_{n+1} - U_n)$  respectively, and lead to descriptions of  $B(Z_{n+1})$  as a projective algebraic variety.

We proceed by analogy with the Schubert calculus for  $F(Z_{n+1})$ . To every flag  $U$  in  $B(Z_{n+1})$  we assign the *support*  $S(U)$ , given by  $\{j \in [n] : U_j \neq Z_j\}$ , and consider the subspace

$$e_Q = \{U \in B(Z_{n+1}) : S(U) = Q\}$$

for each  $Q$  in the Boolean algebra  $\mathcal{B}(n)$ . For example,  $e_\emptyset$  is the singleton consisting of the standard flag.

**Lemma 5.2.** *For all nonempty  $Q$ , the subspace  $e_Q \subset B(Z_{n+1})$  is an open cell of dimension  $2|Q|$ , whose closure  $X_Q$  is the union of all  $e_R$  for which  $R \subseteq Q$  in  $\mathcal{B}(n)$ .*

*Proof.* If  $Q = \cup_j I(j)$ , then  $e_Q$  is homeomorphic to the cartesian product  $\times_j e_{I(j)}$ , so it suffices to assume that  $Q$  is an interval  $[a, b]$ . If  $U$  lies in  $e_{[a,b]}$  then  $U_{a-1} = Z_{a-1}$  and  $U_{b+1} = Z_{b+1}$  certainly both hold; thus  $e_{[a,b]}$  consists of those flags  $U$  for which  $q_j(U)$  is a fixed line  $L$  in  $CP(Z_{a,b+1}) \setminus CP(Z_{a,b})$  for all  $a \leq j \leq b$ . Therefore  $e_{[a,b]}$  is a  $2(b-a+1)$ -cell, as sought. Obviously  $e_R \subset X_{[a,b]}$  for each  $R \subseteq Q$ , so it remains only to observe that the limit of a sequence of flags in  $e_{[a,b]}$  cannot have fewer components satisfying  $U_j = Z_j$ , and must therefore lie in  $e_R$  for some  $R \subseteq [a, b]$ .  $\square$

Clearly  $X_{[n]}$  is  $B(Z_{n+1})$ , so that Lemma 5.2 provides a CW decomposition for  $B_n$  with  $2^n$  cells.

We now prove that all the subvarieties  $X_Q$  are nonsingular, in contrast to the situation for  $F(Z_{n+1})$ .

**Proposition 5.3.** *For any  $Q \subseteq [n]$ , the subvariety  $X_Q$  is diffeomorphic to the cartesian product  $\times_j B(Z_{I(j)+})$ .*

*Proof.* We may define a smooth embedding  $i_Q: \times_j B(Z_{I(j)+}) \rightarrow B(Z_{n+1})$  by choosing the components of  $i_Q(U(1), \dots, U(s))$  to be

$$(5.4) \quad T_k = \begin{cases} Z_{a(j)-1} \oplus U(j)_i & \text{if } k = a(j) + i - 1 \text{ in } I(j) \\ Z_k & \text{if } k \in [n+1] \setminus Q, \end{cases}$$

where  $U(j)_i < Z_{I(j)+}$  for each  $1 \leq i \leq b(j) - a(j) + 1$ ; the resulting flag is indeed bounded, since  $Z_{a(j), a(j)+i-1} < U(j)_i$  holds for all such  $i$  and  $1 \leq j \leq s$ . Any flag  $T$  in  $B(Z_{n+1})$  for which  $S(T) \subseteq Q$  must be of the form (5.4), so that  $i_Q$  has image  $X_Q$ , as required.  $\square$

We may therefore interpret the set

$$\mathcal{X}(n) = \{X_Q : Q \in \mathcal{B}(n)\}$$

as a Boolean algebra of nonsingular subvarieties of  $B(Z_{n+1})$ , ordered by inclusion, on which the support function  $S: \mathcal{X}(n) \rightarrow \mathcal{B}(n)$  induces an isomorphism of Boolean algebras. Moreover, whenever  $Q$  has type  $\omega$  then  $X_Q$  is isomorphic to the cartesian product  $B_1^{\omega_1} B_2^{\omega_2} \dots B_n^{\omega_n}$ , and so may be abbreviated to  $B^\omega$ . In this important sense,  $S$  preserves types. We note that the complex dimension  $|Q|$  of  $X_Q$  may be written as  $|\omega|$ .

The following quartet of lemmas forms the core of our calculus, and is central to computations in §6.

**Lemma 5.5.** *The map  $r_{Q'}: B(Z_{n+1}) \rightarrow \times_{Q'} CP(Z_{h,n+1})$  is transverse to the subvariety  $\times_{Q'} CP(Z_{h+1,n+1})$ , whose inverse image is  $X_Q$ .*

*Proof.* Let  $T$  be a flag in  $B(Z_{n+1})$ . Then  $r_h(T)$  lies  $CP(Z_{h+1,n+1})$  if and only if  $T_{h+1} = T_h \oplus L_h$  for some line  $L_h$  in  $Z_{h+1,n+1}$ . Since  $Z_h < T_{h+1}$ , this condition is equivalent to requiring that  $T_h = Z_h$ , and the proof is completed by allowing  $h$  to range over  $Q'$ .  $\square$

**Lemma 5.6.** *The map  $q_{Q'}: B(Z_{n+1}) \rightarrow \times_{Q'} CP(Z_{h,n+1})$  is transverse to the subvariety  $\times_{Q'} CP(Z_{h+1,n+1})$ , whose inverse image is diffeomorphic to  $B(Z_{Q^\wedge})$ .*

