COMBINATORIAL MODELS FOR COALGEBRAIC STRUCTURES

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Abstract. We introduce a convenient category of combinatorial objects, known as cell-sets, on which we study the properties of the appropriate free abelian group functor. We obtain a versatile generalization of the notion of incidence coalgebra, giving rise to an abundance of coalgebras, Hopf algebras, and comodules, all of whose structure constants are positive integers with respect to certain preferred bases. Our category unifies and extends existing constructions in algebraic combinatorics, providing proper functorial descriptions; it is inspired in part by the notion of CW-complex, and is also geared to future applications in algebraic topology and the theory of formal group laws.

1. Introduction

The theory of coalgebras and Hopf algebras was first developed by algebraic topologists more than fifty years ago. Since the seminal work of Joni and Rota [13], applications to combinatorial mathematics have grown steadily in prominence, motivated by the principle that a diagonal, or coproduct map, is an efficient medium for encoding information about the different ways in which a discrete structure may be split into two pieces. This principle has often been couched in the language of incidence coalgebras, which were first associated to locally finite posets by Goldman and Rota [8] in 1970, and have recently surfaced in the algebraic literature [30]. A comprehensive exposition is given by Schmitt in [28].

Meanwhile, activity in algebraic topology (and increasingly in other fields) has drawn attention to the study of comodules, whose importance has been enhanced by the current wave of interest in quantum groups, culminating in the appearance of texts such as [16], [18], [20], and [29]. We refer readers to these sources for a detailed survey of the state of the coalgebraic art.

Our primary aim in this work is to introduce a corresponding theory of actions and coactions into algebraic combinatorics, based on proper categorical foundations rather than the customized constructions to be found in [4], [12], and [26]. To begin, we introduce the category **Cell** of *cell-sets* and study its simpler internal properties. An appropriate functor $Z_*\colon \mathbf{Cell} \to \mathbf{GAb}$ associates a graded abelian group $Z_*(\mathcal{C})$ to every cell-set $\mathcal{C}$, and transforms combinatorial information encoded within $\mathcal{C}$ to algebraic information in $Z_*(\mathcal{C})$.

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For example, product and coproduct morphisms on $\mathcal{C}$ translate respectively into algebra and coalgebra structures on $Z_*(\mathcal{C})$, which becomes a Hopf algebra whenever the two morphisms exist compatibly (and admit the requisite units and counits). Comodule structures arise in similar fashion. If $\mathcal{C}$ consists of a suitable collection of posets then $Z_*(\mathcal{C})$ is an incidence coalgebra, so that **Cell** allows a functorial approach to incidence coalgebra theory, incorporating more satisfying notions of product and coproduct than were previously available, and giving rise to novel comodules and quotient coalgebras.

Parallels with algebraic topology have motivated several of our constructions, and underlie three further goals that we explain below; but our presentation is unashamedly combinatorial, and reasonably self-contained. To reinforce this theme, we have attempted to provide a steady stream of illustrative examples, which we hope confirm the status of **Cell** as a versatile framework for algebraic combinatorics.

One of our more ambitious goals originates in algebraic features which are common to all our examples. By very definition $Z_*(\mathcal{C})$ has a preferred set of generators, and the structure constants of all our algebras, coalgebras and coactions are necessarily nonnegative integers with respect to such bases; indeed, the same constraint automatically applies to all algebraic homomorphisms induced by morphisms of cell-sets. Similar situations are common in many areas of applied algebra, and we would like to be able to model them combinatorially wherever they arise.

We have been guided by the occurrence of these phenomena in the theory of CW-complexes with cells in even dimensions, as exemplified by the projective spaces, Grassmannians, and flag manifolds of complex geometry. The cellular chain complex of such an X is generated over $\mathbb{Z}$ by its cells, and may be identified as a coalgebra with the integral homology groups $H_*(X)$. The common coproduct is induced by the diagonal map $\Delta\colon X \to X \times X$ and in many standard cases has positive structure constants with respect to the basis of cells. If X is endowed with product and inverse maps (as happens, for example, in the case of a topological group or loop space), then $H_*(X)$ becomes a Hopf algebra, whose product often has positive structure constants as well. Similar observations apply to various generalized homology groups $E_*(X)$, such as complex K-theory, complex cobordism, or Brown-Peterson homology [1]. In 1969, when Adams invited topologists to consider coactions on $E_*(X)$ by Hopf algebroids of homology cooperations, positive structure constants again emerged for the same examples.

The functor Z_* therefore shares several properties with the integral homology functor $H_*\colon \mathbf{ECW} \to \mathbf{GAb}$ (where **ECW** denotes the category of CW-complexes with even dimensional cells), and a considerable portion of our current work is influenced by the analogy. Indeed, a second goal is to pursue the analogy further, and to establish a commutative triangle of functors

$$(1)\qquad \begin{array}{ccc} \mathbf{ECW} & \longrightarrow & \mathbf{Cell} \\ & {\scriptstyle H_*}\searrow \quad \swarrow{\scriptstyle Z_*} & \\ & \mathbf{GAb} & \end{array} \; .$$

Such a program is tantamount to constructing combinatorial models for some of the simpler spaces of complex geometry, in terms which allow computation of the coaction maps associated with the dual Steenrod algebra. This can certainly be managed within the context of cell-sets, and readers who are well-versed in algebraic topology will already recognize many of the bialgabras and Hopf algebras we present below as arising from CW-spaces such as ΩS^3 and $BU(n)$, and spectra such as MU. To fully explain the details, however, it is helpful to consider situations in which our preferred bases are subject to *relations* with positive coefficients. For this extension we must enrich the category of cell-sets so as to include cell-set *pairs*; we may then import mod p coefficients, and make a serious study of antipodes within the combinatorial setting. Since the category of cell-set pairs is of independent interest, we defer the entire discussion until a sequel [23].

Our final goal involves replacing the homology functor H_* by a generalized version E_* in the triangle (1). To preserve commutativity, we must then replace Z_* by some other functor which recognizes the corresponding, more sophisticated combinatorial data encapsulated in the appropriate cell sets. For technical reasons, it is most reasonable to assume that E_* is *complex oriented*, thereby making contact with the theory of formal group laws, the Lazard ring, and associated Hopf algebras. Knowledgable readers will recognize that some of these structures, too, already occur in our examples below.

We expect to return to these matters at a later date, taking note of the work of Christian Lenart, whose thesis [17] already contains applications of related combinatorics to formal group laws and complex oriented homology theory.

We begin in §2 by defining our category **Cell**, whose objects are sets equipped with an equivalence relation and a compatible dimension function. The morphisms are more subtle, and expressed in terms of multisets, although we may avoid their use in practice by choosing appropriate representatives of equivalence classes; hence none of our applications requires anything more than straightforward set theory. We show that **Cell** admits simple constructions such as direct products, and introduce the functor $Z_*\colon \mathbf{Cell} \to \mathbf{GAb}$ by associating to each cell-set $\mathcal{C}$ the free abelian group $Z_*(\mathcal{C})$, generated by the equivalence classes of elements of $\mathcal{C}$ and graded by dimension.

