CHROMATIC POLYNOMIALS OF PARTITION SYSTEMS

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ABSTRACT. The theory of the umbral chromatic polynomial of a simplicial complex provides a combinatorial framework for the study of formal group laws over a commutative, torsion free ring, and our aim in this work is to extend its definition to a class of set systems $\mathcal{P}$, which we label *partition systems*. When suitably evaluated, our polynomial $\chi^{\phi}(\mathcal{P}; \S)$ enumerates *factorized* colorings, as well as *coloring forests* of the partition system by type. These colorings are related to the Mullin-Rota concept of reluctant functions, and whenever $\mathcal{P}$ is a simplicial complex, they reduce to more familiar notions of coloring. Our three main results demonstrate how several properties of the classical chromatic polynomial $\chi(H; x)$, where H is a simple graph, may be generalized. Firstly, we prove that our polynomial $\chi^{\phi}(\mathcal{P}; \S)$ retains the property of being expressible as the characteristic polynomial of an appropriate poset, which holds for $\chi(H; x)$ by virtue of Whitney's original definition. Secondly, we provide two generalizations for the formula describing the chromatic polynomial of a disjoint union of graphs; one of these formulas depends explicitly on the context of partition systems and is not available when we restrict attention to graphs or simplicial complexes. Thirdly, we introduce partition systems with group action, thereby providing a combinatorial interpretation of normalized versions of our polynomials. In the case of the symmetric group acting on the trivial partition system, these are the normalized conjugate Bell polynomials, whose interpretation is a vital prerequisite for extending our framework to formal group laws over arbitrary rings of scalars; here, however, we concentrate solely on combinatorial aspects.

1. INTRODUCTION

The theory of formal group laws is rich and sophisticated, and has extensive applications in areas such as number theory and algebraic topology. Hazewinkel's book [5] gives an encyclopaedic account, and confirms that much of the associated algebra has a distinctly combinatorial flavour. It is therefore of interest to develop a purely combinatorial description for objects such as the Lazard ring, in the hope of finding new approaches to outstanding problems. The second author [8] has suggested that an appropriate setting is provided by the Roman-Rota umbral calculus [12] (with emphasis on the conjugate Bell polynomials $B_n^{\phi}(x)$), and by the related study of the umbral chromatic polynomial $\chi^{\phi}(H; x)$ of a simple graph H. This viewpoint has already proven fruitful in the context of applications to algebraic topology [6], and has been amplified by related advances in algebraic combinatorics, such as Stanley's symmetric function generalization [15] of the classical chromatic polynomial $\chi(H; x)$, which encodes the same information as our $\chi^{\phi}(H; x)$.

Any such H may be recovered from its *independence complex* $\mathcal{I}(\mathcal{H})$, which is the simplicial complex (or down closed set system) whose simplices consist of the independent (or edge-free) subsets of the vertices $V(H)$. Throughout, we identify a graph with its independence complex. Following Wagner [16], we define a coloring of an arbitrary set system $\mathcal{S}$ to be a map from the elements underlying $\mathcal{S}$ to a set of colors, with the property that the maximal monochromatic subsets are all members of $\mathcal{S}$. A coloring of $\mathcal{I}(\mathcal{H})$ is therefore synonymous with the standard notion of a coloring of H, and we may consider extending the umbral chromatic polynomial to $\chi^{\phi}(\mathcal{S};\S)$. The aim of any such generalization is to obtain properties which better reflect those of the classical chromatic polynomial, and this is the central theme of our work.

In fact we do not proceed in full generality, but introduce instead the concept of a partition system $\mathcal{P}$ as the appropriate compromise between competing constraints; certainly any up or down closed set system is such a $\mathcal{P}$. We then define $\chi^{\phi}(\mathcal{P};\S)$ so as to enumerate *factorized* colorings and *coloring forests* of $\mathcal{P}$, after appropriate evaluation; these are defined so as to take account of the fact that colorings of a non-simplicial complex $\mathcal{P}$ are likely to include monochromatic sets which are not in $\mathcal{P}$, and are closely related to the Mullin-Rota theory of reluctant functions [7].

In our first main result, namely Proposition 5.1, we deduce that $\chi^{\phi}(\mathcal{P};\S)$ may be expanded as the characteristic polynomial of an associated poset, thereby generalizing Whitney's original formula

$$\chi(H;x) = x^{n(H)}\, c(L_H;x); \tag{1.1}$$

here $n(H)$ denotes the number of connected components, and L_H the *lattice of contractions* (or *bond lattice*) of H, that is the set of all connected partitions of H, partially ordered by refinement. Such a description is available for the umbral chromatic polynomial of a simplicial complex, but it is important to confirm that it is not destroyed by generalization to $\mathcal{P}$. In passing, we explain how $\chi^{\phi}(\mathcal{P};\S)$ may be computed by means of a deletion-contraction procedure, which represents an analogue of the well-known addition-contraction procedure for graphs.

As our second result, in Propositions 5.9 and 5.15 we derive two product formulas for $\chi^{\phi}(\mathcal{P};\S)$ which generalize the classic

$$\chi(H_1 \sqcup H_2;x) = \chi(H_1;x)\,\chi(H_2;x); \tag{1.2}$$

the second formula depends on the context of partition systems for its very existence. In terms of application to formal group laws, this formula may best be interpreted in the context of Hopf algebras.

The framework of partition systems offers an additional bonus, and our third main result, Corollary 6.4. When $\mathcal{P}$ is equipped with the action of a finite group G, we give a combinatorial interpretation of the normalized polynomial $\chi^{\phi}(\mathcal{P};\S)/|\mathcal{G}|$, where $|G|$ denotes the order of the group. When G is the symmetric group S_n, this yields an interpretation of the normalized conjugate Bell polynomials $B_n^{\phi}(x)/n!$. These polynomials are essential for the study of formal group laws over arbitrary rings of scalars (as opposed to those which are merely torsion free), and we have long sought such an interpretation in order to extend the topological applications. Given the interests of our current readership, however, we offer only the skimpiest comments on formal group laws, and concentrate instead on a purely combinatorial exposition of partition systems with group action.

We should explain that we have previously proposed a definition of $\chi^{\phi}(\mathcal{S};\S)$ for arbitrary set systems in [10]. This differs from the $\chi^{\phi}(\mathcal{P};\S)$ introduced here and has less desirable properties; for example, it generally admits no simple expression as a characteristic polynomial. However, the two polynomials do agree when $\mathcal{P}$ is a simplicial complex, and therefore also for a graph H, by reference to its independence complex.

We now outline the contents of each section.

In §2 we define partition systems and various concepts related to their coloring, such as factorized colorings and coloring forests. In §3 we define our umbral chromatic polynomial $\chi^{\phi}(\mathcal{P};\S)$ in terms of chains of partitions of the vertices. In so doing, we describe how the appropriate evaluation functional yields an enumeration of the factorized colorings and of the coloring forests of $\mathcal{P}$ by type. In §4 we describe the characteristic type polynomial of the poset of divisions by $\mathcal{P}$ and deduce certain of its properties, including a deletion-contraction formula. These lead to our central technical result, Theorem 4.9, which we immediately apply to prove that the umbral chromatic polynomial agrees with the characteristic type polynomial of a poset obtained from $\mathcal{P}$ by complementation. In §5 we obtain properties of $\chi^{\phi}(\mathcal{P};\S)$ by using the latter description, which is especially suitable for the introduction of our promised product formulas. We also discuss the connection with Stanley's symmetric function generalization of the chromatic polynomial of a graph, which is suitably extended to simplicial complexes. Finally, in §6, we consider a finite group G acting on a set system $\mathcal{S}$ which contains all its singletons, and which is therefore a partition system. We define *ordered factorized colorings*, and deduce that the normalized polynomial $\chi^{\phi}(\mathcal{S};\S)/|\mathcal{G}|$ enumerates by type the orbits of G on the set of such colorings of $\mathcal{S}$; we also discuss its expansion in terms of the normalized conjugate Bell polynomials, and the divided powers of x.

We conclude this introduction by outlining certain definitions, conventions, and notation which we shall use below without further comment.

We shall always write $|V|$ for the cardinality of a given set V, and $[n]$ for the set of integers $\{1, 2, \dots, n\}$.

Given any finite non-empty set V of *vertices*, we refer to a collection of subsets $\mathcal{S} \subseteq \in^{\mathcal{V}}$ as a *set system* if $\emptyset \in \mathcal{S}$ and $V = \bigcup_{W \in \mathcal{S}} W$; since $\mathcal{S}$ uniquely determines the vertices, we denote V by $V(\mathcal{S})$ whenever $\mathcal{S}$ is in doubt. We label a partition σ of $V(\mathcal{S})$ which satisfies $\sigma \subseteq \mathcal{S}$ as a *division* by $\mathcal{S}$, and denote the set of such divisions by $\Pi(\mathcal{S})$. Note that $\Pi(2^V)$ is the set of all partitions of V, and is traditionally denoted by $\Pi(V)$. A coloring of $\mathcal{S}$ with at most m colors is therefore a map $f\colon V(\mathcal{S}) \to [\Updownarrow]$ whose kernel is a division by $\mathcal{S}$. In particular, when $\mathcal{S}$ is a simplicial complex, a coloring is characterized by having monochromatic simplices.

The polynomials we define will lie in a certain graded polynomial algebra $\Phi_*[x]$; here Φ_* denotes the graded ring $\mathbb{Z}[\phi_1, \phi_2, \dots]$, the element ϕ_i has degree i, and x has degree 1. More precisely, the polynomials associated with a partition system on n vertices will lie in the submodule $\Phi_n[x]$ of homogeneous polynomials of degree n. These degrees originate (in topological applications) as the dimensions of complex vector spaces, so that the signs which we might expect when commuting elements of odd degree do not arise. By definition, the powers x^n make up the *standard basis* for $\Phi_*[x]$ over Φ_*.

An alternative such basis arises from the Roman-Rota umbral calculus, and is given by the *conjugate Bell polynomials* $B^\phi_n(x)$, which are the universal sequence of polynomials satisfying the binomial identity

$$B^\phi_n(x+y) = \sum_{k=0}^{n} B^\phi_{n-k}(x)\, B^\phi_k(y),$$

(where $B^\phi_0(x) = 1$) for all $n \geq 0$. They are uniquely specified by insisting that $B^\phi_n(0) = 0$ for $n > 0$, and that the formal differential operator

$$\Delta^\phi = D + \phi_1 D^2/2! + \cdots + \phi_{k-1} D^k/k! + \ldots$$

acts such that $\Delta^\phi B^\phi_n(x) = n\, B^\phi_{n-1}(x)$ for all $n > 0$. Thus, for example,

$$B^\phi_1(x) = x, \qquad B^\phi_2(x) = x^2 - \phi_1 x, \qquad \text{and} \qquad B^\phi_3(x) = x^3 - 3\phi_1 x^2 + (3\phi_1^2 - \phi_2)x.$$

In fact Δ^ϕ may be identified with the exponential series for the universal one-dimensional formal group law. For each natural number m, there is a linear map $\Phi_*[x] \to \Phi_*$ defined by applying the operator $(1+\Delta^\phi)^m$ and then setting x to zero. We refer to this as *umbral substitution* $p(x) \mapsto p(m\phi)$, and remark that it acts on the conjugate Bell polynomials in such a way that $B^\phi_n(m\phi) = m(m-1)\ldots(m-n+1)$ for all $n > 0$.

In order to ensure that the *normalized* conjugate Bell polynomials $B^\phi_n(x)/n!$ are closed with respect to multiplication, we must extend Φ_* to the *Lazard ring* L_*, over which the universal formal group law is defined. After rationalization (by applying $\otimes\mathbb{Q}$), both L_* and Φ_* are isomorphic to $\mathbb{Q}[\phi_1, \phi_2, \ldots]$. The free L_*-module with basis the normalized conjugate Bell polynomials is the underlying module of the *covariant bialgebra* of the universal formal group law. We remark that $B^\phi_n(x)/n!$ continues to yield an integer, namely $\binom{m}{n}$, after umbral substitution by $m\phi$.