*Proof.* Let  $T$  be a flag in  $B(Z_{n+1})$  such that  $q_h(T)$  lies  $CP(Z_{h+1,n+1})$ , which occurs if and only if  $T_h = Z_{h-1} \oplus L_h$  for some line  $L_h$  in  $Z_{h+1,n+1}$ . Whenever this equation holds for all  $h$  in some interval  $[a, b]$ , we deduce that  $L_h$  actually lies in  $Z_{b+1,n+1}$ . Thus we may describe  $T$  globally by

$$T_k = Z_{[k-1] \setminus Q} \oplus U_i,$$

where  $U_i$  lies in  $Z_{Q^\wedge}$ , and  $i$  is  $k - |[k-1] \setminus Q|$ . Clearly  $U_{i-1} < U_i$  and  $Z_{\{q_1, \dots, q_{i-1}\}} < U_i$  for all appropriate  $i$ , so that  $U$  lies in  $B(Z_{Q^\wedge})$ . We may now identify the required inverse image with the image of the natural smooth embedding  $j_Q: B(Z_{Q^\wedge}) \rightarrow B(Z_{n+1})$ , as sought.  $\square$

We therefore define  $Y_Q$  to consist of all flags  $T$  for which the line  $T_h - Z_{h-1}$  lies in  $Z_{Q^\wedge}$  for every  $h$  in  $Q'$ . Since  $Y_{[n]}$  is  $B(Z_{n+1})$  (and  $Y_\emptyset$  is the singleton standard flag), the set

$$\mathcal{Y}(n) = \{Y_Q : Q \in \mathcal{B}(n)\}$$

is also a Boolean algebra of nonsingular subvarieties. In this instance, however,  $Y_Q$  is isomorphic to  $B_k$  whenever  $Q$  has cardinality  $k$ , irrespective of type. We may consider  $X_Q$  and  $Y_{Q'}$  to be complementary, insofar as the supports of the constituent flags satisfy  $S(T) \subseteq Q$  and  $Q \subseteq S(T)$  respectively.

**Lemma 5.7.** *For any  $1 \leq m \leq n - h$ , the map  $q_h: B(Z_{n+1}) \rightarrow CP(Z_{h,n+1})$  is transverse to the subvariety  $CP(Z_{h+m,n+1})$ , whose inverse image is diffeomorphic to  $Y_{[h, h+m-1]}$ .*

*Proof.* Let  $T$  be a flag in  $B(Z_{n+1})$  such that  $q_h(T)$  lies  $CP(Z_{h+m,n+1})$ , which occurs if and only if  $T_h = Z_{h-1} \oplus L_h$  for some line  $L_h$  in  $Z_{h+m,n+1}$ . Following the proof of Lemma 5.6 we immediately identify the required inverse image with  $Y_{[h-1] \cup [h+m, n]}$ , as sought.  $\square$

**Lemma 5.8.** *The following intersections in  $B(Z_{n+1})$  are transverse:*

$$X_Q \cap X_R = X_{Q \cap R} \quad \text{and} \quad Y_Q \cap Y_R = Y_{Q \cap R} \quad \text{whenever} \quad Q \cup R = [n],$$

$$\text{and} \quad X_Q \cap Y_R = \begin{cases} X_{Q \cap R, R} & \text{if } Q \cup R = [n] \\ \emptyset & \text{otherwise,} \end{cases}$$

where  $X_{Q \cap R, R}$  denotes the submanifold  $X_{Q \cap R} \subseteq B(Z_{R^\wedge})$ . Moreover,  $m$  copies of  $Y_{\{h\}'}$  may be made self-transverse so that

$$Y_{\{h\}'} \cap \cdots \cap Y_{\{h\}'} = Y_{[h, h+m-1]'}$$

for each  $1 \leq h \leq n$  and  $1 \leq m \leq n - h$ .

*Proof.* The first three formulae follow directly from the definitions, and dimensional considerations ensure that the intersections are transverse. The manifold  $X_{Q \cap R, R}$  is diffeomorphic to  $\times_j Y_{R'(j)}$  as a submanifold of  $B(Z_{n+1})$ , where  $Q = \cup_j I(j)$  and  $R'(j) = R' \cap I(j)$  for each  $1 \leq j \leq s$ .

Since  $Y_{\{h\}'}$  is defined by the single constraint  $U_h = Z_{h-1} \oplus L_h$ , where  $L_h$  is a line in  $Z_{h+1, n+1}$ , we may deform the embedding  $j_{\{h\}'}$  (through smooth embeddings, in fact) to  $m - 1$  further embeddings in which the  $L_h$  is constrained to lie in  $Z_{[h, n+1] \setminus \{h+i-1\}}$ , for each  $2 \leq i \leq m$ . The intersection of the  $m$  resulting images is determined by the single constraint  $L_h < Z_{h+m, n+1}$ , and the result follows by applying Lemma 5.7.  $\square$

We conclude this section with a study of the  $U$ - and double  $U$ -structures on  $B(Z_{n+1})$ , for which a few bundle-theoretic preliminaries are required.

For each  $1 \leq i \leq n$  we consider the complex line bundles  $\gamma_i$  and  $\rho_i$ , classified respectively by the maps  $q_i$  and  $r_i$ . It is consistent to take  $\gamma_0$ ,  $\gamma_{n+1}$ , and  $\rho_0$  to be  $0$ ,  $\mathbb{C}$  and  $\gamma_1$  respectively, from which we deduce that

$$(5.9) \quad \gamma_i \oplus \rho_i \oplus \rho_{i+1} \oplus \cdots \oplus \rho_n \cong \mathbb{C}^{n-i+2}$$

for every  $i$ . Since we may use (5.1) as in [24] to obtain an expression of the form  $\tau \oplus \mathbb{R} \cong (\oplus_{i=2}^{n+1} \gamma_i) \oplus \mathbb{R}$  for the tangent bundle of  $B(Z_{n+1})$ , so (5.9) leads to an isomorphism  $\nu \cong \oplus_{i=2}^n (i-1)\rho_i$ . We refer to the resulting  $U$ -structure as the *basic*  $U$ -structure on  $B(Z_{n+1})$ . We emphasise that these isomorphisms are of real bundles only, and therefore that the basic  $U$ -structure does not arise from the underlying complex algebraic variety. On  $B(Z_2)$ , for example, the basic  $U$ -structure is that of a 2-sphere  $S^2$ , rather than  $CP^1$ . Indeed, the basic  $U$ -structure on  $B(Z_{n+1})$  extends over the 3-disc bundle associated to  $\gamma_1 \oplus \mathbb{R}$  for all values of  $n$ , so that  $B(Z_{n+1})$  represents zero in  $\Omega_{2n}^U$ .