In §3 we consider how the internal combinatorial richness of certain cell-sets gives rise to product and coproduct morphisms, and discuss the resulting coalgebraic and Hopf algebraic structures induced on $Z_*(\mathcal{C})$. By way of illustration, we realize binomial Hopf algebras in one and several variables, together with generalizations based on families of finite graphs [26].

Many of our cell-sets rely on the theory of partially ordered sets for their construction. We establish the appropriate notation and terminology in §4, highlighting the notion of *subgrading*.

We apply these ideas in §5, introducing cell-sets of finite intervals; many of them admit natural coproducts, which induce well-known *incidence coalgebra* structures on their free abelian groups. We formulate these constructions in the context of *interval categories*, and illustrate the theory by realizing various divided power coalgebras in one and several variables, and generalizations thereof.

Such examples often admit product structures, which we discuss in §6; many are closely related to the direct product operation on intervals, and we introduce the concept of a *productive category* to make this relationship precise. We are then able to realize binomial Hopf algebras, Hopf algebras of symmetric functions [6] and their duals [7], and the *Faà di Bruno Hopf algebra* [3]. Formulae for the corresponding antipodes are already familiar in the combinatorial literature [26].

Cell-sets based on *labelled* posets are also highly versatile, and we introduce them in §7. They are indispensable for many of our coactions, and we develop their properties in the context of *arrow categories*. A fundamental example is provided by a certain cell-set $\mathcal{N}$, which realizes the dual of the *Landweber-Novikov algebra* as $Z_*(\mathcal{N})$; this Hopf algebra is well-known to topologists [21], occurs in the theory of group schemes, and bears the same relation to ordinary formal power series as that of the Faà di Bruno algebra to formal Hurwitz series.

The framework of §7 allows us to introduce *quotient* interval cell-sets in §8, which correspond under the free abelian group functor to the construction of quotients of reduced incidence coalgebras.

We make further applications in §9, where we develop an important class of cell-set morphisms which may be informally described as the extraction of various families of embedded chains. The study of these morphisms raises fascinating questions related to the theory of formal groups, and imposes new product structures on some of our earlier coalgebras; in particular, we realize the Hopf algebra of quasi-symmetric functions, whose dual we identified in §6.

Finally, we return to arbitrary cell-sets in §10. Given a cell-set $\mathcal{A}$ equipped with a coproduct, we define its coaction on a cell-set $\mathcal{C}$ to be a morphism with the appropriate properties, and a coaction of $Z_*(\mathcal{A})$ on $Z_*(\mathcal{C})$ ensues. Several examples are driven by the combinatorial considerations of previous sections, and we focus on cases for which $Z_*(\mathcal{A})$ has a Hopf algebra structure, such as the dual Landweber-Novikov algebra and the Faà di Bruno algebra.

Many of the algebras that we study are graded, and involve considering formal sums b of the form $\sum_{k\geq 0} b_k$; in these circumstances, it is convenient to abbreviate the component of the nth power of b in grading k to $(b)^n_k$, and we do so without further comment. All tensor products of such algebras are taken in the graded sense, over $\mathbb{Z}$, although signs do not arise because all nonzero elements are of even grading.

2. The category of cell-sets

We now set up our category **Cell** of cell-sets. We assume given a universal set $\mathcal{U}$ (as in Mac Lane [18], for example), closed under standard operations such as cartesian product and disjoint union, whose elements we shall not normally specify. All our sets will be elements of $\mathcal{U}$, so that any family or system of sets that we might consider will itself be a set.

An object of **Cell** is a triple $(\mathcal{C}, \sim, d)$, where $\mathcal{C}$ is a set, $\sim$ is an equivalence relation on $\mathcal{C}$, and $d\colon \mathcal{C} \to \mathbb{Z}$ is a *dimension function*, with $d(x) = d(y)$ whenever $x \sim y$; we write $\mathcal{C}_n$ for the elements of $\mathcal{C}$ of dimension n. We often abbreviate a triple $(\mathcal{C}, \sim, d)$ to $\mathcal{C}$ whenever $\sim$ and d are understood, and write the equivalence class of $x \in \mathcal{C}$ as $\langle x \rangle$. We refer to our objects as *cell-sets*, and to any object $\mathcal{T}$ consisting of a single element as a *trivial* cell-set. A cell-set $\mathcal{C}$ is *pointed* if $\mathcal{C}_0$ contains a distinguished equivalence class, written as 1, whose representatives are called *base points*; and $\mathcal{C}$ is *connected* if $\mathcal{C}_0$ contains exactly one equivalence class, in which case it may naturally be pointed by taking the elements of $\mathcal{C}_0$ as base points. A trivial cell-set is both connected and pointed.

Many of our cell-sets occur as the object set of some small category $\mathbf{C}$, with equivalence given by isomorphism in the category. We refer to $\mathbf{C}$ as a *cell-category* whenever the objects are equipped with an integer-valued *dimension function* which is constant on isomorphism classes. The set of objects is then a cell-set $\mathcal{C}$, which we call the *object cell-set of* $\mathbf{C}$; we refer to isomorphisms in $\mathbf{C}$ as $\mathcal{C}$-*isomorphisms*. Wherever possible we maintain the convention of denoting cell-categories and their object cell-sets by boldface and calligraphic versions of the same symbol, making occasional exception in situations where an object cell-set arises from distinct cell-categories.

If U and V are two subsets of a general cell-set $\mathcal{C}$, we write $U \sim V$ whenever there exists a bijection $U \leftrightarrow V$ such that corresponding elements are equivalent under $\sim$. We extend this convention to multisets on $\mathcal{C}$ in the obvious manner.

If $\mathcal{C}$ and $\mathcal{D}$ are cell-sets, then a *cell-map* $f\colon \mathcal{C} \to \mathcal{D}$ is a correspondence which assigns a finite multiset $f(x)$ on $\mathcal{D}$ to each element x of $\mathcal{C}$ in such a way that

$$x \sim y \quad \Longrightarrow \quad f(x) \sim f(y). \tag{2}$$

Unless otherwise stated we shall insist that f preserves dimension, in the sense that every element of $f(x)$ has dimension $d(x)$, for all $x \in \mathcal{C}$. The composition of cell-maps $g\colon \mathcal{C} \to \mathcal{D}$ and $f\colon \mathcal{D} \to \mathcal{E}$ is given by

$$fg(x) \;=\; \bigcup_{y \in g(x)} f(y), \tag{3}$$

where $\cup$ denotes the usual union of multisets whenever necessary. The identity cell-map $1_{\mathcal{C}}$ is defined by $1_{\mathcal{C}}(x) = \{x\}$, for all $x \in \mathcal{C}$.