Given a sequence α of elements $(1, \alpha_1, \alpha_2, \ldots)$ in a graded commutative ring R_*, we may specify a ring homomorphism from $\Phi_*[x]$ to $R_*[x]$ by $\phi_i \mapsto \alpha_i$; thus Φ_*, together with the sequence $\phi = (1, \phi_1, \phi_2, \ldots)$, may be interpreted as the universal example. Given a polynomial $P^\phi(x)$ in $\Phi_*[x]$, we shall write $P^\alpha(x)$ for its image in $R_*[x]$. The ring of scalars $k_* := \mathbb{Z}[u]$ and the sequence $\kappa = (1, u, u^2, \ldots)$ play a special rôle, being obtained from the universal example by substituting u^i for each ϕ_i. In terms of formal group laws, they arise from the *multiplicative formal group law*. If we further substitute 1 for u, then the scalars reduce to $\mathbb{Z}$, and we shall abbreviate any polynomial of the form $P^{(1,1,1,\ldots)}(x)$ to $P(x)$. Whenever $P^\phi(x)$ is homogeneous, then $P^\kappa(x)$ is a homogenized version of $P(x)$. For example, replacing each ϕ_i by 1 in $B^\phi_n(x)$ yields $B_n(x)$, which may readily be identified as the falling factorial polynomial $x(x-1)\ldots(x-n+1)$.

For each partition σ of a given set, we define its *type* $\tau^\phi(\sigma)$ to be the monomial $\phi_1^{k_1}\phi_2^{k_2}\ldots$ in Φ_*, where k_i is the number of blocks of σ with $i+1$ elements. The type of a coloring is the type of its kernel. In [10], the authors associated to a simplicial complex $\mathcal{K}$ the polynomial

$$\sum_{\sigma\in\Pi(\mathcal{K})} \tau^{\phi}(\sigma)\, B^{\phi}_{|\sigma|}(x) \tag{1.3}$$

in $\Phi_n[x]$, where $n = |V(\mathcal{K})|$. After umbral substitution by $m\phi$, this polynomial obviously enumerates by type the colorings of $\mathcal{K}$ with at most m colors, and it was therefore referred to in [10] as the *umbral chromatic polynomial* of $\mathcal{K}$. Whenever $\mathcal{K}$ is the independence complex of a graph H, it reduces to the umbral chromatic polynomial $\chi^{\phi}(H\,;x)$ introduced in [11]. Note that our conventions dictate that we write $\chi(H;x)$ for the classical chromatic polynomial of H, and $\chi^{\kappa}(H\,;x)$ for its homogenized version. Considering the latter is natural from the point of view of graded algebras; the combinatorial significance is the following: after umbral substitution by $m\kappa$ (which is a special case of umbral substitution by $m\phi$), $\chi^{\kappa}(H\,;x)$ enumerates colorings of H with at most m colors by the number of colors.

2. Partition Systems and Their Colorings

Given a partition π of a finite non-empty set V, we denote by $\mathrm{Bool}(\pi)$ the Boolean algebra of subsets of V consisting of arbitrary unions of blocks of π. A set system $\mathcal{P}$ satisfying $\pi \subseteq \mathcal{P} \subseteq \mathrm{Bool}(\pi)$ for an arbitrary partition π of V will be called a *partition system*. The blocks of π are the atoms of the poset $(\mathcal{P}, \subseteq)$; we will refer to them as the *atoms* of $\mathcal{P}$. Since π is uniquely determined by $\mathcal{P}$, it is often convenient to denote π by $\mathrm{At}(\mathcal{P})$, and $\mathrm{Bool}(\pi)$ by $\mathrm{Bool}(\mathcal{P})$. The sets belonging to $\mathrm{Non}(\mathcal{P}) := \mathcal{P} \setminus \{\emptyset\} \setminus \mathrm{At}(\mathcal{P})$ will be called *non-atoms*. Rather than considering arbitrary set sytems, we shall henceforth restrict our attention to partition systems, since they provide the most appropriate framework for our constructions. The partition systems $\mathcal{P}$ and $\mathcal{Q}$ are *isomorphic* if there is a bijection $f\colon V(\mathcal{P}) \to V(\mathcal{Q})$ such that $\{f(U) \,:\, U \in \mathcal{P}\} = \mathcal{Q}$.

Any set system which contains every vertex as a singleton is obviously a partition system, with singletons as atoms. Amongst such examples, we shall regularly consider

$$\mathcal{N}_V := \{\{x\} \,:\, x \in V\} \cup \{\emptyset\}\,, \qquad \text{and} \qquad \mathcal{K}_V := 2^V\,.$$

If $V = [n]$, we denote $\mathcal{N}_V$ by $\mathcal{N}_n$ and $\mathcal{K}_V$ by $\mathcal{K}_n$.

Let $\sigma \subseteq \mathrm{Bool}(\mathcal{P})$ be a partition of a set $U \subseteq V(\mathcal{P})$ (so that U necessarily lies in $\mathrm{Bool}(\mathcal{P})$). We define the partition system $\mathcal{P}|\sigma$ to be

$$\{W \in \mathcal{P} \,:\, W \subseteq B \ \text{ for some } \ B \in \sigma\}\,,$$

and call it the *restriction* of $\mathcal{P}$ to σ. If σ is the single block $\{U\}$, we abbreviate the restriction to $\mathcal{P}|U$. Then the set $\bigcup \Pi(\mathcal{P}|U)$, where U ranges over $\mathrm{Bool}(\mathcal{P})$, consists of all divisions of appropriate *subsets* of the vertices by elements of $\mathcal{P}$; this set will be useful below, and we label it $\widetilde{\Pi}(\mathcal{P})$. Given two partition systems $\mathcal{P}$ and $\mathcal{Q}$ with $\mathrm{At}(\mathcal{Q}) = \mathrm{At}(\mathcal{P})$ and $\mathcal{Q} \subseteq \mathcal{P}$, we define the *complement* of $\mathcal{Q}$ in $\mathcal{P}$ to be the partition system

$$\complement_{\mathcal{P}}\mathcal{Q} := \mathcal{P} \setminus \mathrm{Non}(\mathcal{Q})\,.$$

The complement of $\mathcal{P}$ in $\mathrm{Bool}(\mathcal{P})$ will be denoted by $\overline{\mathcal{P}}$, and called, simply, the complement of $\mathcal{P}$. Complementation is the main ingredient for generalizing Whitney's formula (1.1); from this point of view, passing from $\mathcal{P}$ to $\overline{\mathcal{P}}$ is the partition system analogue of passing from a graph (which is identified with its independence complex) to its lattice of contractions (which consists of divisions by sets of vertices inducing connected subgraphs).

We now fix a partition system $\mathcal{P}$ for the rest of this section. We consider the set $\Pi(\mathcal{P})$ with the usual partial ordering by refinement, that is $\pi \leq \sigma$ if every block of π is a subset of some block of σ. The definition of a partition system ensures that the partition $\mathrm{At}(\mathcal{P})$ is the minimum element $\widehat{0}$ (or $\widehat{0}_{\Pi(\mathcal{P})}$ if the context is unclear) of the poset $\Pi(\mathcal{P})$. We are now able to define the concepts of coloring mentioned in §1.

Definition 2.1.

(1) *A* factorized coloring *of $\mathcal{P}$ with at most m colors is a pair (γ, f) consisting of a chain $\gamma = \{\widehat{0}_{\Pi(\mathcal{P})} = \sigma_1 < \sigma_2 < \ldots < \sigma_k\}$ of partitions of $V(\mathcal{P})$ and a coloring f of $\mathcal{P}$ with at most m colors, such that the following conditions are satisfied:*
   (a) *$\sigma_i \in \Pi(\overline{\mathcal{P}})$, for $1 \leq i < k$;*
   (b) *the kernel of f is σ_k.*

(2) *A* coloring forest *of $\mathcal{P}$ with at most m colors is a pair (δ, f) consisting of a set δ with $\mathrm{At}(\mathcal{P}) \subseteq \delta \subseteq \mathrm{Bool}(\mathcal{P}) \setminus \{\emptyset\}$ and a coloring f of $\mathcal{P}$ with at most m colors, such that in the poset $(\delta, \subseteq)$ we have:*
   (a) *the set of elements covered by U is a division by $\overline{\mathcal{P}}|U$ for all non-atoms $U \in \delta$;*
   (b) *the set $\max(\delta)$ of maximal elements of δ is a partition of $V(\mathcal{P})$, and the kernel of f is $\max(\delta)$.*

The name of the first concept comes from viewing the pair (γ, f) as a factorized function

$$\sigma_0 \xrightarrow{f_1} \sigma_1 \xrightarrow{f_2} \sigma_2 \ \ldots \ \xrightarrow{f_k} \sigma_k \xrightarrow{f_{k+1}} [m]\,,$$

where σ_0 is the partition of $V(\mathcal{P})$ into singletons, every function f_i with $1 \leq i \leq k$ sends a block of σ_{i-1} to the block of σ_i containing it, and the composite $f_{k+1} \circ f_k \circ \ldots \circ f_1$ coincides with the coloring f (the last condition assumes that we identify the partition of $V(\mathcal{P})$ into singletons with $V(\mathcal{P})$). The name of the second concept is motivated by the fact that the Hasse diagram of the poset $(\delta, \subseteq)$ is a forest, since every element which is not maximal has a unique cover. Moreover, a coloring forest of $\mathcal{P}$ can be viewed as a forest of rooted trees with colored roots and leaves labelled with symbols corresponding to the atoms of $\mathcal{P}$. The ordinary coloring f with kernel $\ker(f)$ can be viewed as the factorized coloring $(\{\widehat{0}_{\Pi(\mathcal{P})} \leq \ker(f)\}, f)$, and as the coloring forest $(\mathrm{At}(\mathcal{P}) \cup \mathrm{ker(f)}, \mathrm{f})$. It is easy to see that all the factorized colorings and all the coloring forests of a simplicial complex are, in fact, ordinary colorings. We denote by $\Delta(\mathcal{P})$ the collection of sets δ with $\mathrm{At}(\mathcal{P}) \subseteq \delta \subseteq \mathrm{Bool}(\mathcal{P}) \setminus \{\emptyset\}$ satisfying the first condition in the definition of a coloring forest of $\mathcal{P}$ and $\max(\delta) \in \Pi(\mathcal{P})$. Given a chain γ as in the definition of a factorized coloring of $\mathcal{P}$, we associate with it the union of all partitions in γ; this collection of sets lies in $\Delta(\mathcal{P})$, and will be denoted by $\Delta(\gamma)$. We also associate with every factorized coloring (γ, f) the coloring forest $(\Delta(\gamma), f)$. These correspondences are surjective but not injective, as Example 2.3 shows.

In [7], Mullin and Rota defined a *reluctant function* from S to X to be a function from S to the disjoint union $S \sqcup X$, such that only a finite number of terms of the sequence $s, f(s), f(f(s)), \ldots$ are defined. Given a factorized coloring (γ, f) as in Definition 2.1 (1), we can identify it with the reluctant function $\widehat{f}$ from the disjoint union $\bigsqcup_{i=1}^{k} \sigma_i$ to $[m]$, specified by insisting that the restriction of $\widehat{f}$ to σ_i coincide with the function f_{i+1} discussed above. On the other hand, a coloring forest (δ, f) can be identified with the reluctant function $\overline{f}$ from δ to $[m]$ sending a set B to its cover in $(\delta, \subseteq)$ if B is not maximal, and to $f(x)$ for some x in B otherwise (this is a good definition because of the second condition in Definition 2.1 (2)).

We now define the type of the objects considered above. We need the function ζ^{ϕ} from the set of intervals in the lattice of partitions of $V(\mathcal{P})$ to Φ_*; this is defined by: $\zeta^{\phi}(\pi, \sigma) := \tau^{\phi}(\sigma/\pi)$, where σ/π is the *induced partition* on the blocks of π (that is the partition of π whose blocks are the sets $\{B \in \pi \,:\, B \subseteq C\}$ for every block $C \in \sigma$). Note that we have followed convention in abbreviating $g([\pi, \sigma])$ to $g(\pi, \sigma)$, and continue to do so. The type of a chain $\gamma = \{\sigma_1 < \sigma_2 < \ldots < \sigma_k\}$ in $\Pi(\mathcal{P})$ of length $l(\gamma) := k - 1$ is defined by

$$\tau^{\phi}(\gamma) := (-1)^{l(\gamma)-1} \zeta^{\phi}(\sigma_1, \sigma_2) \, \ldots \, \zeta^{\phi}(\sigma_{k-1}, \sigma_k)\,,$$

if $k > 1$, and $\tau^{\phi}(\gamma) := 1$, otherwise. For every δ in $\Delta(\mathcal{P})$, we define its type $\tau^{\phi}(\delta) := (-1)^{s(\delta)} \phi_1^{k_1} \phi_2^{k_2} \ldots$, where $s(\delta)$ is $|\delta \setminus \mathrm{At}(\mathcal{P}) \setminus \max(\delta)|$, and k_i is the number of elements of $(\delta, \subseteq)$ which cover precisely $i + 1$ elements. Let us note that

$$\tau^{\phi}(\Delta(\gamma)) = \pm \tau^{\phi}(\gamma)\,. \tag{2.2}$$

We define the type of a factorized coloring and of a coloring forest by $\tau^{\phi}(\gamma, f) := \tau^{\phi}(\gamma)$ and $\tau^{\phi}(\delta, f) := \tau^{\phi}(\delta)$. Both of these definitions are compatible with the definition of the type of an ordinary coloring.