If we split  $\nu$  so that  $\nu_\ell$  is  $\bigoplus_{i=1}^n i\rho_i$  and  $\nu_r$  is  $\gamma_1$  (appealing to (5.9)), we again refer to the resulting double  $U$ -structure as *basic*; equivalently, we may rewrite  $\nu_\ell$  stably as  $-(\gamma_1 \oplus \cdots \oplus \gamma_n)$ . The basic double  $U$ -structure does not bound, however, as we shall see in Proposition 5.10. Given any cartesian product of manifolds  $B(Z_{n+1})$ , we also refer to the product of basic structures as *basic*.

We may now formulate the fundamental connection between bounded flag manifolds and the Landweber-Novikov algebra.

**Proposition 5.10.** *With the basic double  $U$ -structure,  $B(Z_{n+1})$  represents  $g_n$  in  $\Omega_*^{DU}$ ; if the left and right components of  $\nu$  are interchanged, it represents  $\bar{g}_n$ .*

*Proof.* Applying Proposition 3.5, the image of the double  $U$ -cobordism class of  $B(Z_{n+1})$  is represented by the singular  $U$ -manifold  $\gamma_1: B(Z_{n+1}) \rightarrow BU$  in  $\Omega_{2n}^U(BU_+)$  under the isomorphism  $h$ . Since  $\gamma_1$  lifts to  $CP^\infty$  this class is  $\beta_n$ , as proven in [24]. Appealing to Proposition 3.8 completes the proof for  $g_n$ , and the result follows for  $\bar{g}_n$  by applying the involution  $\chi$ .  $\square$

**Corollary 5.11.** *Under the canonical isomorphism  $G_* \cong S_*$ , the cobordism classes of the basic double  $U$ -manifolds  $X_Q$  give an additive basis for the dual of the Landweber-Novikov algebra, as  $Q$  ranges over finite subsets of  $[\infty]$ .*

*Proof.* By Proposition 4.27, the canonical isomorphism identifies the monomials  $g^\omega$  and  $b^\omega$ . Applying Proposition 5.10 to cartesian products, we deduce that  $X_Q$  (and therefore  $B^\omega$ ) represents  $g^\omega$  whenever  $X_Q$  carries the basic double  $U$ -structure and  $Q$  has type  $\omega$ . The result follows.  $\square$

Henceforth we shall insist that  $B_n$  denotes  $B(Z_{n+1})$  (or any isomorph) only when equipped with the basic double  $U$ -structure.

**Proposition 5.12.** *Both  $\mathcal{X}(n)$  and  $\mathcal{Y}(n)$  are Boolean algebras of basic  $U$ -submanifolds, in which the intersection formulae of Lemma 5.8 respect the basic  $U$ -structures.*

*Proof.* It suffices to prove that the pullbacks in Lemmas 5.5, 5.6 and 5.7 are compatible with the basic  $U$ -structures. Beginning with Lemma 5.5, we note that whenever  $\rho_h$  over  $B(Z_{n+1})$  is restricted by  $i_Q$  to a factor  $B(Z_{I(j)+})$ , we obtain  $\rho_{k+1}$  if  $h = a(j) + k$  lies in  $I(j)$  and  $\gamma_1$  if  $h = a(j) - 1$  (unless  $1 \in Q$ ); for all other values of  $h$ , the restriction is trivial. Since the construction of Lemma 5.5 identifies  $\nu(i_Q)$  with the restriction of  $\bigoplus_h \rho_h$  as  $h$  ranges over  $Q'$ , we infer an isomorphism  $\nu(i_Q) \cong (\times_j \gamma_1) \oplus \mathbb{C}^{n-j-|Q|}$  over  $X_Q$  (unless  $1 \in Q$ , in which case the first  $\gamma_1$  is trivial). Appealing to (5.9), we then verify that this is compatible with the basic structures in the isomorphism  $\nu^{X_Q} \cong (i_Q)^* \nu^{B(Z_{n+1})} \oplus \nu(i_Q)$ , as claimed. The proofs for Lemmas 5.6 and 5.7 are similar, noting that the restriction of  $\rho_h$  to  $Y_Q$  is  $\rho_{k+1}$  if  $h = a(j) + k$  lies in  $Q$ , and is trivial otherwise, and that the restriction of  $\gamma_h$  is  $\gamma_k$  if  $h = q_k$  lies in  $Q$ , and is  $\gamma_{k+1}$  if  $q_k$  is the greatest element of  $Q$  for which  $h > q_k$  (meaning  $\gamma_1$  if  $h < q_1$ , and the trivial bundle if  $h > q_k$  for all  $k$ ). Since the construction of Lemma 5.6 identifies  $\nu(j_Q)$  with the restriction of  $\bigoplus_h \gamma_h$  as  $h$  ranges over  $Q'$ , we infer an isomorphism

$$(5.13) \quad \nu(j_Q) \cong \bigoplus_{j=1}^{s+1} \left( a(j) - b(j-1) - 1 \right) \gamma_{c(j)}$$

over  $Y_Q$ , where  $c(j) = j + \sum_{i=0}^{j-1} (b(i) - a(i))$ . This isomorphism is also compatible with the basic structures in  $\nu^{Y_Q} \cong (j_Q)^* \nu^{B(Z_{n+1})} \oplus \nu(j_Q)$ , once more by appeal to (5.9).  $\square$

The corresponding results for double  $U$ -structures are more subtle, since we are free to choose our splitting of  $\nu(i_Q)$  and  $\nu(j_Q)$  into left and right components.