Our definition of cell-map is related to Henle's notion of morphism between *dissects* [12], and also generalizes Joyal's construction of *decomposition laws* on groupoids [15]; it is simple to verify that **Cell** becomes a category when cell-maps are taken as morphisms. The appearance of multisets is a technical necessity to ensure the properties of composition, but we emphasize that ordinary set theory suffices to describe all examples of cell-maps which we consider. We note that there are unique cell-maps $\emptyset \to \mathcal{C}$ and $\mathcal{C} \to \emptyset$ for any cell-set $\mathcal{C}$, so that $\emptyset$ is a null object (that is, both initial and terminal) in the category **Cell**.

A *subcell-set* $\mathcal{D}$ of $\mathcal{C}$ is a (possibly empty) collection of nonempty, finite multisets on $\mathcal{C}$ such that the dimension function of $\mathcal{C}$ is constant on each $U \in \mathcal{D}$. The equivalence relation on $\mathcal{D}$ is that induced by $\sim$ on multisets of $\mathcal{C}$ and the dimension function is the obvious one. By identifying each element $x \in \mathcal{C}$ with the singleton set $\{x\}$, any subset U of $\mathcal{C}$ can be considered as a subcell-set of $\mathcal{C}$. For notational convenience we shall often make this identification of elements with singleton sets. Whenever $\mathcal{D}$ is a subcell-set of $\mathcal{C}$ the *inclusion* cell-map $i_{\mathcal{D},\mathcal{C}} \colon \mathcal{D} \to \mathcal{C}$, given by $i_{\mathcal{D},\mathcal{C}}(U) = U$ for all U in $\mathcal{D}$, is a monomorphism.

If $\mathcal{C}$ and $\mathcal{D}$ are pointed and $f \colon \mathcal{C} \to \mathcal{D}$ is a cell-map such that $f(u)$ consists entirely of base points whenever u is a base point, then f is a *pointed* cell-map. There is a unique pointed cell-map $\epsilon \colon \mathcal{C} \to \mathcal{T}$ whenever $\mathcal{C}$ is pointed and $\mathcal{T}$ is trivial, but there are many pointed cell-maps $\eta \colon \mathcal{T} \to \mathcal{C}$ in general. Trivial cell-sets are therefore terminal objects in the category **PCell** of pointed cell-sets and pointed cell-maps, but are not initial. Henceforth, we assume that all cell-maps between pointed cell-sets are pointed.

Cell-maps $f, f' \colon \mathcal{C} \to \mathcal{D}$ are *equivalent*, denoted by $f \sim f'$, if $f(x) \sim f'(x)$ for all $x \in \mathcal{C}$; our employment of multisets ensures that $fg \sim f'g'$ whenever $f \sim f'$ and $g \sim g'$ and the composition is defined. We may therefore define categories **Cell**$'$ and **PCell**$'$ by retaining the original objects, but taking equivalence classes of the appropriate cell-maps as morphisms; there are canonical functors **Cell** $\to$ **Cell**$'$ and **PCell** $\to$ **PCell**$'$, each of which is the identity on objects, but maps cell-maps f to their equivalence classes $\langle f \rangle$. Cell-sets $\mathcal{C}$ and $\mathcal{D}$ are *equivalent* if there exist cell-maps $f \colon \mathcal{C} \to \mathcal{D}$ and $g \colon \mathcal{D} \to \mathcal{C}$ such that $gf \sim 1_{\mathcal{C}}$ and $fg \sim 1_{\mathcal{D}}$.

For any cell-set $\mathcal{C}$ and $n \in \mathbb{Z}$, we let $Z_n(\mathcal{C})$ denote the free abelian group $\mathbb{Z}\{\mathcal{C}_n/{\sim}\}$ generated by the set $\mathcal{C}_n/{\sim}$ of equivalence classes of n-dimensional elements of $\mathcal{C}$, and we denote by $Z_*(\mathcal{C})$ the graded abelian group consisting of the sequence $Z_n(\mathcal{C})$. We refer to $Z_*(\mathcal{C})$, somewhat imprecisely, as the *free abelian group on* $\mathcal{C}$; for future reference, we note that the corresponding *free* R*-module on* $\mathcal{C}$, written $R_*(\mathcal{C})$, may be constructed for any commutative ring R with unit.

A cell-map $f\colon \mathcal{C} \to \mathcal{D}$ determines a homomorphism $f_*\colon Z_*(\mathcal{C}) \to Z_*(\mathcal{D})$ by

$$f_*\langle x\rangle = \sum_{y\in f(x)} \langle y\rangle, \qquad \text{for all } x \in \mathcal{C}, \tag{4}$$

so that Z_* is a functor $\mathbf{Cell} \to \mathbf{GAb}$ into the category of graded abelian groups. Because cell-maps f and g are equivalent if and only if $f_* = g_*$, it follows that Z_* factors through the canonical functor $\mathbf{Cell} \to \mathbf{Cell}'$.

If $\mathcal{T}$ is trivial and $\mathcal{C}$ is pointed, then the cell-map $\epsilon\colon \mathcal{C} \to \mathcal{T}$ induces an augmentation $\epsilon_*\colon Z_*(\mathcal{C}) \to Z_*(\mathcal{T}) \cong \mathbb{Z}$. Any cell-map $\eta\colon \mathcal{T} \to \mathcal{C}$ is a right inverse for ϵ, and so η_* includes a copy of the integers as a summand of $Z_*(\mathcal{C})$, called the *augmentation summand*. If $\mathcal{C}$, $\mathcal{D}$ and $f\colon \mathcal{C} \to \mathcal{D}$ are pointed, then f_* preserves augmentations. Thus we have a functor, also denoted by Z_*, from the category $\mathbf{PCell}$ of pointed cell-sets to to the category of augmented $\mathbb{Z}$-graded abelian groups (with augmentation preserving homomorphisms) which factors through the canonical functor $\mathbf{PCell} \to \mathbf{PCell}'$. We obtain a new functor $\widetilde{Z}_*\colon \mathbf{PCell} \to \mathbf{GAb}$ by defining $\widetilde{Z}_*(\mathcal{C})$ as the kernel of the augmentation, and $\widetilde{Z}_*(f)$ as the restriction of f_* to $\widetilde{Z}_*(\mathcal{C})$ for any pointed cell-map f. We refer to $\widetilde{Z}_*(\mathcal{C})$ as the *reduced group* on $\mathcal{C}$.

We may perform standard operations on cell-sets, which are respected by the free abelian group functor in an appropriate sense. For example, given cell-sets $\mathcal{C}$ and $\mathcal{D}$ their *disjoint union* $\mathcal{C} \sqcup \mathcal{D}$ is defined on the disjoint union of their underlying sets by taking the obvious equivalence relation, and the dimension function specified by $(\mathcal{C} \sqcup \mathcal{D})_n = \mathcal{C}_n \sqcup \mathcal{D}_n$; there is therefore an isomorphism

$$Z_*(\mathcal{C} \sqcup \mathcal{D}) \cong Z_*(\mathcal{C}) \oplus Z_*(\mathcal{D}).$$

Similarly, the *direct product* $\mathcal{C} \times \mathcal{D}$ is formed from the cartesian product of the underlying sets, with equivalence relation defined coordinatewise and dimension function given by $d(x, y) = d(x) + d(y)$; thus

$$(\mathcal{C} \times \mathcal{D})_n = \bigcup_k (\mathcal{C}_k \times \mathcal{D}_{n-k}),$$

and there is an isomorphism

$$Z_*(\mathcal{C} \times \mathcal{D}) \cong Z_*(\mathcal{C}) \otimes Z_*(\mathcal{D}), \tag{5}$$

defined by $\langle (x, y)\rangle \mapsto \langle x\rangle \otimes \langle y\rangle$ for all $x \in \mathcal{C}$ and $y \in \mathcal{D}$. If $\mathcal{C}$ and $\mathcal{D}$ are pointed so is $\mathcal{C}\times\mathcal{D}$, by selecting as base points all those pairs (u, v) for which u and v are base points of $\mathcal{C}$ and $\mathcal{D}$ respectively.