Example 2.3. Let us consider vertices [3], and the partition systems

$$\mathcal{Q} := \{\, \emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{1,2\}, \{3,4\}, \{1,2,3\} \,\} \qquad \text{and} \;\; \mathcal{P} := \overline{\mathcal{Q}}\,.$$

There are six *trees* in $\Delta(\mathcal{P})$. They are listed below, and their types are ϕ_3, $-\phi_1\phi_2$, $-\phi_1\phi_2$, ϕ_1^3, $-\phi_1\phi_2$, and ϕ_1^3, respectively. All these trees have only one chain of partitions associated with them, except for the fourth one, which has the following three such chains with types ϕ_1^3, ϕ_1^3, and $-\phi_1^3$, respectively:

$$\widehat{0} \,<\, \{\,\{1,2\},\{3\},\{4\}\,\} \,<\, \{\,\{1,2\},\{3,4\}\,\} \,<\, \{\,\{1,2,3,4\}\,\}\,,$$
$$\widehat{0} \,<\, \{\,\{1\},\{2\},\{3,4\}\,\} \,<\, \{\,\{1,2\},\{3,4\}\,\} \,<\, \{\,\{1,2,3,4\}\,\}\,,$$
$$\widehat{0} \,<\, \{\,\{1,2\},\{3,4\}\,\} \,<\, \{\,\{1,2,3,4\}\,\}\,.$$

Clearly, the trees and chains discussed above can only be paired with monochromatic colorings of $\mathcal{P}$ in order to obtain coloring forests and factorized colorings.

Proposition 2.4. *Given a forest δ in $\Delta(\mathcal{P})$, the sum of types of all chains γ with $\Delta(\gamma) = \delta$ coincides with the type of δ.*

The proposition follows from (2.2) and slightly modified versions of two lemmas in [4]. For the sake of completeness, we state these lemmas here, and define the concepts involved. For any finite poset P, a *filtration* F of P is a chain $\{\emptyset = I_0 \subset I_1 \subset \ldots \subset I_k = P\}$ of lower order ideals such that $I_j \setminus I_{j-1}$ is an antichain for all $1 \leq j \leq k$. The number k is the *length* of the filtration, and is denoted by $l(F)$. We now assume that the forest δ is such that $\max(\delta) \neq \widehat{0}_{\Pi(\mathcal{P})}$, and denote by C the set of chains γ associated with δ.

Lemma 2.5. *There is a length-preserving bijection between C and the set of filtrations of the poset $(\delta \setminus \mathrm{At}(\mathcal{P}) \setminus \max(\delta)) \cup \{\mathrm{V}(\mathcal{P})\}$, ordered by inclusion.*

Lemma 2.6. *For any finite poset P, we have that*

$$\sum_F (-1)^{l(F)-1} = (-1)^{|P|-1},$$

where the summation is over all filtrations F of P.

3. The Umbral Chromatic Polynomial

In this section we define the umbral chromatic polynomial of $\mathcal{P}$ as a polynomial enumerating factorized colorings and coloring forests by type. For this purpose, we need certain results concerning the Möbius type function (as considered in [10] and [11]) for a subposet P of the lattice of partitions of a finite set V, ordered by refinement.

For such P, the *incidence algebra* $\Phi_*(P)$ is the free Φ_*-module generated by all functions from the collection of intervals in P to the ring Φ_*, with multiplication specified by the *convolution* formula

$$(f_1 * f_2)(\pi,\sigma) := \sum_{\pi\le\rho\le\sigma} f_1(\pi,\rho)\, f_2(\rho,\sigma)\,.$$

The identity of $\Phi_*(P)$ is the function δ which is defined, using the Kronecker delta, by $\delta(\pi,\sigma) := \delta_{\pi,\sigma}$. The function ζ^ϕ in $\Phi_*(P)$ is defined as above.

We write $C(P)$ for the set of chains between a minimal and a maximal element of P. The function ζ^ϕ has a convolution inverse, which is denoted by μ^ϕ (or μ^ϕ_P if there is possible ambiguity); it is called the *Möbius type function* of P, and if $\pi \ne \sigma$ it is given by

$$\mu^\phi(\pi,\sigma) = -\sum_\gamma \tau^\phi(\gamma)\,, \tag{3.1}$$

where the summation ranges over $C(\pi,\sigma)$. Observe that both $\zeta^\phi(\pi,\sigma)$ and $\mu^\phi(\pi,\sigma)$ lie in $\Phi_{|\pi|-|\sigma|}$.

The classical Möbius function $\mu(\pi,\sigma)$ (or $\mu_P(\pi,\sigma)$) is obtained by setting each ϕ_i to 1. Not only does this conform with our chosen notation, but it also suggests that we might generalize certain standard properties of $\mu(\pi,\sigma)$ to $\mu^\phi(\pi,\sigma)$. Thus we may establish the following two lemmas by straightforward adaptation of the proofs in [14].

Lemma 3.2. *Consider a subposet Q of the lattice of partitions of a set W, where $W\cap V = \emptyset$, and identify the pair $(\sigma,\sigma')\in P\times Q$ with $\sigma\cup\sigma'$; then*

$$\mu^\phi_{P\times Q}(\pi\cup\pi',\sigma\cup\sigma') = \mu^\phi_P(\pi,\sigma)\,\mu^\phi_Q(\pi',\sigma') \qquad \textit{in} \qquad \Phi_*\,.$$

Lemma 3.3. *If Q is a subposet of the interval $[\pi,\sigma]$ which contains both π and σ, then*

$$\mu^\phi_Q(\pi,\sigma) = \sum (-1)^k \mu^\phi_P(\pi,\pi_1)\,\ldots\,\mu^\phi_P(\pi_k,\sigma)\,,$$

where the summation ranges over all chains $\{\pi<\pi_1<\ldots<\pi_k<\sigma\}\in C(\pi,\sigma)$ for which $\pi_i\notin Q$.

Let us now return to $\mathcal{P}$. Let $\mathcal{Q}$ be another partition system with $V(\mathcal{Q}) = V(\mathcal{P})$ and $\mathrm{At}(\mathcal{Q}) = \mathrm{At}(\mathcal{P})$. Given a partition $\sigma\in\widetilde{\Pi}(\mathcal{P})$, we define its *Möbius type* $\nu^\phi_{\mathcal{Q}}(\sigma)$ with respect to $\mathcal{Q}$ by

$$\sum_\pi \mu^\phi_{\Pi(\mathcal{Q}|\sigma)}(\widehat{0},\pi)\,\zeta^\phi(\pi,\sigma)\,,$$

where the summation ranges over $\Pi(\mathcal{Q}|\sigma)$. We remark that whenever $\sigma\not\subseteq\mathcal{Q}$, then $\nu^\phi_{\mathcal{Q}}(\sigma) = -\mu^\phi_{\Pi(\mathcal{Q}|\sigma)\cup\{\sigma\}}(\widehat{0},\sigma)$; otherwise, $\nu^\phi_{\mathcal{Q}}(\sigma)$ is 1 if σ is contained in $\mathrm{At}(\mathcal{Q})$, and 0 if it is not.

In fact, the Möbius type can be viewed as a map from $\widetilde{\Pi}(\mathcal{P})$ to the ring Φ_*. We shall refer to any such map w into a commutative ring as a *weight*, and say that w is multiplicative if $w(\sigma_1\cup\sigma_2) = w(\sigma_1)\,w(\sigma_2)$ for any $\sigma_1,\sigma_2\in\widetilde{\Pi}(\mathcal{P})$ with disjoint vertex sets.

We may now define the umbral chromatic polynomial.

Proposition 3.4. *There exists a polynomial in $\Phi_{|\mathrm{At}(\mathcal{P})|}[x]$ with the property that after umbral substitution by $m\phi$, it enumerates by type the factorized colorings and the coloring* *forests of $\mathcal{P}$ with at most m colors. This polynomial can be expressed in interpolated form as follows:*

$$\chi^\phi(\mathcal{P}\,;x) := \sum_\sigma \nu^\phi_{\mathcal{P}}(\sigma)\, B^\phi_{|\sigma|}(x)\,,$$

where the summation ranges over the poset $\Pi(\mathcal{P})$ of divisions by $\mathcal{P}$.

Proof. Let f be a coloring of $\mathcal{P}$ whose kernel $\ker(f)$ has n blocks. According to (3.1),

$$\sum_\gamma \tau^\phi(\gamma, f) = \nu^\phi_{\mathcal{P}}(\ker(f))\,,$$

where the summation ranges over all chains γ for which (γ, f) is a factorized coloring. Since there are $m(m-1)\dots(m-n+1)$ colorings of $\mathcal{P}$ with at most m colors having the same kernel as f, the proposition follows. The fact that the polynomial $\chi^\phi(\mathcal{P}\,;x)$ also enumerates coloring forests of $\mathcal{P}$ by type follows from Proposition 2.4. ☐

We call $\chi^\phi(\mathcal{P}\,;x)$ the *umbral chromatic polynomial* of $\mathcal{P}$. Clearly, if $\mathcal{K}$ is a simplicial complex then $\nu^\phi_{\mathcal{K}}(\sigma) = \tau^\phi(\sigma)$, so $\chi^\phi(\mathcal{P}\,;x)$ coincides with the polynomial (1.3). On the other hand, we can obtain various types of chromatic polynomials of the partition system $\mathcal{P}$ from $\chi^\phi(\mathcal{P}\,;x)$ by replacing the umbra ϕ with another umbra. In particular, we obtain $\chi(\mathcal{P};x)$ and $\chi^\kappa(\mathcal{P}\,;x)$, which we will call the classical chromatic polynomial of $\mathcal{P}$, and its homogenized version. After appropriate evaluation, the latter enumerates factorized colorings and coloring forests of $\mathcal{P}$ by the type τ^κ.

We now present a computational result on Möbius types; in order to state it, we need to generalize the lattice of contractions of a graph. Further information on Möbius types will be given in Theorem 4.8 below.

Observe that the independence complex of the disjoint union of two graphs can be obtained from the independence complexes of the constituent components by the operation known as *join*. We denote this operation on partition systems by

$$\mathcal{P}_1 \vee \mathcal{P}_2 := \{U_1 \sqcup U_2 \;:\; U_1 \in \mathcal{P}_1,\; U_2 \in \mathcal{P}_2\}\,.$$

We call a partition system *join-connected* if it is not isomorphic to the join of two partition systems, and write $\mathcal{P}_c$ for the partition system consisting of those sets $U \in \overline{\mathcal{P}}$ for which $\mathcal{P}|U$ is join-connected. It is easy to see that $\Pi(\mathcal{I}(H)_c) = L_H$ for any graph H with independence complex $\mathcal{I}(H)$. In general, finding an alternative description of $\mathcal{P}_c$, which is easier to grasp than the one given above, amounts to finding such a description for join-connectivity; however, we have not been able to do this. Let us note that $\overline{\mathcal{P}}$ may be much larger than $\mathcal{P}_c$, whence a summation ranging over $\Pi(\overline{\mathcal{P}})$ may contain many more terms than one ranging over $\Pi(\mathcal{P}_c)$.

Proposition 3.5.