**Corollary 5.14.** *The same results hold for double  $U$ -structures with respect to the splittings  $\nu(i_Q)_\ell = 0$  and  $\nu(i_Q)_r = \nu(i_Q)$ , and  $\nu(j_Q)_\ell = \nu(j_Q)$  and  $\nu(j_Q)_r = 0$ .*

*Proof.* One extra fact is required in the calculation for  $i_Q$ , namely that  $\gamma_1$  on  $B(Z_{n+1})$  restricts trivially to  $X_Q$  (or to  $\gamma_1$  if  $1 \in Q$ ).  $\square$

At this juncture we may identify the inclusions of  $X_Q$  in  $F(Z_{n+1})$  with certain of the desingularizations introduced by Bott and Samelson [4]; for example,  $X_{[n]}$  provides the desingularization of the Schubert variety  $X_{(n+1,1,2,\dots,n)}$ . Moreover, the corresponding  $U$ -cobordism classes form the cornerstone of Bressler and Evens's calculus for  $\Omega_U^*(F(Z_{n+1}))$ . In both of these applications, however, the underlying complex manifold structures suffice. The basic  $U$ -structures become vital when investigating the Landweber-Novikov algebra (and could also have been used in [5], although a different calculus would result). We leave the details to interested readers.

## 6. COMPUTATIONS AND FORMULAE

In this section we study the normal characteristic numbers of bounded flag manifolds, and deduce formulae for the actions of various cohomology operations on the corresponding bordism classes. These allow us to provide our promised geometrical realization of many of the algebraic structures of the Landweber-Novikov algebra, and its dual and quantum double.

We begin by recalling the CW decomposition of  $B(Z_{n+1})$  resulting from Lemma 5.2, and noting that the cells  $e_Q$  define a basis for the cellular chain complex. Since they occur only in even dimensions, the corresponding homology classes  $x_Q^H$  form a  $\mathbb{Z}$ -basis for the integral homology groups  $H_*(B(Z_{n+1}))$  as  $Q$  ranges over  $\mathcal{B}(n)$ . Applying  $\text{Hom}_{\mathbb{Z}}$ , we obtain a dual basis  $Hd(x_Q^H)$  for the integral cohomology groups  $H^*(B(Z_{n+1}))$ ; we delay clarifying the cup product structure until after Theorem 6.2 below.

We introduce the complex bordism classes  $x_Q$  and  $y_Q$  in  $\Omega_{2|Q|}^U(B(Z_{n+1}))$ , represented respectively by the inclusions  $i_Q$  and  $j_Q$  of the subvarieties  $X_Q$  and  $Y_Q$  with their basic  $U$ -structures. By construction, the fundamental class in  $H_{2|Q|}(X_Q)$  maps to  $x_Q^H$  in  $H_{2|Q|}(B(Z_{n+1}))$  under  $i_Q$ ; thus  $x_Q$  maps to  $x_Q^H$  under the homomorphism  $\Omega_*^U(B(Z_{n+1})) \rightarrow H_*(B(Z_{n+1}))$  induced (as described in §3) by the Thom class  $t^H$ . The Atiyah-Hirzebruch spectral sequence for  $\Omega_*^U(B(Z_{n+1}))$  therefore collapses, and the classes  $x_Q$  form an  $\Omega_*^U$ -basis as  $Q$  ranges over  $\mathcal{B}(n)$ . The classes  $x_{[n]}$  and  $y_{[n]}$  coincide, since they are both represented by the identity map. They constitute the *basic fundamental class* in  $\Omega_{2n}^U(B(Z_{n+1}))$ , with respect to which the Poincaré duality isomorphism is given by

$$Pd(w) = w \cap x_{[n]}$$

in  $\Omega_{2(n-d)}^U(B(Z_{n+1}))$ , for any  $w$  in  $\Omega_U^{2d}(B(Z_{n+1}))$ .

An alternative source of elements in  $\Omega_U^2(B(Z_{n+1}))$  is provided by the Chern classes

$$x_i = c_1(\gamma_i) \quad \text{and} \quad y_i = c_1(\rho_i)$$

for each  $1 \leq i \leq n$ . It follows from (5.9) that

$$(6.1) \quad x_i = -y_i - y_{i+1} - \cdots - y_n$$

for every  $i$ . Given  $Q \subseteq [n]$ , we write  $\prod_Q x_h$  as  $x^Q$  and  $\prod_Q y_h$  as  $y^Q$  in  $\Omega_U^{2|Q|}(B(Z_{n+1}))$ , where  $h$  ranges over  $Q$  in both products.

We may now discuss the implications of our intersection results of Lemma 5.8 for the structure of  $\Omega_U^*(B(Z_{n+1}))$ . It is convenient (but by no means necessary) to use Quillen's geometrical interpretation of cobordism classes, which provides a particularly succinct description of cup and cap products and Poincaré duality, and is conveniently summarized in [5].

**Theorem 6.2.** *The complex bordism and cobordism of  $B(Z_{n+1})$  satisfy*

- (1)  $Pd(x^{Q'}) = y_Q$  and  $Pd(y^{Q'}) = x_Q$ ;
- (2) the elements  $\{y_Q : Q \subseteq [n]\}$  form an  $\Omega_*^U$ -basis for  $\Omega_*^U(B(Z_{n+1}))$ ;
- (3)  $Hd(x_Q) = x^Q$  and  $Hd(y_Q) = y^Q$ ;
- (4) there is an isomorphism of rings

$$\Omega_U^*(B(Z_{n+1})) \cong \Omega_*^U[x_1, \dots, x_n]/(x_i^2 = x_i x_{i+1}),$$

where  $i$  ranges over  $[n]$  and  $x_{n+1}$  is interpreted as 0.

*Proof.* For (1), we apply Lemma 5.6 and Proposition 5.12 to deduce that  $x^{Q'}$  in  $\Omega_U^{2|Q'|}(B(Z_{n+1}))$  is the pullback of the Thom class under the collapse map onto  $M(\nu(j_Q))$ . Hence  $x^{Q'}$  is represented geometrically by the inclusion  $j_Q: Y_Q \rightarrow B(Z_{n+1})$ , and therefore  $Pd(x^{Q'})$  is represented by the same singular  $U$ -manifold in  $\Omega_{2|Q|}(B(Z_{n+1}))$ . Thus  $Pd(x^{Q'}) = y_Q$ . An identical method works for  $Pd(y^{Q'})$ , by applying Lemma 5.5. For (2), we have already shown that the  $x_Q$  form an  $\Omega_*^U$ -basis for  $\Omega_*^U(B(Z_{n+1}))$ . Thus by (1) the  $y^Q$  form a basis for  $\Omega_U^*(B(Z_{n+1}))$ , and therefore so do the  $x^Q$  by (6.1); the proof is concluded by appealing to (1) once more. To establish (3), we remark that the cap product  $x^Q \cap x_R$  is represented geometrically by the fiber product of  $j_{Q'}$  and  $i_R$ , and is therefore computed by the intersection theory of Lemma 5.8. Bearing in mind the crucial fact that each basic  $U$ -structure bounds (except in dimension zero!), we obtain