If $f\colon \mathcal{C} \to \mathcal{D}$ and $f'\colon \mathcal{C}' \to \mathcal{D}'$ are cell-maps, their direct product $(f \times f')\colon (\mathcal{C} \times \mathcal{C}') \to (\mathcal{D} \times \mathcal{D}')$ is defined by letting $(f \times f')(x, y)$ be the cartesian product $f(x) \times f'(y)$ of the multisets $f(x)$ and $f'(y)$. The natural isomorphisms of cell-sets $(\mathcal{C} \times \mathcal{C}) \times \mathcal{C} \leftrightarrow \mathcal{C} \times (\mathcal{C} \times \mathcal{C})$ and $\mathcal{C} \times \mathcal{T} \leftrightarrow \mathcal{C} \leftrightarrow \mathcal{T} \times \mathcal{C}$, together with the *switch* automorphism τ of $\mathcal{C} \times \mathcal{C}$ given by $\tau(x, y) = (y, x)$, then bestow a symmetric monoidal structure [18] on $\mathbf{Cell}$.

3. Products and Coproducts on Cell-Sets

We now consider cell-sets which are equipped with deeper combinatorial structure, and discuss the corresponding enhancement of their free abelian groups.

A *product* on $\mathcal{C}$ is a cell-map $\mu\colon \mathcal{C}\times\mathcal{C}\to\mathcal{C}$ which satisfies $\mu(1_{\mathcal{C}}\times\mu)\sim\mu(\mu\times 1_{\mathcal{C}})$, and so is associative up to equivalence. If $\mathcal{C}$ admits a product μ, then by appeal to (5) we see that μ induces an associative product on $Z_*(\mathcal{C})$. If $\mathcal{C}$ is pointed and μ satisfies $\mu(u,x)\sim\mu(x,u)\sim x$ for all base points u and arbitrary x in $\mathcal{C}$, then $Z_*(\mathcal{C})$ is a ring with augmentation and has identity element 1. In this case, if η is any of the equivalent cell-maps $\mathcal{T}\to\mathcal{C}$, where $\mathcal{T}$ is trivial, then $\eta_*\colon\mathbb{Z}\to Z_*(\mathcal{C})$ is the corresponding unit map. If $\mathcal{C}$ and $\mathcal{D}$ are cell-sets with respective products $\mu_{\mathcal{C}}$ and $\mu_{\mathcal{D}}$, and $f\colon\mathcal{C}\to\mathcal{D}$ is a cell-map which satisfies $f\mu_{\mathcal{C}}\sim\mu_{\mathcal{D}}(f\times f)$, and so commutes with products up to equivalence, then f_* is a ring homomorphism. We remark that $\mathcal{C}\times\mathcal{D}$ admits the product $\mu_{\mathcal{C}\times\mathcal{D}}=(\mu_{\mathcal{C}}\times\mu_{\mathcal{D}})(1_{\mathcal{C}}\times\tau\times 1_{\mathcal{D}})$.

A *coproduct* on $\mathcal{C}$ is a cell-map $\delta\colon\mathcal{C}\to\mathcal{C}\times\mathcal{C}$ which satisfies $(1_{\mathcal{C}}\times\delta)\delta\sim(\delta\times 1_{\mathcal{C}})\delta$, and so is coassociative up to equivalence; whenever such a coproduct exists, we again appeal to (5) to see that δ induces a coassociative coproduct on $Z_*(\mathcal{C})$. If $\mathcal{C}$ is pointed and δ is such that $\delta(x)$ always contains unique elements (u,x) and (x,v) for some base points u and v (these elements may coincide if x is a base point), then the augmentation ϵ_* is a counit in this coalgebra. If $\mathcal{C}$ and $\mathcal{D}$ are cell-sets with respective coproducts $\delta_{\mathcal{C}}$ and $\delta_{\mathcal{D}}$, and $f\colon\mathcal{C}\to\mathcal{D}$ is a cell-map which satisfies $\delta_{\mathcal{D}}f\sim(f\times f)\delta_{\mathcal{C}}$, and so commutes with coproducts up to equivalence, then f_* is a coalgebra homomorphism. Of course $\mathcal{C}\times\mathcal{D}$ admits the coproduct $\delta_{\mathcal{C}\times\mathcal{D}}=(1_{\mathcal{C}}\times\tau\times 1_{\mathcal{D}})(\delta_{\mathcal{C}}\times\delta_{\mathcal{D}})$.

Suppose that a cell-set $\mathcal{C}$ admits both a product μ and a coproduct δ. These structures may be related by the formula $\delta\mu\sim\mu_{\mathcal{C}\times\mathcal{C}}(\delta\times\delta)$, so that μ respects δ up to equivalence, or by $(\mu\times\mu)\delta_{\mathcal{C}\times\mathcal{C}}\sim\delta\mu$, so that δ respects μ up to equivalence. The two alternatives are actually identical, and equivalent to the condition

$$\delta\mu\sim(\mu\times\mu)(1_{\mathcal{C}}\times\tau\times 1_{\mathcal{C}})(\delta\times\delta). \tag{6}$$

Whenever μ and δ satisfy (6), we say that they are *compatible.* In this case $Z_*(\mathcal{C})$ is a graded bialgebra with product μ_* and coproduct δ_*.

If $\mathcal{C}$ is pointed we assume that μ and δ are pointed cell-maps as above, so that η respects δ and ϵ respects μ; thus η_* and ϵ_* provide a unit and counit respectively for the bialgebra $Z_*(\mathcal{C})$. If $\mathcal{C}$ is connected we may then construct an antipode, and so ensure that $Z_*(\mathcal{C})$ is actually a Hopf algebra. We outline this construction below, by way of contrast with our future work [23] in which an antipode may be induced by a corresponding map of *cell-set pairs.*

We assume given a connected bialgebra H, with structure maps μ, δ, η and ϵ. The set of $\mathbb{Z}$-linear endomorphisms $\mathrm{Hom}(H,H)$ is a $\mathbb{Z}$-algebra, with the usual *convolution* product of endomorphisms f and g given by

$\mu(f \otimes g)\delta$. The identity for this product is the composition $\eta\epsilon$, which is in general different from the identity map 1_H.