1. *The weight $\nu^\phi_{\mathcal{Q}}$ is multiplicative.*
2. *If $\sigma \in \widetilde{\Pi}(\mathcal{P})$, and $\Pi(\mathcal{Q}|\sigma)$ has a unique maximal element $\widehat{1}$ and is non-trivial, then $\nu^\kappa_{\mathcal{Q}}(\sigma) = 0$.*
3. *If $\sigma \in \widetilde{\Pi}(\mathcal{P})$, then $\nu^\kappa_{\mathcal{P}}(\sigma) = \nu^\kappa_{\mathcal{P}_c}(\sigma)$.*

Proof. (1) Let $\sigma_1, \sigma_2 \in \widetilde{\Pi}(\mathcal{P})$ such that $V(\sigma_1) \cap V(\sigma_2) = \emptyset$, and denote $\sigma_1 \cup \sigma_2$ by σ. Write D_1 for $\Pi(\mathcal{Q}|\sigma_1)$, and D_2 for $\Pi(\mathcal{Q}|\sigma_2)$. Clearly, the poset $\Pi(\mathcal{Q}|\sigma)$ is isomorphic to $D_1 \times D_2$. Using Lemma 3.2, we deduce

$$\nu^\phi_{\mathcal{Q}}(\sigma) = \sum_{(\pi_1,\pi_2) \in D_1 \times D_2} \mu^\phi_{D_1 \times D_2}(\widehat{0}_{D_1} \cup \widehat{0}_{D_2}, \pi_1 \cup \pi_2)\, \zeta^\phi(\pi_1 \cup \pi_2, \sigma_1 \cup \sigma_2)$$

$$= \left(\sum_{\pi_1 \in D_1} \mu^\phi_{D_1}(\widehat{0}_{D_1}, \pi_1)\, \zeta^\phi(\pi_1, \sigma_1) \right) \left(\sum_{\pi_2 \in D_2} \mu^\phi_{D_2}(\widehat{0}_{D_2}, \pi_2)\, \zeta^\phi(\pi_2, \sigma_2) \right) = \nu^\phi_{\mathcal{Q}}(\sigma_1)\, \nu^\phi_{\mathcal{Q}}(\sigma_2)\,.$$

(2) We may assume that $\sigma \notin \widetilde{\Pi}(\mathcal{Q})$, since otherwise $\nu^\phi_{\mathcal{Q}}(\sigma) = 0$. We can pair each chain $\{\widehat{0} < \sigma_1 < \ldots < \sigma_n < \sigma\}$ in $\Pi(\mathcal{Q}|\sigma) \cup \{\sigma\}$ for which $\sigma_n \neq \widehat{1}$ with the chain $\{\widehat{0} < \sigma_1 < \ldots < \sigma_n < \widehat{1} < \sigma\}$. The contribution to $\nu^\kappa_{\mathcal{Q}}(\sigma)$ of each pair is 0, whence $\nu^\kappa_{\mathcal{Q}}(\sigma) = 0$.

(3) We need the concept of *coclosure operator* on a poset P (see e.g. [9]), which is a function $x \mapsto \overline{x}$ from P into itself such that: (1) $\overline{x} \leq x$, (2) $\overline{\overline{x}} = \overline{x}$, and (3) $x \leq y$ implies $\overline{x} \leq \overline{y}$, for all $x, y \in P$. An element x of P is *closed* if $\overline{x} = x$. Consider the poset $P := \Pi(\overline{\mathcal{P}}|\sigma) \cup \{\sigma\}$, and define the coclosure operator $\pi \mapsto \overline{\pi}$ by $\overline{\sigma} = \sigma$, and by letting the partition $\overline{\pi}$ be obtained from π by splitting every block B into the sets of vertices of the join-connected components of $\mathcal{P}|B$ whenever $\pi \neq \sigma$. Obviously, the subposet of coclosed elements is $Q = \Pi(\mathcal{P}_c|\sigma) \cup \{\sigma\}$. According to [13], we have that $\mu^\kappa_Q(\widehat{0}, \sigma) = \sum \mu^\kappa_P(\pi, \sigma)$, where the summation is over all π such that $\overline{\pi} = \widehat{0}$ (we have used the fact that $\mu^\kappa_P(\pi, \sigma) = u^{|\pi| - |\sigma|}\, \mu_P(\pi, \sigma)$). But whenever $\overline{\pi} = \widehat{0}$ and $U \in \pi$, we have $U \in \overline{\mathcal{P}}$ and $\mathcal{P}|U = \mathrm{Bool}(\mathrm{At}(\mathcal{P}|\mathrm{U}))$. Thus $U \in \mathcal{P}$, which is possible if and only if $U \in \mathrm{At}(\mathcal{P})$; hence $\pi = \widehat{0}$, which means that $\mu^\kappa_Q(\widehat{0}, \sigma) = \mu^\kappa_P(\widehat{0}, \sigma)$. □

According to Propositions 3.5 (3) and 3.4, we can define the classical chromatic polynomial $\chi^\kappa(\mathcal{P}\,; x)$ in terms of $\mathcal{P}_c$. On the other hand, we shall see in §5 that it is possible to state the analogue of Whitney's result (1.1) for $\chi^\kappa(\mathcal{P}\,; x)$ in terms of $\mathcal{P}_c$, although we were only able to state it in terms of $\overline{\mathcal{P}}$ for $\chi^\phi(\mathcal{P}\,; x)$.

We conclude this section with a reference to Stanley's symmetric function X_H, which generalizes the chromatic polynomial of a graph H (see [15]). We use the same notation for symmetric functions as in [15], namely p_λ, m_λ, and $\widetilde{m}_\lambda$ for the power sum, monomial, and augmented monomial symmetric functions corresponding to the partition $\lambda = \langle 1^{r_1} 2^{r_2} \ldots \rangle$ of a positive integer, where the latter is defined by

$$\widetilde{m}_\lambda := r_1! r_2! \ldots m_\lambda\,.$$

Given a partition σ of a set V, we denote by $\lambda(\sigma)$ the partition of $|V|$ whose parts are the sizes of the blocks of σ. Stanley's definition of X_H can be easily extended in order to associate a symmetric function $X_{\mathcal{K}}$ with an arbitrary simplicial complex $\mathcal{K}$ with vertices $V = \{v_1, \ldots, v_n\}$. Thus, we define

$$X_{\mathcal{K}} = X_{\mathcal{K}}(x) := \sum_f x^f\,, \tag{3.6}$$

where the summation ranges over all colorings of $\mathcal{K}$, and $x^f := x_{f(v_1)} \dots x_{f(v_n)}$. Clearly, $X_H = X_{\mathcal{I}(H)}$. Following Stanley, we observe that

$$X_{\mathcal{K}} = \sum_{\sigma \in \Pi(\mathcal{K})} \widetilde{m}_{\lambda(\sigma)} \,. \tag{3.7}$$

Let us now denote by $\Phi\mathbb{Q}_*$ the rationalization $\Phi_* \otimes \mathbb{Q}$, and let us consider the $\mathbb{Q}$-linear map from the space $\Lambda_{\mathbb{Q}}$ of symmetric functions with rational coefficients to $\Phi\mathbb{Q}_*[x]$ specified by

$$\widetilde{m}_{\lambda(\sigma)} \mapsto \tau^{\phi}(\sigma)\, B^{\phi}_{|\sigma|}(x) \,, \tag{3.8}$$

where σ is a partition of $[n]$. Clearly, this map preserves the gradings, is injective, but it is neither surjective, nor an algebra map. Comparing (3.7) with the formula for the umbral chromatic polynomial in Proposition 3.4, we easily see that the above map sends $X_{\mathcal{K}}$ to $\chi^{\phi}(\mathcal{K}\,; x)$.

4. The Characteristic Type Polynomial

In §5 we shall study the properties of the umbral chromatic polynomial by relating it to a *characteristic type polynomial*, so in this section we introduce the latter, and determine some of its properties. It is defined for an arbitrary partition system $\mathcal{P}$ by

$$c^{\phi}(\mathcal{P}\,; x) := \sum_{\sigma} \mu^{\phi}_{\Pi(\mathcal{P})}(\widehat{0}, \sigma)\, x^{|\sigma|} \,, \tag{4.1}$$

where the summation takes place over the poset of divisions $\Pi(\mathcal{P})$. Such a polynomial is used in [9] and [10], but in the context of posets of partitions of a given set, rather than set systems as here. If $\mathcal{K}$ is a simplicial complex, and all the maximal partitions of $\Pi(\mathcal{K})$ have the same number of blocks n, then the substitution $\phi_i \mapsto 1$ maps $c^{\phi}(\mathcal{K}\,; x)$ to the characteristic polynomial of the poset $\Pi(\mathcal{K})$, multiplied by x^n. We therefore have

$$c^{\phi}(\mathcal{N}_n\,; x) = x^n \qquad \text{and} \qquad c^{\phi}(\mathcal{K}_n\,; x) = B^{\phi}_n(x) \,. \tag{4.2}$$

The characteristic type polynomial is well-behaved with respect to the disjoint union $\mathcal{P}_1 \cdot \mathcal{P}_2$ of partition systems $\mathcal{P}_1$ and $\mathcal{P}_2$, according to the formula

$$c^{\phi}(\mathcal{P}_1 \cdot \mathcal{P}_2\,; x) = c^{\phi}(\mathcal{P}_1\,; x)\, c^{\phi}(\mathcal{P}_2\,; x) \,. \tag{4.3}$$

This follows from Lemma 3.2, and the fact that the poset $\Pi(\mathcal{P}_1 \cdot \mathcal{P}_2)$ is isomorphic to $\Pi(\mathcal{P}_1) \times \Pi(\mathcal{P}_2)$.

We now present a deletion/contraction identity which enables us to compute $c^{\phi}(\mathcal{P}\,; x)$ recursively. We use it as our main tool in proofs by induction, such as those of Theorem 4.8 and Theorem 4.9. Let us first introduce deletion and contraction for partition systems.

Given $U \in \mathrm{Non}(\mathcal{P})$, we define the *deletion* of U to be the partition system $\mathcal{P} \setminus \{U\}$, abbreviated to $\mathcal{P} \setminus U$. Given a partition $\sigma \subseteq \mathrm{Bool}(\mathcal{P})$ of a set $U \subseteq V(\mathcal{P})$, we define the partition system $\mathcal{P}/\sigma$ to be

$$\{W \in \mathcal{P} \,:\, B \subseteq W \ \text{ or } \ B \cap W = \emptyset\,, \ \text{ for all } \ B \in \sigma\} \cup \sigma \,,$$

and call it the *contraction* of $\mathcal{P}$ through σ; once more, we abbreviate $\mathcal{P}/\{U\}$ to $\mathcal{P}/U$. The restriction and the contraction of a partition π through a set $U \in \mathrm{Bool}(\pi)$ is defined in a similar way to the restriction and the contraction of a partition system, namely:

$$\pi|U := \{B \in \pi \,:\, B \subseteq U\}\,, \qquad \pi/U := \{B \in \pi \,:\, B \cap U = \emptyset\} \cup \{U\}\,.$$

Note that even if all the atoms of a partition system are singletons, not all the atoms of a contraction of it are (except for the trivial case, when we are contracting through atoms).

We can transform an arbitrary partition system $\mathcal{P}$ into one which has only singleton atoms by defining $\mathrm{Sing}(\mathcal{P})$ to be

$$\{\mathrm{At}(\mathcal{P}|\mathrm{U}) \,:\, \mathrm{U} \in \mathcal{P}\}\,.$$

We may then define the *strong contraction* of $\mathcal{P}$ through σ, as $\mathrm{Sing}\,(\mathcal{P}/\sigma)$. Note that for any partition system $\mathcal{P}$ we have

$$\chi^{\phi}(\mathcal{P}\,;x) = \chi^{\phi}(\mathrm{Sing}(\mathcal{P})\,;x) \qquad \text{and} \qquad c^{\phi}(\mathcal{P}\,;x) = c^{\phi}(\mathrm{Sing}(\mathcal{P})\,;x)\,.$$

Theorem 4.4. *If $U \in \mathrm{Non}(\mathcal{P})$ is arbitrary, then*

$$c^{\phi}(\mathcal{P}\,;x) = c^{\phi}(\mathcal{P} \setminus U\,;x) + \mu^{\phi}_{\Pi(\mathcal{P}|U)}(\widehat{0},\{U\})\, c^{\phi}(\mathcal{P}/U\,;x)\,;$$

moreover, the identity still holds if we replace contraction by strong contraction.

Proof. For simplicity, we write P, Q, R, and S for $\Pi(\mathcal{P})$, $\Pi(\mathcal{P}\setminus U)$, $\Pi(\mathcal{P}/U)$, and $\Pi(\mathcal{P}|U)$ respectively.