$$(6.3) \quad \langle x^Q, x_R \rangle = \delta_{Q,R}$$

and therefore that  $Hd(x_Q) = x^Q$ , as sought. The result for  $Hd(y_Q)$  follows similarly. To prove (4) we note that it suffices to obtain the product formula  $x_i^2 = x_i x_{i+1}$ , since we have already demonstrated that the monomials  $x^Q$  form a basis in (2). Now  $x_i$  and  $x_{i+1}$  are represented geometrically by  $Y_{\{i\}}$  and  $Y_{\{i+1\}}$  respectively, and products are represented by intersections; according to Lemma 5.8 (with  $m = 2$ ), both  $x_i^2$  and  $x_i x_{i+1}$  are therefore represented by the same subvariety  $Y_{\{i, i+1\}}$ , so long as  $1 \leq i < n$ . When  $i = n$  we note that  $x_n$  pulls back from  $CP^1$ , so that  $x_n^2 = 0$ , as required.  $\square$

For any  $Q \subseteq [n]$ , we obtain the corresponding structures for the complex bordism and cobordism of  $X_Q$  by applying the Künneth formulae to Theorem 6.2. Using the same notation as in  $B(Z_{n+1})$  for any cohomology

class which restricts along (or homology class which factors through) the inclusion  $i_Q$ , we deduce, for example, a ring isomorphism

$$(6.4) \quad \Omega_U^*(X_Q) \cong \Omega_*^U[x_i : i \in Q]/(x_i^2 = x_i x_{i+1}),$$

where  $x_i$  is interpreted as 0 for all  $i \notin Q$ .

The relationship between the classes  $x_i$  and  $y_i$  in  $\Omega_U^*(B(Z_{n+1}))$  is described by (6.1), but may be established directly by appeal to the third formula of Lemma 5.8, as in the proof of Theorem 6.2; for example, we deduce immediately that  $x_i y_i = 0$  for all  $1 \leq i \leq n$ . When applied with arbitrary  $m$ , the fourth formula of Lemma 5.8 simply iterates the quadratic relations, and produces nothing new.

Of course we may extend the results of Theorem 6.2 and its corollaries to any complex oriented spectrum  $E$ . We define  $x_Q^E$  and  $y_Q^E$  in  $E_{2|Q|}(B(Z_{n+1}))$  to be the respective images of the  $E$ -homology fundamental classes of  $X_Q$  and  $Y_Q$ , and  $x_E^Q$  and  $y_E^Q$  in  $E^{2|Q|}(B(Z_{n+1}))$  to be the appropriate monomials in the  $E$ -cohomology Chern classes of the  $\gamma_i$  and  $\rho_i$  respectively. We apply the Thom class  $t^E$  to deduce that  $x_E^{Q'}$  and  $y_E^Q$ , and  $y_E^{Q'}$  and  $x_E^Q$ , are Poincaré dual; that  $x_Q^E$  and  $x_E^Q$ , and  $y_Q^E$  and  $y_E^Q$ , are  $\text{Hom}_{E_*}$ -dual; and that there is an isomorphism of rings

$$(6.5) \quad E^*(B(Z_{n+1})) \cong E_*[x_1^E, \dots, x_n^E]/((x_i^E)^2 = x_i^E x_{i+1}^E),$$

where  $i$  ranges over  $[n]$  and  $x_{n+1}^E$  is zero. In particular (6.5) applies to integral cohomology, and completes the study begun at the start of the section. The analogue of (6.4) is immediate.

We may substitute any doubly complex oriented spectrum  $D$  for  $E$  in (6.5), on the understanding that left or right Chern classes must be chosen consistently throughout. Duality, however, demands extra care and attention, and we take our cue from the universal example. We have to consider the choice of splittings provided by Corollary 5.14, and the failure of formulae such as (6.3) because the manifolds  $B_n$  are no longer double  $U$ -boundaries.

We are particularly interested in the left and right Chern classes  $x_\ell^Q, y_\ell^Q, x_r^Q$  and  $y_r^Q$  in  $\Omega_{DU}^{2|Q|}(B_n)$ , and we seek economical geometric descriptions of their Poincaré duals. We continue to write  $x_R$  and  $y_R$  in  $\Omega_{2|R|}^{DU}(B_n)$  for the homology classes represented by the respective inclusions of  $X_R$  and  $Y_R$  with their basic double  $U$ -structures.

**Proposition 6.6.** *In  $\Omega_{2(n-|Q|)}^{DU}(B_n)$ , we have that*

$$\text{Pd}(x_\ell^{Q'}) = y_Q \quad \text{and} \quad \text{Pd}(y_r^{Q'}) = x_Q,$$

whilst  $\text{Pd}(x_r^{Q'})$  and  $\text{Pd}(y_\ell^{Q'})$  are represented by the inclusion of  $Y_Q$  and  $X_Q$  with the respective double  $U$ -structures

$$(\nu^{Y_Q} - (\nu(j_Q) \oplus j_Q^* \gamma_1)) \oplus (\nu(j_Q) \oplus j_Q^* \gamma_1) \quad \text{and} \quad (\nu^{X_Q} - i_Q^* \gamma_1) \oplus i_Q^* \gamma_1,$$

for all  $n \geq 0$ .

*Proof.* The first two formulae follow at once from Corollary 5.14, by analogy with (1) of Theorem 6.2. The second two formulae require the interchange of the left and right components of the normal bundles of  $j_Q$  and  $i_Q$  respectively.  $\square$

We extend to  $X_Q$  in the obvious fashion.

Corresponding results for general  $D$  are immediate, so long as we continue to insist that  $x_Q^D$  and  $y_Q^D$  in  $D_{2|Q|}^{DU}(B_n)$  are induced from the universal example by the Thom class  $t^D$ .