Lemma 7. *The function*

$$\chi = \sum_{n\geq 0}(\eta\epsilon - 1_H)^n$$

acts as an antipode for H, which therefore becomes a Hopf algebra.

Proof. Since H is connected we have a bialgebra filtration $H_0 \subseteq H_1 \subseteq \cdots$, with H_0 (the coradical of H) isomorphic to $\mathbb{Z}$. Thus, in particular, $\eta\epsilon|H_0 = 1_{H_0}$ or, equivalently, $H_0 \subseteq \ker(\eta\epsilon - 1_H)$. Since $\delta(H_n) \subseteq \bigoplus_{i+j=n} H_i \otimes H_j$, it follows that $H_{n-1} \subseteq \ker(\eta\epsilon - 1_H)^n$, for all $n \geq 1$, and therefore the power series $\sum_{n\geq 0}(\eta\epsilon - 1_H)^n$ is a well-defined convolution inverse for $1_H = \eta\epsilon - (\eta\epsilon - 1_H)$ in $\mathrm{Hom}(H, H)$. □

The proof of Lemma 7 works equally well over an arbitrary commutative ring R with unit, and therefore also applies to the free R-module $R_*(\mathcal{C})$. Motivated by this result, we refer to any connected cell-set with compatible product and coproduct as a *Hopf cell-set*. We decree that Hopf cell-sets $\mathcal{C}$ and $\mathcal{D}$ are *equivalent* if there is an equivalence of cell-sets $f\colon \mathcal{C} \to \mathcal{D}$ and $g\colon \mathcal{D} \to \mathcal{C}$ such that the cell-maps f and g commute with products and coproducts, up to equivalence. In this case, the maps $f_*\colon Z_*(\mathcal{C}) \to Z_*(\mathcal{D})$ and $g_*\colon Z_*(\mathcal{D}) \to Z_*(\mathcal{C})$ are inverse Hopf algebra isomorphisms.

We now give some fundamental examples of Hopf cell-sets.

Example 8. Let $\mathcal{S}$ be the collection of all finite sets (in our universe $\mathcal{U}$), define $U \sim V$ whenever the cardinalities $|U|$ and $|V|$ are equal, and set $d(U) = |U|$ for the dimension function. Then $\mathcal{S}$ is a connected cell-set with base point the empty set $\emptyset$, and the free abelian group $Z_*(\mathcal{S})$ has one generator x_k in each dimension k. We define a product μ on $\mathcal{S}$ by letting $\mu(U, V)$ be the disjoint union $U \sqcup V$, which induces $x_j x_k = x_{j+k}$ on $Z_*(\mathcal{S})$. We define a coproduct δ by

$$\delta(U) = \{(V, W) : \ V \cup W = U \text{ and } V \cap W = \emptyset\},$$

which in turn induces

$$\delta_*(x_k) = \sum_{i=0}^{k} \binom{k}{i} x_i \otimes x_{k-i} \tag{9}$$

on $Z_*(\mathcal{S})$. The product and coproduct on $\mathcal{S}$ are compatible, so that $\mathcal{S}$ is a Hopf cell-set and the Hopf algebra $Z_*(\mathcal{S})$ is isomorphic to the polynomial algebra $\mathbb{Z}[x]$, with coproduct determined by $\delta_*(x) = x \otimes 1 + 1 \otimes x$ and antipode by $\chi(x) = -x$. □

Example 10. A *partition* is a finite set σ whose elements, called *blocks*, are pairwise disjoint, nonempty finite sets. If V is the union of the blocks of σ we call σ a *partition of* V. We make the collection $\mathcal{S}_\pi$ of all partitions into a cell-set by defining σ and τ as equivalent whenever there exists a bijection $f\colon \sigma \to \tau$ which preserves cardinalities, and by setting $d(\sigma)$ equal to $\sum_{B\in\sigma}|B|$. The cell-set $\mathcal{S}_\pi$ is connected, having the empty partition $\emptyset$ as base point. If σ is a partition of V and $U \subseteq V$, the restriction $\sigma|U$ of σ to U is the partition of U given by

$$\sigma|U = \{B \cap U \colon B \in \sigma \text{ and } B \cap U \neq \emptyset\}.$$

We define a coproduct δ on $\mathcal{S}_\pi$ by setting

$$\delta(\sigma) = \{(\sigma|U, \sigma|W) : U \cup W = V \text{ and } U \cap W = \emptyset\},$$

whenever σ is a partition of V; and a product μ by

$$\mu(\sigma, \tau) = \sigma \sqcup \tau.$$

The product and coproduct on $\mathcal{S}_\pi$ are compatible and thus $\mathcal{S}_\pi$ is a Hopf cell-set. The Hopf algebra $Z_*(\mathcal{S}_\pi)$ is the polynomial algebra $\mathbb{Z}[x_1, x_2, \ldots]$, where x_i has dimension i, the monomial $\prod_{B\in\sigma} x_{|B|}$ denotes the equivalence class of the partition σ, and the coproduct is given by (9). The expression for the antipode χ of $Z_*(\mathcal{S}_\pi)$ is considerably more complicated than that for the antipode of $Z_*(\mathcal{S})$. Two formulae for χ are

$$\chi(x_n) = \sum_{k=0}^{n} (-1)^k k! B_{n,k}(x_1, x_2, \ldots),$$

where the $B_{n,k}(x_1, x_2, \ldots)$ are the partial Bell polynomials, and

$$\chi(x_n) = (-1)^n \det\left(\binom{n-i+1}{n-j} x_{j-i+1}\right)_{1\le i,j\le n},$$

which both may be found in [26]. □

The following example generalizes the previous two.

Example 11. Suppose G is a graph with vertex set $V(G)$ and edge set $E(G)$. If $U \subseteq V(G)$, then the *induced subgraph* $G|U$ is the graph having vertex set U and edge set consisting of all edges of G which have both end-vertices contained in U.

Suppose $\mathcal{G}$ is a family of finite graphs which is closed under formation of disjoint unions and induced subgraphs. We make $\mathcal{G}$ into a cell-set by letting equivalence be given by graph isomorphism, and setting the dimension of a graph equal to the cardinality of its vertex set. We define a product on $\mathcal{G}$ by letting $\mu(G, H)$ be the disjoint union of the graphs G and H, and a coproduct δ by setting

$$\delta(G) = \{(G|U, G|W) : U \cup W = V(G) \text{ and } U \cap W = \emptyset\}.$$

It follows from the general antipode formula for incidence Hopf algebras, given in [24], that the antipode χ of $Z_*(\mathcal{G})$ satisfies

$$\chi\langle G\rangle = \sum_{\pi} (-1)^{|\pi|} |\pi|! \langle G|\pi\rangle,$$

where the sum is over all partitions π of the vertex set $V(G)$, and $G|\pi$ denotes the graph obtained from G by deleting all edges whose end-vertices lie in different blocks of π. It was shown in [26] that the elements

$$\lambda\langle G\rangle = \sum_{\pi}(-1)^{|\pi|-1}(|\pi|-1)!\langle G|\pi\rangle$$

of $Z_*(\mathcal{G})$ are primitive for all $G \in \mathcal{G}$, and that $Z_*(\mathcal{G})$ is polynomial on the set of primitive indeterminants $\{\lambda\langle G\rangle : G \in \mathcal{G} \text{ is connected}\}$. □

The simplicity of this approach contrasts favorably with that of [24], where the Hopf algebras $Z_*(\mathcal{G})$ were introduced as incidence Hopf algebras of hereditary families of partially ordered sets. The Hopf cell-sets of Examples 8 and 10 correspond to the special cases in which $\mathcal{G}$ is the family of all graphs with no edges, and all disjoint unions of complete graphs, respectively. It was shown in [22] that a well-known identity of Tutte [31] is equivalent to the fact that the map which sends a graph to its chromatic polynomial induces a Hopf algebra map from $Z_*(\mathcal{G})$ to the Hopf algebra $\mathbb{Z}[x]$ of Example 8.