Consider an arbitrary partition $\sigma \in P$, for which three possible cases arise. Firstly, if no block of σ contains U, then $\mu^{\phi}_P(\widehat{0}_P,\sigma) = \mu^{\phi}_Q(\widehat{0}_P,\sigma)$. Secondly, if one block of σ is U itself, then

$$\mu^{\phi}_P(\widehat{0}_P,\sigma) = \mu^{\phi}_S(\widehat{0}_S,\widehat{1}_S)\,\mu^{\phi}_P(\widehat{0}_R,\sigma) = \mu^{\phi}_P(\widehat{0}_P,\widehat{0}_R)\,\mu^{\phi}_P(\widehat{0}_R,\sigma)\,, \tag{4.5}$$

as follows from the poset isomorphisms $[\widehat{0}_P,\sigma] \cong [\widehat{0}_S,\widehat{1}_S] \times [\widehat{0}_R,\sigma] \cong [\widehat{0}_P,\widehat{0}_R] \times [\widehat{0}_R,\sigma]$ by using Lemma 3.2. Thirdly, if one block of σ strictly contains U, then by Lemma 3.3

$$\mu^{\phi}_Q(\widehat{0}_P,\sigma) = \sum (-1)^k \mu^{\phi}_P(\widehat{0}_P,\sigma_1)\,\ldots\,\mu^{\phi}_P(\sigma_k,\sigma)\,, \tag{4.6}$$

where $\sigma_1 < \sigma_2 < \ldots < \sigma_k$ all have U as a block, and $\sigma_k < \sigma$. Using (4.5), we deduce that the terms of the form $\mu^{\phi}_P(\widehat{0}_P,\sigma_1)\,\mu^{\phi}_P(\sigma_1,\sigma)$ with $\sigma_1 \neq \widehat{0}_R$ cancel with terms $\mu^{\phi}_P(\widehat{0}_P,\widehat{0}_R)\,\mu^{\phi}_P(\widehat{0}_R,\sigma_1)\,\mu^{\phi}_P(\sigma_1,\sigma)$, and so on. Hence, (4.6) becomes

$$\mu^{\phi}_Q(\widehat{0}_P,\sigma) = \mu^{\phi}_P(\widehat{0}_P,\sigma) - \mu^{\phi}_S(\widehat{0}_S,\widehat{1}_S)\,\mu^{\phi}_P(\widehat{0}_R,\sigma)\,. \tag{4.7}$$

The required formula follows by considering each of these cases in turn. □

We may now give another method for computing $\nu^{\phi}_{\overline{\mathcal{P}}}(\sigma)$, where $\sigma \in \Pi(\mathcal{P})$, which may also be regarded as a complementation formula for the Möbius type function. It expresses $\nu^{\phi}_{\overline{\mathcal{P}}}(\sigma) = -\mu^{\phi}_{\Pi(\overline{\mathcal{P}}|\sigma)\cup\{\sigma\}}(\widehat{0},\sigma)$ in terms of the Möbius type function of the poset $\Pi(\mathcal{P}|\sigma)$, and so is valuable when this poset is smaller than $\Pi(\overline{\mathcal{P}}|\sigma) \cup \{\sigma\}$. We must first introduce the sequence $\overline{\phi} = (1 = \overline{\phi}_0, \overline{\phi}_1, \overline{\phi}_2, \ldots)$ of elements in Φ_*. Let Π_n denote the lattice of partitions of the set $[n]$ (or, equivalently, $\mathit{\Pi}(\mathcal{K}_n)$), and let $\overline{\phi}_n := \mu^{\phi}_{\mathit{\Pi}_{n+1}}(\widehat{0},\widehat{1})$. Then the formal divided power series $\sum_{n=1}^{\infty} \overline{\phi}_{n-1}\, X^n/n!$ is the substitutional inverse of $\sum_{n=1}^{\infty} \phi_{n-1}\, X^n/n!$; the elements $\overline{\phi}_n$ can be expressed in terms of ϕ_i by the Lagrange inversion formula.

Theorem 4.8. *Let $\mathcal{P}$ be an arbitrary partition system; then for every partition $\sigma \in \mathit{\Pi}(\mathcal{P})$, we have that*

$$\nu^{\phi}_{\mathcal{P}}(\sigma) = \mu^{\overline{\phi}}_{\mathit{\Pi}(\mathcal{P})}(\widehat{0},\sigma)\,.$$

Proof. Write V for $V(\mathcal{P})$. According to Lemma 3.2 and Proposition 3.5 (1), we may assume that $\sigma = \{V\}$, which means that $V \in \mathcal{P}$. We use induction with respect to $|\mathrm{Non}(\mathcal{P})|$, which starts at 1 (we assume that $|\mathrm{At}(\mathcal{P})| > 1$). If $|\mathrm{Non}(\mathcal{P})| > 1$, we choose $U \in \mathrm{Non}(\mathcal{P}) \setminus \{V\}$, set $\mathcal{Q} := \overline{\mathcal{P}} \cup \{U, V\}$, and observe that $\overline{\mathcal{P} \setminus U} = \overline{\mathcal{P}} \cup \{U\}$, that $\overline{\mathcal{P}|U} = \overline{\mathcal{P}}|U$, and that $\overline{\mathcal{P}/U} = (\overline{\mathcal{P}} \cup \{U\})/U$. We then employ the inductive hypothesis, applying (4.7) twice in the process; for clarity, it helps to set $\omega^{\phi}_{\mathcal{P}}(\sigma) := \mu^{\phi}_{\mathit{\Pi}(\mathcal{P})}(\widehat{0},\sigma)$ for any $\sigma \in \mathit{\Pi}(\mathcal{P})$, thereby yielding

$$\begin{aligned}
\omega^{\overline{\phi}}_{\mathcal{P}}(\{V\}) &= \omega^{\overline{\phi}}_{\mathcal{P}\setminus U}(\{V\}) + \omega^{\overline{\phi}}_{\mathcal{P}|U}(\{U\})\, \omega^{\overline{\phi}}_{\mathcal{P}/U}(\{V\}) \\
&= \nu^{\phi}_{\overline{\mathcal{P}\setminus U}}(\{V\}) + \nu^{\phi}_{\overline{\mathcal{P}|U}}(\{U\})\, \nu^{\phi}_{\overline{\mathcal{P}/U}}(\{V\}) \\
&= \omega^{\phi}_{\mathcal{Q}}(\{V\}) + \omega^{\phi}_{\mathcal{Q}|U}(\{U\})\, \omega^{\phi}_{\mathcal{Q}/U}(\{V\}) \\
&= -\omega^{\phi}_{\overline{\mathcal{P}}\cup\{V\}}(\{V\}) = \nu^{\phi}_{\mathcal{P}}(\{V\})\,,
\end{aligned}$$

as sought. □

We now apply Theorem 4.4 to prove a complementation formula for the characteristic type polynomial, remarking that similar such formulas were obtained in a very general context for the matching polynomial in [18]. Our formula expresses the characteristic type polynomial of a partition systems $\mathcal{Q}$ in terms of divisions by the complement of $\mathcal{Q}$ in a partition system containing it.

Theorem 4.9. *If $\mathcal{P}$ and $\mathcal{Q}$ are partition systems such that* $\mathrm{At}(\mathcal{P}) = \mathrm{At}(\mathcal{Q})$ *and $\mathcal{Q} \subseteq \mathcal{P}$, then*

$$c^{\phi}(\mathcal{Q}\,; x) = \sum_{\sigma} \nu^{\phi}_{\mathcal{Q}}(\sigma)\, c^{\phi}(\mathcal{P}/\sigma\,; x)\,,$$

where the summation is taken over the poset of divisions by $\complement_{\mathcal{P}}\mathcal{Q}$.

Proof. Let us denote $\complement_{\mathcal{P}}\mathcal{Q}$ by $\mathcal{R}$. We will prove our theorem by induction on $|\mathrm{Non}(\mathcal{R})|$, noting that the induction starts at 0 because then $\mathcal{Q} = \mathcal{P}$, whilst $\mathit{\Pi}(\mathcal{R}) = \{\mathrm{At}(\mathcal{P})\}$, $\nu^{\phi}_{\mathcal{P}}(\mathrm{At}(\mathcal{P})) = 1$, and $\mathcal{P}/\mathrm{At}(\mathcal{P}) = \mathcal{P}$.

If $\mathrm{Non}(\mathcal{R}) \neq \emptyset$, we choose $W \in \mathrm{Non}(\mathcal{R})$ to be minimal with respect to inclusion, and let $\overline{W} := V(\mathcal{P}) \setminus W$. Then the inductive hypothesis yields

$$c^{\phi}(\mathcal{Q}\,; x) = \sum_{\sigma} \nu^{\phi}_{\mathcal{Q}}(\sigma)\, c^{\phi}((\mathcal{P} \setminus W)/\sigma\,; x)\,, \tag{4.10}$$

where the summation is taken over $\Pi(\mathcal{R} \setminus W)$. Given $\sigma \in \Pi(\mathcal{R} \setminus W)$, we have

$$(\mathcal{P} \setminus W)/\sigma = \begin{cases} (\mathcal{P}/\sigma) \setminus W & \text{if} \quad \sigma \leq \{W, \overline{W}\} \\ \mathcal{P}/\sigma & \text{otherwise} . \end{cases}$$

Consider a partition $\sigma \in \Pi(\mathcal{R} \setminus W)$ satisfying $\sigma \leq \{W, \overline{W}\}$. The choice of W implies that $\sigma|W = \mathrm{At}(\mathcal{P})|\mathrm{W}$; hence $(\mathcal{P}/\sigma)|W = \mathcal{P}|W$. Combining this fact with Theorem 4.4, we deduce

$$c^\phi((\mathcal{P}/\sigma) \setminus W \,; x) = c^\phi(\mathcal{P}/\sigma \,; x) - \mu^\phi_{\Pi(\mathcal{P}|W)}(\widehat{0}, \{W\})\, c^\phi((\mathcal{P}/\sigma)/W \,; x) \,. \tag{4.11}$$

Now $\sigma/W \in \Pi(\mathcal{R})$ and $(\mathcal{P}/\sigma)/W = \mathcal{P}/(\sigma/W)$. Recalling the choice of W again, we observe that $\nu^\phi_{\mathcal{Q}}(\sigma|W) = 1$ and $\mu^\phi_{\Pi(\mathcal{P}|W)}(\widehat{0}, \{W\}) = \mu^\phi_{\Pi((\mathcal{Q}|W)\cup\{W\})}(\widehat{0}, \{W\}) = -\nu^\phi_{\mathcal{Q}}(\{W\})$. Using these facts, and applying Proposition 3.5 (1) twice, we obtain

$$\nu^\phi_{\mathcal{Q}}(\sigma)\, \mu^\phi_{\Pi(\mathcal{P}|W)}(\widehat{0}, \{W\}) = -\nu^\phi_{\mathcal{Q}}(\sigma|W)\, \nu^\phi_{\mathcal{Q}}(\sigma|\overline{W})\, \nu^\phi_{\mathcal{Q}}(\{W\}) = -\nu^\phi_{\mathcal{Q}}(\sigma/W) \,. \tag{4.12}$$

According to (4.11) and (4.12), we may replace each term in the right-hand side of (4.10) corresponding to a partition $\sigma \leq \{W, \overline{W}\}$ with

$$\nu^\phi_{\mathcal{Q}}(\sigma)\, c^\phi(\mathcal{P}/\sigma \,; x) + \nu^\phi_{\mathcal{Q}}(\sigma/W)\, c^\phi(\mathcal{P}/(\sigma/W) \,; x) \,.$$

It remains only to define a bijection between the sets $\{\sigma \in \Pi(\mathcal{R} \setminus W) : \sigma \leq \{W, \overline{W}\}\}$ and $\{\sigma \in \Pi(\mathcal{R}) \,:\, W \in \sigma\}$; we take $\sigma \mapsto \sigma/W$, with inverse $\pi \mapsto (\mathrm{At}(\mathcal{P})|\mathrm{W}) \cup (\pi|\overline{\mathrm{W}})$, thereby concluding the induction. ☐

5. Properties of the Umbral Chromatic Polynomial

In this section, we begin by establishing our promised relation between the umbral chromatic polynomial and the characteristic type polynomial, generalizing the main result of [10] in passing, and enabling us to investigate further properties of the former. All the results in this section follow directly from those in the previous one, and mainly from Theorem 4.9.

Proposition 5.1. *For any partition system $\mathcal{P}$, we have*

$$\chi^\phi(\mathcal{P} \,; x) = c^\phi(\overline{\mathcal{P}} \,; x) \,.$$

Proof. This follows from Theorem 4.9 by considering the partition systems $\overline{\mathcal{P}} \subseteq \mathcal{K}_{V(\mathcal{P})}$, and noting that $c^\phi(\mathcal{K}_{V(\mathcal{P})}/\sigma \,; x) = c^\phi(\mathcal{K}_{|\sigma|} \,; x) = B^\phi_{|\sigma|}(x)$, according to (4.2). The right-hand side of the identity in Theorem 4.9 is precisely $\chi^\phi(\mathcal{P} \,; x)$. ☐

Proposition 5.1 reduces to Whitney's formula (1.1) after replacing the umbra ϕ with κ, as implied by the proof of Corollary 5.2. In fact, it reduces to a generalization of (1.1) to a formula for the *classical* chromatic polynomial of a partition system (or its homogenized version).