We apply Proposition 6.6 to compute the effect of the normal bundle map

$$\nu_\ell \times \nu_r: X_Q \rightarrow BU \times BU$$

in  $\Omega_{DU}^*(\ )$ -theory, for any  $Q \subseteq [n]$ . By Corollary 5.11, this suffices to describe  $\nu_\ell \times \nu_r$  on our monomial basis for  $G_*$ . To ease computation, we consider an alternative  $E_*$ -basis  $(\bar{\beta}^E)^\psi$  for  $E_*(BU)$  (given any complex oriented spectrum  $E$ ); this is defined as the image of the standard basis  $(\beta^E)^\psi$  under the homomorphism  $\perp_*$ , induced by the involution  $\perp: BU \rightarrow BU$  of orthogonal complementation. Since  $\perp_*$  acts by reciprocating the formal sum  $\beta^E$ , each  $\bar{\beta}_n^E$  is an integral homogeneous polynomial in  $\bar{\beta}_1^E, \dots, \bar{\beta}_n^E$ . Moreover,  $\perp$  is a map of H-spaces with respect to Whitney sum, so that the relations

$$(6.7) \quad (\bar{\beta}_1^E)^{\psi_1} (\bar{\beta}_2^E)^{\psi_2} \dots (\bar{\beta}_n^E)^{\psi_n} = (\bar{\beta}^E)^\psi$$

continue to hold, and the elements  $\bar{\beta}_n^E$  again form a polynomial basis. Since  $\perp^*$  is an involution, the dual basis  $\bar{c}_\psi^E$  for  $E^*(BU)$  is obtained by applying  $\perp^*$  to  $c_\psi^E$ . We investigate the normal bundle map in terms of the basis  $\bar{\beta}^\psi \otimes \beta^\omega$  for  $\Omega_{DU}^*(BU \times BU)$ .

Fixing the subset  $Q = \cup_j I(j)$  of  $[n]$ , we consider the set  $H(Q)$  of non-negative integer sequences  $h$  of the form  $(h_1, \dots, h_n)$ , where  $h_i = 0$  for any  $i \notin Q$ ; for any such sequence  $h$ , we set  $|h| = \sum_i h_i$ . Whenever  $h$  satisfies  $\sum_{i=l}^{b(j)} h_i \leq b(j) - l + 1$  for all  $a(j) \leq l \leq b(j)$ , we define the subset  $hQ \subseteq Q$  by

$$\left\{ m : \sum_{i=l}^m h_i < m - l + 1 \text{ for all } a(j) \leq l \leq m \leq b(j) \right\};$$

otherwise, we set  $hQ = Q$ . For each  $j$  we write  $I(j) \cap hQ$  as  $I(j, h)$ , and whenever  $I(j, h)$  is nonempty we denote its minimal element by  $m(j, h)$ ; we then define subsets

$$A(h) = \{m(j, h) : m(j, h) = a(j)\} \quad \text{and} \quad M(h) = \{m(j, h) : m(j, h) > a(j)\}$$

of  $hQ$ . We identify the subset  $K(Q) \subseteq H(Q)$  of sequences  $k$  for which  $k_i$  is nonzero only if  $i = a(j)$  for some  $1 \leq j \leq s$ . Finally, it is convenient to partition  $K(Q)$  and  $H(Q)$  into compatible blocks  $K(Q, \theta)$  and  $H(Q, \theta)$  for every indexing sequence  $\theta$ ; each block consists of those sequences  $k$  or  $h$  which have  $\theta_i$  entries  $i$  for each  $i \geq 1$ , and all other entries zero. Thus, for example,  $|h| = |\theta|$  for all  $h$  in  $H(Q, \theta)$ . Any such block will be empty whenever  $\theta$  is incompatible with  $Q$  in the appropriate sense.

With this data, and for each  $k$  in  $K(Q)$  and  $h$  in  $H(kQ)$ , we follow the notation of Lemma 5.8 and write  $X_{hkQ, h[n]}^{k+1}$  for the manifold  $X_{hkQ, h[n]}$

equipped with the double  $U$ -structure

$$\nu_\ell = -\left( \bigoplus_{khQ} \gamma_i \oplus \bigoplus_{A(h)} k_{a(j)} \gamma_{m(j,h)} \right) \quad \text{and}$$

$$\nu_r = \bigoplus_{A(h)} (k_{a(j)} + 1) \gamma_{m(j,h)} \oplus \bigoplus_{M(h)} \gamma_i;$$

we note that  $m(j, h) = a(j) + k_{a(j)}$  for each  $m(j, h)$  in  $A(h)$ .

**Theorem 6.8.** *When applied to the basic fundamental class in  $\Omega_{2|Q|}^{DU}(X_Q)$ , the normal bundle map yields*

$$\sum_{\theta, \omega} \left( \sum_{K(Q, \omega)} \sum_{H(kQ, \theta)} g(k; h) \right) \bar{\beta}^\theta \otimes \beta^\omega,$$

where the first summation ranges over all  $\theta$  and  $\omega$  such that  $|\theta| + |\omega| \leq |Q|$ , and  $g(k; h) \in \Omega_{2(|Q|-|k|-|h|)}^{DU}$  is represented by  $X_{hkQ, h[n]}^{k+1}$  for all  $k \in K(Q)$  and  $h \in H(kQ)$ .