Other Hopf algebras arising from graphs, which also fit neatly into the framework of cell-sets, were studied in [25], [26] and [27].

4. Partially ordered sets

As we shall explain, a wide variety of cell-sets with product and coproduct structures may be assembled from partially ordered sets (or *posets* for short). We therefore devote this section to establishing some of the basic notation and terminology that are required for their application.

A *subposet* of a poset P is a subset Q of P which is partially ordered in such a way that $x \leq y$ in Q implies $x \leq y$ in P. A subposet Q is *induced* if it has the induced partial ordering, given by insisting that $x \leq y$ in Q if and only if $x \leq y$ in P. A subposet of a poset P is called *spanning* if its sets of minimal and maximal elements are respectively contained in the sets of minimal and maximal elements of P. An important type of subposet is the *subinterval* $[x, y] = \{z : x \leq z \leq y\}$, defined for all $x \leq y$ in P. If P has unique minimal and maximal elements x and y, then $P = [x, y]$ is an *interval*, and for all $z \in P$ we denote the subintervals $[x, z]$ and $[z, y]$ by P_z and P^z respectively.

If x and y are elements of a poset P we say that y *covers* x if $x < y$ and there is no element z of P such that $x < z < y$. A chain $x_0 < x_1 < \cdots < x_n$ in a finite poset P is *saturated* if x_i covers x_{i-1}, for $1 \leq i \leq n$. A *maximal* chain in P is a saturated, spanning chain. The *length* of a finite chain is defined to be one less than its cardinality, and a finite interval is *graded* if all of its maximal chains have the same length.

A *subgrading* on a finite poset P is a function $\rho\colon P \to \mathbb{N}$ for which $\rho(x) < \rho(y)$ whenever $x < y$ in P, and we call P *subgraded* whenever it is equipped with a specific subgrading ρ. Any subposet Q of a subgraded poset P has the *induced subgrading*, given by the restriction $\rho|Q$, and is therefore subgraded.

A graded interval P has a canonical subgrading ρ_k, for every integer k, determined by the conditions

$$\rho_k(x) = \begin{cases} k & \text{if } x \text{ is minimal} \\ \rho_k(w)+1 & \text{whenever } x \text{ covers } w. \end{cases} \tag{12}$$

Equivalently, ρ_k is determined by setting $\rho_k(z)$ equal to k plus the common length of all maximal chains in P_z, for all $z \in P$. On the other hand, if a finite interval P has a subgrading which satisfies (12), it follows that P is graded. Unless we specifically state otherwise, we will always assume that a graded interval is equipped with the canonical subgrading $\rho = \rho_0$, to which we refer as the *rank function* of P. Thus we may regard graded intervals as special instances of subgraded intervals.

Proposition 13. *Every finite subgraded poset may be embedded in a graded interval with the induced subgrading.*

Proof. We let P be an arbitrary poset with subgrading ρ, and construct a graded interval as follows. We begin by identifying those pairs $x < y$ in P for which y covers x and $\rho(y) > \rho(x)+1$; we connect each such x to the corresponding y by a chain of length $\rho(y) - \rho(x)$, whose nonextremal elements are disjoint from P and from all other such chains. If P has more than one minimal element x we select an integer m, less than each $\rho(x)$, and then adjoin a new minimal element w, which we connect to each x by mutually disjoint chains of length $\rho(x) - m$. Finally, if necessary, we adjoin a new maximal element z by the corresponding procedure. The resulting interval $[w, z]$ is graded, and clearly contains P as a subposet; its canonical subgrading ρ_m induces the subgrading ρ by construction. □

The *direct product* of posets P and Q is the cartesian product $P \times Q$, with componentwise ordering. If P and Q are graded, so is $P \times Q$; the same remark applies if they are subgraded, by choosing $\rho(x, y) = \rho(x) + \rho(y)$ for all $(x, y) \in P \times Q$.

Example 14. A poset in which any two elements are comparable is called a *chain*, or a *linearly ordered* set. Any finite chain is graded; in particular, the set $[n] = \{1, 2, \ldots, n\}$, with the usual ordering, is a chain of rank $n-1$. □

Example 15. If V is a finite set we denote by $B(V)$ the collection of all subsets of V, ordered by inclusion. The interval $B(V)$ is graded, with rank function given by $\rho(U) = |U|$ for all $U \subseteq V$. Hence the rank of $B(V)$ is $|V|$. If $U \subseteq W$ in $B(V)$, there is a canonical isomorphism between the subinterval $[U, W]$ and $B(W \setminus U)$, given by the correspondence $X \leftrightarrow X \setminus U$. A poset which is isomorphic to $B(V)$, for some set V, is called a *Boolean algebra.* □

Example 16. Suppose V is a finite set and that σ is a partition of V. The Boolean algebra $B(\sigma)$ is subgraded, with

$$\rho(U) = \sum_{B \in U} |B|, \tag{17}$$

for all subsets U of σ. There is a canonical isomorphism from $B(\sigma)$ onto the subposet $B[\sigma] \subseteq B(V)$ consisting of all unions of blocks of σ, which maps each $U \in B(\sigma)$ to the set

$$\bigcup_{B \in U} B \subseteq V.$$

Under this isomorphism, (17) corresponds to the induced subgrading on $B[\sigma]$. The subposet $B[\sigma]$ is a sub-Boolean algebra (which is equivalent to being a spanning sublattice) of $B(V)$, and every sub-Boolean algebra of $B(V)$ is equal to $B[\sigma]$ for some partition σ of V. □

Example 18. For any finite set V the set $\Pi(V)$ of all partitions of V is partially ordered by setting $\sigma \leq \pi$ whenever each block of σ is a subset of some block of π. This is the *refinement* ordering of partitions, which makes $\Pi(V)$ into a graded poset (and lattice) of rank $|V| - 1$. The correspondence $\sigma \leftrightarrow B[\sigma]$ defines an anti-isomorphism between $\Pi(V)$ and the lattice of all sub-Boolean algebras of $B(V)$, ordered by inclusion. □

Example 19. A *delineation* (or *linear partition* [14]) of a finite set V is a set J of disjoint chains whose underlying sets form a partition π_J of V. The set $\Delta(V)$ of all delineations of V is partially ordered by setting $J \leq K$ whenever each element of J is a subinterval of some element of K. The poset $\Delta(V)$ is graded, has a unique minimal element consisting of singleton chains, and one maximal element $\{L\}$ for each linear ordering L of V. For every such L we denote the spanning subinterval $\Delta(V)_{\{L\}}$ of $\Delta(V)$ by $\Omega(L)$. The correspondence $J \mapsto \pi_J$ determines an order preserving map $\theta_V \colon \Delta(V) \to \Pi(V)$ which preserves gradings. The restriction of θ_V to $\Omega(L)$ is a poset embedding onto a spanning subposet of $\Pi(V)$. □

Whenever $J \leq K$ in $\Delta(V)$, the delineation K induces a delineation of J which we denote by K/J. In particular, if $\{L\}$ is maximal in $\Delta(V)$, then any $J = \{J_1, \ldots, J_k\}$ in $\Omega(L)$ has a linear order determined by L; hence we write J as the sequence $(J_1, \ldots, J_k)$.