Corollary 5.2. *For any partition system $\mathcal{P}$, we have*

$$\chi^{\kappa}(\mathcal{P}\,;x) = c^{\kappa}(\mathcal{P}_c\,;x)\,.$$

Proof. Let us define the coclosure operator (cf. the proof of Proposition 3.5 (3)) $\sigma \mapsto \overline{\sigma}$ on $\Pi(\overline{\mathcal{P}})$, where the partition $\overline{\sigma}$ is obtained from σ by splitting every block B into the sets of vertices of the join-connected components of $\mathcal{P}|B$. Using a similar argument to the one in the proof of Proposition 3.5 (3), we obtain

$$\mu^{\kappa}_{\Pi(\overline{\mathcal{P}})}(\widehat{0},\sigma) = \begin{cases} \mu^{\kappa}_{\Pi(\mathcal{P}_c)}(\widehat{0},\sigma) & \text{if } \sigma \in \mathcal{P}_c \\ 0 & \text{otherwise}\,. \end{cases}$$

Hence $c^{\kappa}(\overline{\mathcal{P}}\,;x) = c^{\kappa}(\mathcal{P}_c\,;x)$, so we may now apply Proposition 5.1. □

We cannot use the map specified by (3.8) to transform the formula in Proposition 5.1 into a formula for the symmetric function $X_{\mathcal{K}}$ associated with the simplicial complex $\mathcal{K}$ in (3.6), because this map is not surjective. However, there is an analogue of Proposition 5.1 for $X_{\mathcal{K}}$, which generalizes Theorem 2.6 in [15], and which provides a different generalization of Whitney's formula. We present these results below.

Proposition 5.3. *We have*

$$X_{\mathcal{K}} = \sum_{\sigma\in\Pi(\mathcal{K}_c)} \mu_{\Pi(\mathcal{K}_c)}(\widehat{0},\sigma)\, p_{\lambda(\sigma)}\,.$$

Proof. We only have to adapt Stanley's proof to the context of simplicial complexes. Given σ in $\Pi(\mathcal{K}_c)$, we define

$$X_{\sigma} := \sum_{f} x^{f}\,, \tag{5.4}$$

where the summation ranges over all functions f from $V(\mathcal{K})$ to $\mathbb{N}$ satisfying $\overline{\ker(f)} \leq \sigma \leq \ker(f)$; here we have used the same coclosure operator as in the proof of Corollary 5.2. Given *any* $f\colon V(\mathcal{K}) \to \mathbb{N}$, there is a unique σ in $\Pi(\mathcal{K}_c)$, namely $\overline{\ker(f)}$, such that f is one of the maps appearing in the sum (5.4). It follows that for any π in $\Pi(\mathcal{K}_c)$, we have

$$p_{\lambda(\pi)} = \sum_{\sigma\geq\pi} X_{\sigma}\,.$$

By Möbius inversion,

$$X_{\pi} = \sum_{\sigma\geq\pi} p_{\lambda(\sigma)}\, \mu_{\Pi(\mathcal{K}_c)}(\pi,\sigma)\,.$$

But $X_{\widehat{0}} = X_{\mathcal{K}}$ (note that it is essential for $\mathcal{K}$ to be a simplicial complex), and the proof follows. □

Proposition 5.5. *We have*

$$\chi^{\phi}(\mathcal{K}\,;x) = \sum_{\sigma\in\Pi(\mathcal{K}_c)} \mu_{\Pi(\mathcal{K}_c)}(\widehat{0},\sigma)\, P^{\phi}_{\lambda(\sigma)}(x)\,,$$

where

$$P^{\phi}_{\lambda(\sigma)}(x) := \sum_{\pi\geq\sigma} \tau^{\phi}(\sigma,\pi)\, x^{|\pi|}\,, \qquad \tau^{\phi}(\sigma,\pi) := \sum_{\sigma\leq\rho\leq\pi} \zeta^{\phi}(\widehat{0},\rho)\, \mu^{\phi}_{\Pi(V)}(\rho,\pi)\,,$$

and $V := V(\mathcal{K})$.

Proof. We apply the map specified by (3.8) to the formula in Proposition 5.3. To this end, we compute the images of the power sum symmetric functions under this map. According to Theorem 1 in [3], we have

$$p_{\lambda(\sigma)} = \sum_{\pi\geq\sigma} \widetilde{m}_{\lambda(\pi)}\,.$$

Combining this result with (4.2), we find that the map specified by (3.8) sends $p_{\lambda(\sigma)}$ to

$$\begin{aligned}\sum_{\rho\geq\sigma} \tau^{\phi}(\rho)\, B^{\phi}_{|\rho|}(x) &= \sum_{\rho\geq\sigma} \zeta^{\phi}(\widehat{0},\rho) \sum_{\sigma\leq\rho\leq\pi} \mu^{\phi}_{\Pi(V)}(\rho,\pi)\, x^{|\pi|}\\ &= \sum_{\pi\geq\sigma} x^{|\pi|} \sum_{\sigma\leq\rho\leq\pi} \zeta^{\phi}(\widehat{0},\rho)\, \mu^{\phi}_{\Pi(V)}(\rho,\pi) = P^{\phi}_{\lambda(\sigma)}(x)\,.\end{aligned}$$

□

Let us note that $\tau^{\phi}(\sigma,\sigma) = \tau^{\phi}(\sigma)$ and $\tau^{\kappa}(\sigma,\pi) = 0$ unless $\sigma = \pi$, because we can pair the chains from σ to π contributing to $\tau^{\kappa}(\sigma,\pi)$ such that the contribution of each pair is 0. Hence, after replacing ϕ by κ, Proposition 5.5 reduces to a special case of Corollary 5.2, and to Whitney's formula if $\mathcal{K}$ is the independence complex of a graph.

We can immediately deduce from Theorem 4.8 another formula relating the umbral chromatic polynomial to the characteristic type polynomial of a partition system. To state our formula in a nice way, we recall from [7] the *umbral notation*, according to which we write $p(B^{\phi}(x))$ for the image of the polynomial $p(x)$ under the Φ_*-linear operator on $\Phi_*[x]$ mapping x^n to $B^{\phi}_n(x)$.

Proposition 5.6. *For any partition system* $\mathcal{P}$*, we have*

$$\chi^{\phi}(\mathcal{P}\,; x) = c^{\overline{\phi}}(\mathcal{P}\,; B^{\phi}(x))\,.$$

A deletion/contraction procedure for the umbral chromatic polynomial follows easily from Proposition 5.1.

Proposition 5.7. *Given any set* $U \in \mathrm{Non}(\mathcal{P})$*, we have*

$$\chi^{\phi}(\mathcal{P}\,; x) = \chi^{\phi}(\mathcal{P}\setminus U\,; x) + \nu^{\phi}_{\overline{\mathcal{P}}}(\{U\})\, \chi^{\phi}(\mathcal{P}/U\,; x)\,;$$

moreover, the identity still holds if we replace contraction by strong contraction.

Proof. Apply Proposition 5.1, Theorem 4.4, and the fact that $\overline{\mathcal{P}\setminus U} = \overline{\mathcal{P}} \cup \{U\}$ and $\overline{\mathcal{P}/U} = (\overline{\mathcal{P}} \cup \{U\})/U$. □

Proposition 5.7 provides an analogue of the well-known addition-contraction procedure for graphs (see [8]). There is no known deletion-contraction formula for the umbral chromatic polynomial of a graph H, which we could use to obtain a similar formula for X_H.

Not even $X_{\mathcal{K}}$ with $\mathcal{K}$ a simplicial complex has an obvious deletion-contraction formula (deducible from Proposition 5.7, for instance). One way around this problem is to define $X_{\mathcal{P}}$ for $\mathcal{P}$ an arbitrary partition system as in (3.6), with summation ranging over all colorings of $\mathcal{P}$; thus (3.7) also holds, with summation now ranging over $\Pi(\mathcal{P})$. Since $\Pi(\mathcal{P})$ is the disjoint union of $\Pi(\mathcal{P}\setminus\!\setminus U)$ and $\Pi(\mathcal{P}/U)$, where $U \in \mathrm{Non}(\mathcal{P})$ and

$$\mathcal{P}\setminus\!\setminus U := \mathcal{P} \setminus \{W \in \mathcal{P} \,:\, U \subseteq W\}\,,$$

we have

$$X_{\mathcal{P}} = X_{\mathcal{P}\setminus\!\setminus U} + X_{\mathcal{P}/U}\,. \tag{5.8}$$

As was observed in [11], formula (1.2) does not extend in a straightforward way to the umbral chromatic polynomial of a graph, although such a generalization was attempted in [8]. We offer here a superior version, as a special case of formula (5.10) for the umbral chromatic polynomial of a join of two partition systems (which corresponds to the disjoint union of graphs, after identification via the independence complex). To state our result, it is helpful to define an operation $\oplus$ by means of

$$\mathcal{P}_1 \oplus \mathcal{P}_2 := \overline{\overline{\mathcal{P}_1} \vee \overline{\mathcal{P}_2}}\,,$$

for any partition systems $\mathcal{P}_1$ and $\mathcal{P}_2$. Simultaneously, we replace ϕ by κ, and show that (1.2) *does* generalize to the *classical* chromatic polynomial for partition systems (in homogeneous form).

Proposition 5.9. *For arbitrary partition systems $\mathcal{P}_1$ and $\mathcal{P}_2$, we have*

$$\chi^{\phi}(\mathcal{P}_1 \vee \mathcal{P}_2\,; x) = \chi^{\phi}(\mathcal{P}_1\,; x)\chi^{\phi}(\mathcal{P}_2\,; x) - \sum_{\sigma} \nu^{\phi}_{\overline{\mathcal{P}_1}\cdot\overline{\mathcal{P}_2}}(\sigma)\,\chi^{\phi}((\mathcal{P}_1 \vee \mathcal{P}_2)/\sigma\,; x)\,, \tag{5.10}$$

where the summation is taken over non-$\widehat{0}$ divisions by the complement of the disjoint union $\overline{\mathcal{P}_1}\cdot\overline{\mathcal{P}_2}$ in $\overline{\mathcal{P}_1}\oplus\overline{\mathcal{P}_2}$. After replacing ϕ by κ, formula (5.10) reduces to

$$\chi^{\kappa}(\mathcal{P}_1 \vee \mathcal{P}_2\,; x) = \chi^{\kappa}(\mathcal{P}_1\,; x)\,\chi^{\kappa}(\mathcal{P}_2\,; x)\,. \tag{5.11}$$

Proof. According to Proposition 5.1 and (4.3), we have

$$\chi^{\phi}(\mathcal{P}_1 \vee \mathcal{P}_2\,; x) = c^{\phi}(\overline{\mathcal{P}_1}\oplus\overline{\mathcal{P}_2}\,; x)\,, \tag{5.12}$$

$$\text{and}\qquad \chi^{\phi}(\mathcal{P}_1\,; x)\,\chi^{\phi}(\mathcal{P}_2\,; x) = c^{\phi}(\overline{\mathcal{P}_1}\,; x)\,c^{\phi}(\overline{\mathcal{P}_2}\,; x) = c^{\phi}(\overline{\mathcal{P}_1}\cdot\overline{\mathcal{P}_2}\,; x)\,. \tag{5.13}$$

Substituting $\overline{\mathcal{P}_1}\cdot\overline{\mathcal{P}_2}$ for $\mathcal{Q}$ and $\overline{\mathcal{P}_1}\oplus\overline{\mathcal{P}_2}$ for $\mathcal{P}$ in Theorem 4.9, we get

$$c^{\phi}(\overline{\mathcal{P}_1}\cdot\overline{\mathcal{P}_2}\,; x) = \sum_{\sigma} \nu^{\phi}_{\overline{\mathcal{P}_1}\cdot\overline{\mathcal{P}_2}}(\sigma)\,c^{\phi}((\overline{\mathcal{P}_1}\oplus\overline{\mathcal{P}_2})/\sigma\,; x)\,,$$

where the summation is taken over divisions by the complement of $\overline{\mathcal{P}_1}\cdot\overline{\mathcal{P}_2}$ in $\overline{\mathcal{P}_1}\oplus\overline{\mathcal{P}_2}$. Identity (5.10) follows from the above relations by noting that $\overline{(\overline{\mathcal{P}_1}\oplus\overline{\mathcal{P}_2})/\sigma} = (\mathcal{P}_1\vee\mathcal{P}_2)/\sigma$, and then applying Proposition 5.1 once more.