*Proof.* We compute the coefficient of  $\bar{\beta}_{h_1} \dots \bar{\beta}_{h_n} \otimes \beta_{k_1} \dots \beta_{k_n}$  by repeatedly applying Proposition 6.6, bearing in mind that the product structure in  $\Omega_{DU}^*(B_n)$  allows us to replace any  $x_i^m$  (either left or right) by  $x^{[i, i+m-1]}$  when  $[i, i+m-1] \subseteq Q$ , and zero otherwise; indeed, the definitions of  $H(Q)$  and  $K(Q)$  are tailored exactly to these relations. The computation is straightforward, although the bookkeeping demands caution, and yields  $\sum_{K(Q, \omega)} \sum_{H(kQ, \psi)} g(k; h)$ . We conclude by amalgamating the coefficients of those monomials  $\bar{\beta}_{h_1} \dots \bar{\beta}_{h_n}$  and  $\beta_{k_1} \dots \beta_{k_n}$  which give  $\bar{\beta}^\theta =$  and  $\beta^\omega$  respectively.  $\square$

Readers may observe that our expression in §5 for  $\nu_\ell$  as the sum of line bundles  $\bigoplus_{i=1}^n i\rho_i$  appears to circumvent the need to introduce the classes  $\bar{\beta}_n$ . However, it contains  $n(n+1)/2$  summands rather than  $n$ , and their Chern classes  $y_i$  are algebraically more complicated than the  $x_i$  used above, by virtue of (6.1). These two factors conspire to make the alternative calculations less attractive, and it is an instructive exercise to reconcile the two approaches in simple special cases. The apparent dependence of Theorem 6.8 on  $n$  is illusory (and solely for notational convenience), since  $k_i$  and  $h_i$  are zero whenever  $i$  lies in  $Q'$ .

By combining Corollary 5.11 with (4.32) we may read off the values of the double cobordism operations  $\bar{s}_\theta \otimes s_\omega$  on the monomial basis for  $G_*$ . The  $\bar{s}_\theta$  are occasionally referred to in the literature as *tangential* Landweber-Novikov operations, and may be expressed in terms of the original  $s_\psi$  by applying  $\perp_*$  and (6.7). We write

$$(6.9) \quad \bar{\beta}^\theta = \sum_{\psi} \lambda_{\theta, \psi} \beta^\psi,$$

where the  $\lambda_{\theta, \psi}$  are integers, the summation ranges over sequences  $\psi$  for which  $|\psi| = |\theta|$  and  $\sum \psi_i \geq \sum \theta_i$ , and the equation holds good for both the left and the right  $\beta$ s. We illustrate the procedure in the following important special cases.

**Corollary 6.10.** *Up to double  $U$ -cobordism, the actions of  $S_\ell^*$  and  $S_r^*$  on monomial generators of  $G_*$  are represented respectively by*

$$s_{\psi,\ell}(X_Q) = \sum_{\theta} \sum_{H(Q,\theta)} \lambda_{\theta,\psi} X_{hQ,h[n]} \quad \text{and} \quad s_{\omega,r}(X_Q) = \sum_{K(Q,\omega)} X_{kQ}^{k+1},$$

where the first summation ranges over sequences  $\theta$  for which  $|\theta| = |\psi|$  and  $\sum \theta_i \leq \sum \psi_i$ .

*Proof.* For  $s_{\psi,\ell}(X_Q)$ , we need the coefficient of  $\beta^\psi \otimes 1$  in Theorem 6.8. This is obtained by first setting  $k = 0$ , so that  $kQ = Q$ , then collecting together monomials  $\bar{\beta}_{h_1} \dots \bar{\beta}_{h_n} \otimes 1$  into the appropriate  $\beta^\theta$ s and applying (6.9). For  $s_{\omega,r}(X_Q)$  we set  $h = 0$ , so that  $hQ = Q$ , and apply (6.9) in the corresponding fashion.  $\square$

We expect this result to provide a purely geometrical confirmation of Corollary 4.29, that  $G_*$  is closed under the action of  $S^* \otimes S^*$  on  $\Omega_*^{DU}$ ; however, it remains to show that  $X_{kQ}^{k+1}$  lies in  $G_*$ ! We confirm that this is the case after Proposition 6.15(2) below, but would prefer a more explicit proof.

We may specialize Corollary 6.10 to the cases when  $\psi$  and  $\omega$  are of the form  $\epsilon(k)$  for some integer  $0 \leq k \leq |Q|$ , or when  $Q = [n]$  (so that we are dealing with polynomial generators of  $G_*$ ), or both. We obtain

$$(6.11) \quad s_{\epsilon(k),\ell}(X_Q) = - \sum_j \sum_{i=a(j)}^{b(j)-k+1} Y_{Q \setminus [i, i+k-1]}$$

$$\text{and} \quad s_{\epsilon(k),r}(X_Q) = \sum_j X_{Q \setminus [a(j), a(j)+k-1]}^{k+1},$$

where the summations range over those  $j$  for which  $b(j) - a(j) \geq k - 1$ ,

$$(6.12) \quad s_{\psi,\ell}(X_{[n]}) = \sum_{\theta} \sum_{H([n],\theta)} \lambda_{\theta,\psi} Y_{h[n]}$$

$$\text{and} \quad s_{\omega,r}(X_{[n]}) = \begin{cases} X_{[k+1,n]}^{k+1} & \text{when } \omega = \epsilon(k) \\ 0 & \text{otherwise,} \end{cases}$$

and

$$(6.13) \quad s_{\epsilon(k),\ell}(X_{[n]}) = - \sum_{i=1}^{n-k+1} Y_{[i-1] \cup [i+k,n]}.$$

These follow from the respective facts;  $\lambda_{\theta,\epsilon(k)} = -1$  when  $\theta = \epsilon(k)$ , and is zero otherwise;  $K(Q, \epsilon(k))$  consists solely of sequences containing a single nonzero entry  $k$  in some position  $a(j)$ ; and  $K([n], \omega)$  is empty unless  $\omega = \epsilon(k)$  for some  $0 \leq k \leq n$ .