If L is a linear ordering of V, the set of *cuts* of L consists of the edges

$$L_{\text{cut}} = \{(x, y) \colon y \text{ covers } x \text{ in } L\}$$

of the Hasse diagram for L. We may define a mapping $c \colon \Omega(L) \to B(L_{\text{cut}})$ by letting $c(J)$ be the set

$$\{(\max J_i, \min J_{i+1}) \colon 1 \leq i \leq k - 1\}$$

of cuts of L corresponding to the delineation $J = (J_1, \ldots, J_k) \in \Omega(L)$. Since refinement of delineations corresponds to reverse inclusion of cut sets, the map c is an anti-isomorphism of posets. Hence $\Omega(L)$ is a Boolean algebra of rank $|V| - 1$.

A closely related poset is the set $\mathrm{Sp}(L)$ of all spanning chains in L, partially ordered by refinement. If C is the spanning chain $x_0 < \cdots < x_k$, we let $\omega(C) = \{J_1, \ldots, J_k\}$ be the delineation of L_{cut} defined by

$$J_i = \{(x, y) \in L_{\text{cut}} \colon x_{i-1} \leq x \text{ and } y \leq x_i\}, \tag{20}$$

where each J_i has the linear ordering induced by L. The map $\omega\colon \mathrm{Sp}(L) \to \Delta(L_{\mathrm{cut}})$ is an isomorphism onto the spanning subinterval of $\Delta(L_{\mathrm{cut}})$ determined by the linear ordering of L_{cut} which is induced by L.

5. Interval Cell-Sets and Categories

In this section we extend the notion of incidence coproduct to the framework of cell-sets. Since many of our examples are actually object cell-sets, we devote considerable attention to formulating the constructions in terms of category theory, where they assume a helpful degree of generality.

By a *subgraded interval cell-set* we mean a cell-set $\mathcal{C}$ of finite subgraded intervals, whose dimension function is determined for all $P = [a, b]$ in $\mathcal{C}$ by

$$d(P) = \rho(b) - \rho(a), \tag{21}$$

where ρ denotes the subgrading of P. It follows that $d(P) \geq 0$, that $d(P) = 0$ if and only if $|P| = 1$, and

$$d(P) = d(P_x) + d(P^x), \tag{22}$$

for all $x \in P$ such that $P_x, P^x \in \mathcal{C}$. If $\mathcal{C}$ consists entirely of graded intervals and ρ is the canonical rank function, we refer to $\mathcal{C}$ as a *graded interval cell-set*. We abbreviate either term to *interval cell-set* when there is no danger of ambiguity, or when we wish to make common reference to both.

An interval cell-set $\mathcal{C}$ is *closed* if it is closed under formation of subintervals and its equivalence relation is *order compatible* in the sense of [24]; that is, whenever $P \sim Q$ in $\mathcal{C}$ there exists a bijection $f\colon P \to Q$ such that

$$P_x \sim Q_{f(x)} \qquad \text{and} \qquad P^x \sim Q^{f(x)} \tag{23}$$

for all $x \in P$.

Proposition 24. *Whenever $\mathcal{C}$ is a closed interval cell-set, then the relation $\delta\colon \mathcal{C} \to \mathcal{C} \times \mathcal{C}$ given by*

$$\delta(P) = \big\{(P_x, P^x)\colon x \in P\big\} \tag{25}$$

is a coassociative cell-map, and hence a coproduct on $\mathcal{C}$.

Proof. The fact that the equivalence relation on $\mathcal{C}$ is order compatible implies that δ satisfies Condition (2), and it follows from (22) that δ preserves dimension. Hence δ is a cell-map, which is coassociative by transitivity of $\leq$. □

The cell-map δ is the *incidence* coproduct, and the coalgebra $Z_*(\mathcal{C})$ is the *incidence coalgebra* of $\mathcal{C}$. The induced coproduct δ_* on $Z_*(\mathcal{C})$ is determined by

$$\delta_*\langle P\rangle = \sum_{x\in P} \langle P_x\rangle \otimes \langle P^x\rangle,$$

and a counit is given by

$$\epsilon_*\langle P\rangle = \begin{cases} 1 & \text{if } |P| = 1, \\ 0 & \text{otherwise,} \end{cases} \tag{26}$$

which is induced by the augmentation ϵ when $\mathcal{C}$ is connected.

The majority of interval cell-sets which we encounter occur as object cell-sets of certain cell-categories of intervals, and the categorical point of view proves extremely useful for many of our constructions. We therefore continue our development within the category **Int** of finite intervals and order preserving maps. In particular, we make extensive reference to the full subcategory $\mathbf{I}_g$ of finite graded intervals, and to the category $\mathbf{I}_s$ of finite subgraded intervals and *dimension preserving* maps $f\colon P \to Q$, which satisfy

$$\rho(y) - \rho(x) = \rho(f(y)) - \rho(f(x)),$$

for all $x \leq y$ in P. We emphasize that the objects of $\mathbf{I}_s$ are intervals equipped with additional structure, namely a subgrading; since the morphisms respect this structure, $\mathbf{I}_s$ is neither a subcategory of **Int**, nor contains $\mathbf{I}_g$ as a subcategory.

If $f\colon P \to Q$ is a morphism in **Int**, we write $\widehat{f}$ for the map $P \to f(P)$ obtained from f by restricting its codomain to $f(P)$. We then have the canonical factorization

$$f = i_{f(P),Q} \cdot \widehat{f} \tag{27}$$

of f into an inclusion composed with a surjection in **Int**. A *subgraded interval category* is a subcategory of $\mathbf{I}_s$ in which every morphism retains the canonical factorization (27), and which is equipped with the dimension function (21). We define a *graded interval category* by analogy. As before, we abbreviate either term to *interval category* when there is no danger of ambiguity, or if the distinction is unimportant. Subgraded and graded interval categories are cell-categories, whose object cell-sets are interval cell-sets and graded interval cell-sets respectively.