To deduce (5.11) from (5.10), we prove that given any non-$\widehat{0}$ division σ by the complement of $\overline{\mathcal{P}_1}\cdot\overline{\mathcal{P}_2}$ in $\overline{\mathcal{P}_1}\oplus\overline{\mathcal{P}_2}$, the poset $\Pi((\overline{\mathcal{P}_1}\cdot\overline{\mathcal{P}_2})|\sigma)$ has a non-$\widehat{0}$ maximum element, and then apply

Proposition 3.5 (2). We produce the relevant maximum as follows: consider the partition $\pi \in \Pi(\overline{\mathcal{P}_1} \cdot \overline{\mathcal{P}_2})$ obtained from $(\sigma|V(\mathcal{P}_1)) \cup (\sigma|V(\mathcal{P}_2))$ by splitting into atoms each block which does not belong to either $\overline{\mathcal{P}_1}$ or $\overline{\mathcal{P}_2}$. By construction, π is the maximum of $\Pi(\overline{\mathcal{P}_1}\cdot\overline{\mathcal{P}_2})$, so we only have to show that it is different from $\widehat{0}$. By hypothesis, σ has a block W which intersects both $V(\mathcal{P}_1)$ and $V(\mathcal{P}_2)$, and this block cannot belong to $\mathcal{P}_1 \vee \mathcal{P}_2 = \overline{\overline{\mathcal{P}_1} \oplus \overline{\mathcal{P}_2}}$; hence $W \cap V(\mathcal{P}_i) \in \mathrm{Non}(\overline{\mathcal{P}_i})$ for $i = 1$ or $i = 2$, establishing that $\pi \neq \widehat{0}$. □

We may apply Proposition 5.9 to recover a more systematic version of a result of [8]. Consider graphs H_1 and H_2, and denote their disjoint union by $H_1 \sqcup H_2$ and $\overline{\mathcal{I}(H_i)}$ by $\mathcal{A}(H_i)$. For every non-$\widehat{0}$ division σ by the complement of $\mathcal{A}(H_1)\cdot\mathcal{A}(H_2)$ in $\mathcal{A}(H_1 \sqcup H_2)$, construct the graph $M_\sigma(H_1, H_2)$ with vertices the blocks of σ, and with edges joining either a non-singleton block to another block, or two singleton blocks if the corresponding vertices of $H_1 \sqcup H_2$ are adjacent.

Corollary 5.14. *The umbral chromatic polynomial of $H_1 \sqcup H_2$ is given by*

$$\chi^\phi(H_1 \sqcup H_2\,; x) = \chi^\phi(H_1\,; x)\,\chi^\phi(H_2\,; x) - \sum_\sigma \nu^\phi_{\mathcal{A}(H_1)\cdot\mathcal{A}(H_2)}(\sigma)\,\chi^\phi(M_\sigma(H_1, H_2)\,; x)\,,$$

where the summation is taken over all divisions σ specified in the construction.

Proof. The stated formula follows from (5.10) by replacing $\mathcal{P}_i$ with $\mathcal{I}(H_i)$, and observing that

$$\mathcal{I}(H_1 \sqcup H_2) = \mathcal{I}(H_1) \vee \mathcal{I}(H_2)\,, \qquad \mathcal{A}(H_1 \sqcup H_2) = \mathcal{A}(H_1) \oplus \mathcal{A}(H_2)\,,$$
$$\text{and} \qquad \mathrm{Sing}\,(\mathcal{I}(H_1 \sqcup H_2)/\sigma) = \mathcal{I}(M_\sigma(H_1, H_2))$$

for any division σ of the stated form. □

We recall that in [8], the divisions by $\mathcal{A}(H)$ were called the *admissible* partitions of $V(H)$, and those by the appropriate σ were labelled as *mixed* partitions of $V(H_1 \sqcup H_2)$.

We have seen that the umbral chromatic polynomial does not behave well with respect to the join of partition systems. However, the context of partition systems allows us to define an operation (multiplication) $\odot$ with respect to which the umbral chromatic polynomial is multiplicative. The definition is as follows:

$$\mathcal{P}_1 \odot \mathcal{P}_2 := \overline{\overline{\mathcal{P}_1} \cdot \overline{\mathcal{P}_2}}\,,$$

for any partition systems $\mathcal{P}_1$ and $\mathcal{P}_2$. Note that neither the family of graphs (identified with their independence complexes), nor the family of simplicial complexes are closed with respect to this operation. Let us also note that $\mathcal{P}_1 \vee \mathcal{P}_2 \subseteq \mathcal{P}_1 \odot \mathcal{P}_2$. The multiplicativity of the umbral chromatic polynomial with respect to $\odot$ is an immediate consequence of Proposition 5.1 and (4.3).

Proposition 5.15. *We have that*

$$\chi^\phi(\mathcal{P}_1 \odot \mathcal{P}_2\,; x) = \chi^\phi(\mathcal{P}_1\,; x)\,\chi^\phi(\mathcal{P}_2\,; x)\,.$$

This product formula for the the umbral chromatic polynomial of partition systems clearly reduces to (5.11), after replacing the umbra ϕ with κ.

6. Group Actions on Partition Systems

Let $\mathcal{S}$ be a partition system with singleton atoms, and let G be a subgroup of the group of automorphisms of $\mathcal{S}$ (which is a subgroup of the symmetric group on $V(\mathcal{S})$). We intend to give a combinatorial interpretation for the normalized polynomial $\chi^\phi(\mathcal{S}\,;x)/|G|$ in $\varPhi\mathbb{Q}_*[x]$.

We follow convention by writing the orbit and stabilizer of an element x under the action of G by $G(x)$ and G_x respectively. Thus $|G(x)| = |G|/|G_x|$ for all x. Given a partition $\sigma \in \varPi(\mathcal{S})$, we shall also let $G|\sigma$ denote the blockwise stabilizer $\bigcap_{B\in\sigma} G_B$ of σ.

Consider the graded ring $H_* := \mathbb{Z}[b_1, b_2, \dots]$, where b_i has degree i, and the sequence b of elements $(1 = b_0, b_1, b_2, \dots)$ therein. This notation is derived from topological applications, where H_* is a homology ring. We identify H_* with a subring of $\varPhi\mathbb{Q}_*$ via the inclusion $b_i \mapsto \phi_i/(i+1)!$, whence $\varPhi_*$ may be interpreted as a subring of H_*. Let $H_*\{x\}$ be the divided power algebra over H_*, meaning the free H_*-module generated by symbols $x^i/i!$, with obvious multiplication laws; we let $x^i/i!$ have degree i. Our identifications allow us to interpret $\varPhi_*[x]$ as a subring of $H_*\{x\}$, and $H_*\{x\}$ as a subring of $\varPhi\mathbb{Q}_*[x]$. Such algebras are a fundamental ingredient in the study of formal group laws over rings which may have torsion, and are described more fully in that context by Adams in [1]. It is straightforward to check that the normalized conjugate Bell polynomials $\beta^b_n(x) := B^\phi_n(x)/n!$ form an alternative basis for $H_*\{x\}$ over H_*, and we aim to establish that all $\chi^\phi(\mathcal{S}\,;x)/|G|$ (of which the $\beta^b_n(x)$ are a special case) lie in $H_{|V(\mathcal{S})|}\{x\}$. It is important to understand that when the universal ring H_* is replaced by a ring with torsion, we can no longer embed the corresponding divided power algebra into its rationalization. Thus we must replace the polynomials corresponding to $B^\phi_n(x)/n!$ by purely formal quotients; it is these that we realize combinatorially by considering set systems equipped with an automorphism group.

For our promised combinatorial interpretation, we need the additional concept of *ordered coloring*; we note that the corresponding concept for graphs differs from the one already appeared in the literature. For us, an ordered coloring of $\mathcal{S}$ is a pair (λ, f), where f is a coloring of $\mathcal{S}$, λ is a bijection from $[|V(\mathcal{S})|]$ to $V(\mathcal{S})$, and $f \circ \lambda$ is non-decreasing. We can interpret an ordered coloring as proceeding step-by-step, so that the colors are used in increasing order. An *ordered factorized coloring* of $\mathcal{S}$ is a triple (λ, γ, f), where (λ, f) is an ordered coloring, and (γ, f) is a factorized coloring for which $\lambda^{-1}(U)$ is an interval (in $\mathbb{N}$) for any block U of a partition in the chain γ. The type of such a coloring is defined by $\tau^b(\lambda, \gamma, f) := \tau^b(\gamma, f) \in H_*$. If $\mathcal{S}$ is a simplicial complex, then all the ordered factorized colorings are, of course, ordered colorings. We can define ordered coloring forests in a similar way, and we can view them as colorings based on forests of *plane* trees.

For any factorized coloring (γ, f) with $\tau^\phi(\gamma, f) = \pm\phi_1^{k_1}\phi_2^{k_2}\dots$, there are $(2!)^{k_1}(3!)^{k_2}\dots$ ways of choosing λ such that (λ, γ, f) is ordered. The types of all these colorings are equal, and their sum is $\tau^\phi(\gamma, f)$. Hence, when we regard $\chi^\phi(\mathcal{S}\,;m\phi)$ as an element of H_*, it enumerates by type the ordered factorized colorings of $\mathcal{S}$ with at most m colors. The group G acts on these colorings by $(g, (\lambda, \gamma, f)) \mapsto (g \circ \lambda, g\gamma, f \circ g^{-1})$, and each orbit has precisely $|G|$ elements. Therefore, $\chi^\phi(\mathcal{S}\,;m\phi)/|G| \in H_*$ enumerates by type τ^b the orbits of G on the set of ordered factorized colorings of $\mathcal{S}$ with at most m colors. It also enumerates orbits on the set of ordered coloring forests.

We may give an alternative statement of these facts in terms of the orbits of G simply on the set of factorized colorings. Given such a coloring (γ, f) with $\tau^\phi(\gamma, f) = \phi_1^{k_1}\phi_2^{k_2}\dots$, there are $(2!)^{k_1}(3!)^{k_2}\dots|G(\gamma, f)|/|G| = (2!)^{k_1}(3!)^{k_2}\dots/|G_{(\gamma,f)}|$ orbits of G on the set of ordered factorized colorings which map to $G(\gamma, f)$ via the map $G(\lambda, \gamma, f) \mapsto G(\gamma, f)$. Hence, $\chi^\phi(\mathcal{S}\,; m\phi)/|G|$ enumerates the orbits of G on the set of factorized colorings of $\mathcal{S}$ with at most m colors, each orbit $G(\gamma, f)$ giving a contribution of $\tau^\phi(\gamma, f)/|G_{(\gamma,f)}|$. If $\gamma = \{\widehat{0} < \sigma\}$, then $G_{(\gamma,f)} = G|\sigma$; if, in addition, $G|\sigma$ is the direct product of symmetric groups acting on the blocks of σ, then the contribution of the orbit $G(\gamma, f)$ to $\chi^\phi(\mathcal{S}\,; m\phi)/|G|$ is $\tau^b(\gamma, f)$. We remark that since $\chi^\phi(\mathcal{S}\,; m\phi)/|G|$ lies in H_* for all m, the polynomial $\chi^\phi(\mathcal{S}\,; x)/|G|$ must lie in $H_{|V(\mathcal{S})|}\langle\beta_i^b(x)\rangle$, and hence in $H_{|V(\mathcal{S})|}\{x\}$.

Example 6.1. Let $\mathcal{S} := \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{1, 2\}, \{2, 3\}, \{1, 3\}, \{1, 4\}, \{2, 4\}, \{3, 4\}, \{1, 2, 3\}\}$, and $G := \langle(1, 2), (2, 3)\rangle \cong S_3$. We have $\chi^\phi(\mathcal{S}\,; x)/|G| = (B_4^\phi(x) + 6\phi_1 B_3^\phi(x) + (\phi_2 + 3\phi_1^2)B_2^\phi(x))/6$. Hence, $\chi^\phi(\mathcal{S}\,; 2\phi) = (2\phi_2 + 6\phi_1^2)/6 = 2b_2 + 4b_1^2$. A transversal of the orbits of G on the set of ordered colorings of $\mathcal{S}$ with at most 2 colors is represented by $\{(1234, 1112),\ (4123, 2221),\ (1234, 1122),\ (2143, 1122),\ (3412, 2211),\ (4321, 2211)\}$, where we expressed the map λ by the word $\lambda(1)\lambda(2)\lambda(3)\lambda(4)$, and f by $f(1)f(2)f(3)f(4)$.

Our next goal is to give a combinatorial interpretation for the coefficients of the polynomial $\chi^\phi(\mathcal{S}\,; x)/|G|$ with respect to the bases $\{\beta_i^b(x) \,:\, i \geq 0\}$ and $\{x^i/i! \,:\, i \geq 0\}$ of $H_*\{x\}$. Our interpretation uses *preferential arrangements* of $V(\mathcal{S})$. Recall that a preferential arrangement of a finite set V is a pair (σ, λ), where σ is a partition of V, and λ is a bijection from $[|\sigma|]$ to σ, inducing a linear order on σ. Let $A(\mathcal{S})$ denote the set of preferential arrangements (σ, λ) of $V(\mathcal{S})$ with $\sigma \in \Pi(\mathcal{S})$. This set can be partially ordered by setting $(\pi, \lambda') \leq (\sigma, \lambda)$ if $\pi \leq \sigma$, and only adjacent blocks are amalgamated in order to obtain σ from π. Clearly, the obvious action of G on $A(\mathcal{S})$ preserves the ordering, whence we have an induced poset structure on the orbits $A(\mathcal{S})/G$ (an orbit is $\leq$ than another orbit if and only if it contains an element which is $\leq$ than some element of the second or
bit).