We may rewrite the proof of Theorem 6.8 by describing the duality in more algebraic fashion. We suppose that  $g^\omega$  is represented by the variety  $X_Q$ , where  $Q = \cup_j I(j)$  and  $I(j) = [a(j), a(j) + t(j) - 1]$  for some sequence of integers  $(t(1), \dots, t(s))$  containing  $\omega_i$  entries  $i$  for each  $i \geq 1$  (which requires

that  $s = \sum_i \omega_i$ ). We compute the image in  $\Omega_{2|Q|}^{DU}(BU \times BU)$  of the basic fundamental class to be

$$(6.14) \quad \sum_{k_j, h_{j,1}, \dots, h_{j,t(j)}} \prod_{j=1}^s g_{t(j)-(k_j+h_{j,1}+\dots+h_{j,t(j)})} (\beta_\ell)_{h_{j,1}}^{-1} \cdots (\beta_\ell)_{h_{j,t(j)}}^{-1} \otimes \beta_{k_j, \ell},$$

where the summation ranges over all  $k_j + h_{j,1} + \dots + h_{j,t(j)} \leq t(j)$  such that  $h_{j,m} + \dots + h_{j,t(j)} \leq t(j) - m + 1$  for all  $2 \leq m \leq t(j)$  and  $1 \leq j \leq s$ . We may convert to our preferred basis for  $\Omega_*^{DU}(BU \times BU)$  by using Lemma 3.11(3) to express  $\beta_{k_j, \ell}$  in terms of the  $\beta_{n,r}$ .

Before summarizing our conclusions, we consider two fascinating applications of the proofs of Theorem 6.8 and (6.14).

**Proposition 6.15.** *We have that*

- (1) *the map  $\gamma_h: B_n \rightarrow CP^\infty$  represents either of the expressions*

$$\sum_{k=0}^{n+1-h} g_{n-k} \beta_{k, \ell} \quad \text{or} \quad \sum_{k=0}^n \sum_{j=0}^{n+1-h} g_{n-j} (g)_{j-k}^k \beta_{k, r}$$

*in  $\Omega_{2n}^{DU}(CP^\infty)$ , for each  $1 \leq h \leq n$ ;*

- (2) *the manifold  $X_{[k+1, n]}$  represents  $(g)_{n-k}^{k+1}$  in  $\Omega_{2(n-k)}^{DU}$  when equipped with the double  $U$ -structure*

$$(\nu - (k+1)\gamma_1) \oplus (k+1)\gamma_1,$$

*for all  $0 \leq k \leq n$ .*

*Proof.* For (1), we note that the coefficient of  $\beta_{k, \ell}$  in the first expression is given by  $\langle x_{h, \ell}^k, x_{[n]} \rangle$ ; setting  $Q = [n]$  in (6.14) and concentrating on the terms  $x_{h, \ell}^k$ , we obtain  $g_{n-k}$  when  $1 \leq k \leq n - h + 1$ , and zero otherwise, as required. To convert the result into the second expression, we apply 3.11(3).

For (2), the second expression identifies  $(g)_{n-k}^{k+1}$  as  $Pd(x_{1,r}^k)$  in  $B_n$ . Concentrating on the terms  $x_{1,r}^k$  in Theorem 6.8 (or (6.12)), we deduce that this is represented by  $X_{[k+1, n]}^{k+1}$ , as sought.  $\square$

The formulae in (1) reflect the fact that the map  $\gamma_h$  factors through the  $(n - h + 1)$ -skeleton of  $CP^\infty$ . Similar arguments for arbitrary  $Q$  generalize (2) to show that  $X_{kQ}^{k+1}$  represents  $\prod_j (g)_{t(j)-k_j}^{k_j+1}$ , which lies in  $G_*$  (as we claimed after Corollary 6.10, and is also implicit in Theorem 6.16 below).

We conclude by summarizing the results that have motivated our entire work, using the canonical isomorphism to identify  $G_*$  and  $G_* \otimes G_*$  with  $S_*$  and  $S_* \otimes S_*$  respectively. We note that monomial generators of  $G_* \otimes G_*$  may be expressed as double  $U$ -cobordism classes of pairs of basic double  $U$ -manifolds  $(X_Q, X_R)$ , for appropriate subsets  $Q$  and  $R \subseteq [n]$ .

**Theorem 6.16.** *In the dual of the Landweber-Novikov algebra, the coproduct  $\delta$  and antipode  $\chi$  are induced by the maps*

$$X_Q \mapsto \sum_{\omega} \left( \sum_{K(Q, \omega)} X_{kQ}^{k+1}, X_{Q \setminus kQ} \right) \quad \text{and} \quad X_Q \mapsto \chi(X_Q)$$

up to double  $U$ -cobordism; similarly, in the quantum double  $D(S^*)$ , the commutation law is induced by

$$s_\omega X_Q = \sum_{\substack{\theta \\ \omega' + \omega'' = \omega}} \left( \sum_{K(Q, \omega'')} \sum_{H(k, Q, \theta)} \lambda_{\theta, \omega'} X_{hkQ, h[n]}^{k+1} \right),$$

where  $\theta$  ranges over those sequences for which  $\lambda_{\theta, \omega'}$  is nonzero.

*Proof.* For  $\delta$ , we apply Proposition 6.15(2) to Corollary 4.18, noting that  $K(Q, \omega)$  is empty unless  $\omega$  is compatible with  $Q$ ; for  $\chi$ , we refer in addition to Proposition 5.10. For the commutation law, we apply (6.9) to Theorem 6.8 then appeal to the definition.  $\square$

Referring back to (6.13), we deduce the following special case of the commutation law

$$s_{\epsilon(k)} X_{[n]} = - \sum_{i=1}^{n-k+1} Y_{[i-1] \cup [i+k, n]} + X_{[k+1, n]}^{k+1} + X_{[n]} s_{\epsilon(k)}$$

for any  $k \leq n$  and up to double  $U$ -cobordism. Taken with Corollary 6.10, our results provide geometric confirmation of the formulae of (4.24) and (4.31), once we have identified  $Y_{[i-1] \cup [i+k, n]}$  with  $b_{n-k}$  and  $X_{[k+1, n]}^{k+1}$  with  $(b)_{n-k}^{k+1}$  under the canonical isomorphism.

Intriguingly, we may represent the elements of  $\Omega_*^{DU} \otimes_{\Omega_*^U} \Omega_*^{DU}$  by *threefold*  $U$ -manifolds, so that  $\delta$  is induced on a manifold  $M$  by modifying the double  $U$ -structure from  $\nu_\ell \oplus \nu_r$  to  $\nu_\ell \oplus 0 \oplus \nu_r$ . The theory of multi  $U$ -cobordism is especially interesting, and has applications to the study of iterated doubles and the Adams-Novikov spectral sequence; we reserve our development of these ideas for the future.

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