Lemma 6.2. *Given a sequence of polynomials $p_n(x)$ in $\Phi\mathbb{Q}_*[x]$, a map $w\colon \Pi(\mathcal{S}) \to \Phi\mathbb{Q}_*$, which is constant on the orbits of G on $\Pi(\mathcal{S})$, and an arbitrary transversal $\mathcal{T}$ of $A(\mathcal{S})/G$, we have that*

$$\sum_{(\sigma,f)\in\mathcal{T}} \frac{w(\sigma)}{|G|\sigma|}\,\frac{p_{|\sigma|}(x)}{|\sigma|!} = \frac{1}{|G|}\sum_{\sigma\in\Pi(\mathcal{S})} w(\sigma)\,p_{|\sigma|}(x)\,.$$

Proof. It suffices to observe that

$$\begin{aligned}\sum_{(\sigma,f)\in\mathcal{T}} \frac{w(\sigma)}{|G|\sigma|}\,\frac{p_{|\sigma|}(x)}{|\sigma|!} &= \frac{1}{|G|}\sum_{(\sigma,f)\in\mathcal{T}} |G(\sigma, f)|\,w(\sigma)\,\frac{p_{|\sigma|}(x)}{|\sigma|!}\\ &= \frac{1}{|G|}\sum_{(\sigma,f)\in A(\mathcal{S})} w(\sigma)\,\frac{p_{|\sigma|}(x)}{|\sigma|!}\\ &= \frac{1}{|G|}\sum_{\sigma\in\Pi(\mathcal{S})} w(\sigma)\,p_{|\sigma|}(x)\,. \quad \square\end{aligned}$$

Now consider an arbitrary poset P of partitions of V, assume that P contains the partition into singletons, and let G be a permutation group on V which also permutes P. Consider the poset $A(P)$ of all preferential arrangements (σ, λ) of V with $\sigma \in P$. Let $\widehat{A}(P) := A(P) \sqcup \{\widehat{0}\}$. Clearly, $A(\mathit{\Pi}(\mathcal{S})) = A(\mathcal{S})$, and we denote $A(\mathcal{S}) \sqcup \{\widehat{0}\}$ by $\widehat{A}(\mathcal{S})$. By insisting that $G(\widehat{0}) = \{\widehat{0}\}$, we obtain a poset action of G on $\widehat{A}(P)$, and hence an induced poset structure on the set of orbits $\widehat{A}(P)/G$.

Consider the H_*-incidence algebra of the poset $\widehat{A}(P)/G$, and in particular the element ζ^b, defined by $\zeta^b(G(\pi, \lambda'), G(\sigma, \lambda)) := \zeta^b(\pi, \sigma)$, where $\pi \leq \sigma$, $\zeta^b(\widehat{0}, \widehat{0}) := 1$, and

$$\zeta^b(\widehat{0}, G(\sigma, \lambda)) := \begin{cases} -1 & \text{if } \sigma = \widehat{0} \\ 0 & \text{otherwise}. \end{cases}$$

It is easy to see that the convolution inverse of ζ^b exists; it will be denoted by $\mu^b_{\widehat{A}(P)/G}$, or simply μ^b when the context is clear.

Theorem 6.3. *For any $(\sigma, \lambda) \in A(P)$, we have that*

$$\frac{\mu_P^\phi(\widehat{0}, \sigma)}{|G|\sigma|} = \mu^b_{\widehat{A}(P)/G}(\widehat{0}, G(\sigma, \lambda)),$$

so the former lies in H_*.

Proof. We proceed by induction on the maximum length of chains in $C(\widehat{0}, \sigma)$, the case 0 being clear. For $\sigma \neq \widehat{0}$, we have

$$\begin{aligned}
\frac{\mu_P^\phi(\widehat{0}, \sigma)}{|G|\sigma|} &= -\frac{1}{|G|\sigma|} \sum_{\widehat{0} \leq \pi < \sigma} \mu_P^\phi(\widehat{0}, \pi)\, \zeta^\phi(\pi, \sigma) \\
&= -\frac{1}{|G|\sigma|} \sum_{\widehat{0} \leq \pi < \sigma} |G|\pi|\, \mu^b(\widehat{0}, G(\pi, \lambda'(\pi, \sigma)))\, \zeta^\phi(\pi, \sigma) \\
&= -\frac{1}{|G|\sigma|} \sum_{(\pi', \lambda') < (\sigma, \lambda)} |G|\pi'|\, \mu^b(\widehat{0}, G(\pi', \lambda'))\, \zeta^b(G(\pi', \lambda'), G(\sigma, \lambda)) \\
&= -\sum_{(\pi'', \lambda'') \in \mathcal{T}} \mu^b(\widehat{0}, G(\pi'', \lambda''))\, \zeta^b(G(\pi'', \lambda''), G(\sigma, \lambda)) = \mu^b(\widehat{0}, G(\sigma, \lambda)),
\end{aligned}$$

where $\lambda'(\pi, \sigma)$ gives an ordering of the blocks of π such that $(\pi, \lambda'(\pi, \sigma)) < (\sigma, \lambda)$, and $\mathcal{T}$ is a transversal of the set of orbits $\{O \in A(P)/G \,:\, O < G(\sigma, \lambda)\}$. The first and the last equality follow from the definition of μ^ϕ and μ^b as convolution inverses. The second follows by induction, and the third is a consequence of the fact that

$$\zeta^\phi(\pi', \sigma) = \sum_{\lambda' :\, (\pi', \lambda') < (\sigma, \lambda)} \zeta^b(\pi', \sigma).$$

For each $(\pi', \lambda') < (\sigma, \lambda)$ we have $|(G|\sigma)(\pi', \lambda')| = |G|\sigma|/|G|\pi'|$; therefore, the fourth equality follows from showing that the map from the set of orbits of $G|\sigma$ on $\{(\pi', \lambda') \,:\, (\pi', \lambda') < (\sigma, \lambda)\}$ to the set $\{O \in A(P)/G \,:\, O < G(\sigma, \lambda)\}$, given by $(G|\sigma)(\pi', \lambda') \mapsto G(\pi', \lambda')$, is a bijection. Injectivity follows from the fact that (π_1', λ_1'), $(\pi_2', \lambda_2') < (\sigma, \lambda)$ and $g(\pi_1', \lambda_1') = (\pi_2', \lambda_2')$ imply $g \in G|\sigma$. Surjectivity follows from the chain of implications: $G(\pi'', \lambda'') < G(\sigma, \lambda) \;\Rightarrow\; g(\pi'', \lambda'') < (\sigma, \lambda)$ for some $g \in G \;\Rightarrow\; (G|\sigma)(g(\pi'', \lambda'')) \mapsto G(\pi'', \lambda'')$. □

Given $\sigma \in \Pi(\mathcal{S})$, we define an analogue of the Möbius type by

$$\nu^b_{\mathcal{S},G}(\sigma) := -\mu^b_{(\widehat{A}(\overline{\mathcal{S}}) \cup G(\sigma,\lambda))/G}(\widehat{0}, G(\sigma, \lambda)),$$

where λ gives an arbitrary ordering of the blocks of σ. Lemma 6.2 and Theorem 6.3 immediately imply the following result.

Corollary 6.4. *If $\mathcal{T}$ is an arbitrary transversal of $A(\mathcal{S})/G$, then*

$$\frac{\chi^\phi(\mathcal{S}\,;x)}{|G|} = \sum_{(\sigma,\lambda)\in\mathcal{T}} \nu^b_{\mathcal{S},G}(\sigma)\, \beta^b_{|\sigma|}(x) \qquad \textit{and} \tag{6.5}$$

$$\frac{c^\phi(\mathcal{S}\,;x)}{|G|} = \sum_{(\sigma,\lambda)\in\mathcal{T}} \mu^b_{\widehat{A}(\mathcal{S})/G}(\widehat{0}, G(\sigma, \lambda))\, \frac{x^{|\sigma|}}{|\sigma|!} \tag{6.6}$$

in $H_{|V(\mathcal{S})|}\{x\}$.

We can now combine (6.6) with Proposition 5.1 in order to express $\chi^\phi(\mathcal{S}\,;x)/|G|$ in terms of divided powers of x. Another useful result is the following. Let $\mathcal{IK}_n$ denote the set system consisting of all the sets in $\mathcal{K}_n$ which are intervals (in $\mathbb{N}$).

Corollary 6.7. *We have*

$$\beta^b_n(x) = \sum_{\sigma\in\Pi(\mathcal{IK}_n)} \mu^b_{\Pi(\mathcal{IK}_n)}(\widehat{0}, \sigma)\, \frac{x^{|\sigma|}}{|\sigma|!}\,.$$

Proof. This is immediate from (4.2) and (6.6) with $\mathcal{S} = \mathcal{K}_n$ and $G = S_n$, since the poset $A(\mathcal{K}_n)/S_n$ is isomorphic to $\Pi(\mathcal{IK}_n)$. □

The poset $\Pi(\mathcal{IK}_n)$ is, in fact, isomorphic to $(\mathcal{K}_{n-1}, \subseteq)$, so it is a Boolean algebra. Corollary 6.7 provides an expression for the conjugate Bell polynomials with less terms than (4.2), since $(\mathcal{K}_{n-1}, \subseteq)$ is smaller than the lattice $\Pi_n = \Pi(\mathcal{K}_n)$; in fact, it is much smaller for large n.

References

[1] J. F. Adams. *Stable Homotopy and Generalised Homology.* Chicago Univ. Press, Chicago, 1972.

[2] G. D. Birkhoff. A determinant formula for the number of ways of coloring a map. *Ann. of Math.*, 14(2):42–46.

[3] P. Doubilet. On the foundations of combinatorial theory VII: Symmetric functions through the theory of distribution and occupancy. *Studies Appl. Math.*, LI:377–396, 1972.

[4] M. Haiman and W. Schmitt. Incidence algebra antipodes and Lagrange inversion in one and several variables. *J. Combin. Theory Ser. A*, 50:172–185, 1989.

[5] M. Hazewinkel. *Formal Groups and Applications.* Academic Press, New York, 1978.

[6] J. Hunton and N. Ray A rational approach to Hopf rings. *J. Pure Appl. Algebra*, 101:313–333, 1995.

[7] R. Mullin and G.-C. Rota. On the foundations of combinatorial theory III: theory of binomial enumeration. Graph Theory and Its Applications, pages 168–213. Academic Press, New York, 1970.

[8] N. Ray. Tutte algebras of graphs and formal group theory. *Proc. London Math. Soc.*, 65:23–45, 1992.
[9] N. Ray and W. Schmitt. Coclosure operators and chromatic polynomials. *Proc. Natl. Acad. Sci. USA*, 87:4685–4687, 1990.
[10] N. Ray, W. Schmitt, and C. Wright. Colouring simplicial complexes. *Ars Combin.*, 29A:161–169, 1990.
[11] N. Ray and C. Wright. Colourings and partition types: a generalised chromatic polynomial. *Ars Combin.*, 25B:277–286, 1988.
[12] S. Roman. *The Umbral Calculus* Academic Press, Orlando, FL, 1984.
[13] G.-C. Rota. On the foundations of combinatorial theory I: theory of Möbius functions. *Z. Wahrsch. Verw. Gebiete*, 2:340–368, 1964.
[14] R. P. Stanley. *Enumerative Combinatorics*. Wadsworth & Brooks/Cole, Monterey, CA, 1986.
[15] R. P. Stanley. A symmetric function generalization of the chromatic polynomial of a graph. *Adv. Math.*, 111:166, 1995.
[16] D. Wagner. The partition polynomial of a finite set system. *J. Combin. Theory Ser. A*, 56:138–159, 1991.
[17] H. Whitney. A logical expansion in mathematics. *Bull. Amer. Math. Soc.*, 38:572–579, 1932.
[18] T. Zaslavsky. Generalized matchings and generalized Hermite polynomials. In A. Hajnal, L. Lovász, and V. T. Sos, editors, *Finite and infinite sets*, volume 37 of *Coll. Math. Soc. Janos Bolyai*, pages 851–865. North-Holland, 1984.

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