We describe an interval category $\mathbf{C}$ as *closed* if the inclusion $i_{R,P}$ is a morphism in $\mathbf{C}$ whenever P is an object of $\mathbf{C}$ and R is a subinterval of P. Thus $\mathbf{I}_g$ and $\mathbf{I}_s$ are themselves closed interval categories. If $\mathbf{C}$ is full in either of $\mathbf{I}_s$ or $\mathbf{I}_g$, then every morphism retains the canonical factorization (27), and we refer to $\mathbf{C}$ as a *full interval category.* If $\mathbf{C}$ is a full interval category then the interval cell-set $\mathcal{C}$ is necessarily connected, and $\mathbf{C}$ is closed if and only if $\mathcal{C}$ is closed under the formation of subintervals.

Proposition 28. *The object cell-set of a closed interval category is a closed interval cell-set.*

Proof. Suppose that $\mathbf{C}$ is a closed interval category and that P belongs to $\mathcal{C}$. If R is a subinterval of P, then R also belongs to $\mathcal{C}$, since the inclusion $i_{R,P}$ lies in $\mathbf{C}$. To confirm order compatibility, we consider a $\mathcal{C}$-isomorphism $f\colon P \to Q$. The restriction $f|R$ belongs to $\mathbf{C}$ because it is equal to the composition $f \cdot i_{R,P}$. Hence, by (27), the map $\widehat{f|R}\colon R \to f(R)$ belongs to $\mathbf{C}$, and is easily seen to be a $\mathcal{C}$-isomorphism. In particular, $\widehat{f|P_x}$ and $\widehat{f|P^x}$ are $\mathcal{C}$-isomorphisms, and we deduce that

$$P_x \sim Q_{f(x)} \qquad \text{and} \qquad P^x \sim Q^{f(x)} \tag{29}$$

for all $x \in P$, as sought. □

Example 30. Let **L** denote the full, graded interval category having all finite nonempty chains as objects. The free abelian group $Z_*(\mathcal{L})$ has a single generator β_k in each nonnegative dimension k, which is represented by the chain $[k+1]$. The cell-set $\mathcal{L}$ is closed, and therefore has incidence coproduct δ given by (25), which induces the divided powers coproduct

$$\delta_*(\beta_k) = \sum_{i=0}^{k} \beta_i \otimes \beta_{k-i}.$$

on $Z_*(\mathcal{L}) = \mathbb{Z}\{\beta_0, \beta_1, \ldots\}$. We usually write β_0 as 1, since it generates the augmentation summand of $Z_*(\mathcal{L})$. □

For our next example we recall that a *composition* of a nonnegative integer n consists of a sequence of positive integers $a = (a_1, \ldots, a_k)$ which satisfies $a_1 + \cdots + a_k = n$.

Example 31. Let $\mathbf{L}^+$ denote the interval category of all finite subgraded chains and dimension preserving maps. If L is the subgraded chain $x_0 < \cdots < x_k$, we write $a(L)$ for the composition $(a_1, \ldots, a_k)$ of $d(L)$ given by $a_i = d([x_{i-1}, x_i])$, for $1 \leq i \leq k$. The free abelian group $Z_*(\mathcal{L}^+)$ has one generator β_a of dimension n for each composition a of every $n \geq 0$, which is represented by any subgraded chain L with $a(L) = a$. The interval cell-set $\mathcal{L}^+$ is closed, and the coproduct on $Z_*(\mathcal{L}^+)$ induced by the incidence coproduct is given by

$$\delta_*(\beta_a) = \sum_{cd=a} \beta_c \otimes \beta_d,$$

where cd denotes the concatenation of the compositions c and d. □

For any $T \subseteq \mathbb{N}$, a *T-composition* is a composition $(a_1, \ldots, a_k)$ such that $a_i \in T$ for all $1 \leq i \leq k$, and a *T-chain* is a subgraded chain L for which $a(L)$ is a T-composition. The subcell-set $\mathcal{L}^T$ of $\mathcal{L}^+$ consisting of all T-chains is closed, and the subcoalgebra $Z_*(\mathcal{L}^T)$ of $Z_*(\mathcal{L}^+)$ has the same description as that of $Z_*(\mathcal{L}^+)$, except that all indices are restricted to T-compositions. In particular, $\mathcal{L}^{\{1\}} = \mathcal{L}$ and $\mathcal{L}^{\mathbb{N}} = \mathcal{L}^+$.

Example 32. For any direct product of finite chains $P = L_1 \times \cdots \times L_k$, let J_P denote the set $\{i \colon |L_i| > 1\}$ of indices of nontrivial factors of P. Given $q \geq 1$, let $\mathbf{L}(q)$ denote the full, graded interval category having as objects all such P satisfying $|J_P| \leq q$. Note that $\mathcal{L}(1) = \mathcal{L}$, and that we have inclusions $\mathcal{L}(q-1) \to \mathcal{L}(q)$, for all q. Intervals $P = L_1 \times \cdots \times L_k$ and $Q = M_1 \times \cdots \times M_r$ in $\mathcal{L}(q)$ are equivalent if and only if they are isomorphic posets, which is true if and only if there exists a bijection $f \colon J_P \to J_Q$ such that $|L_i| = |M_{f(i)}|$, for all $i \in J_P$. The free abelian group $Z_*(\mathcal{L}(q))$ has a single generator β_u for each nonincreasing sequence of nonnegative integers $u = (u_1, \ldots, u_q)$, which is represented by the product $[u_1 + 1] \times \cdots \times [u_q + 1]$ and has dimension $|u| = u_1 + \cdots + u_q$. The interval cell-set $\mathcal{L}(q)$ is closed and thus has incidence coproduct δ; the coproduct induced on $Z_*(\mathcal{L}(q))$ is given by

$$\delta_*(\beta_u) = \sum \beta_{s'} \otimes \beta_{t'}, \tag{33}$$

where the summation is over all ordered pairs of sequences of nonnegative integers s and t such that $s_i + t_i = u_i$ for $1 \leq i \leq q$, and s' denotes the sequence s rearranged in nonincreasing order. □

We may combine the previous two examples, supposing that the set of compositions of all nonnegative integers is ordered lexicographically.

Example 34. Given any $T \subseteq \mathbb{N}$ and $q \geq 1$, let $\mathbf{L}^T(q)$ denote the interval category consisting of direct products P of T-chains such that $|J_P| \leq q$, with all dimension preserving, order preserving maps as morphisms. The free abelian group $Z_*(\mathcal{L}^T(q))$ has a single generator β_α, for each nonincreasing sequence of T-compositions $\alpha = (\alpha_1, \dots, \alpha_q)$. The dimension of β_α is the sum of the elements of all of the α_i. The interval cell-set $\mathcal{L}^T(q)$ is closed and thus has incidence coproduct δ; the coproduct induced on $Z_*(\mathcal{L}^T(q))$ is given by

$$\delta_*(\beta_\alpha) = \sum \beta_{\sigma'} \otimes \beta_{\tau'}, \tag{35}$$

where the summation takes place over all ordered pairs of sequences of T-compositions $\sigma = (\sigma_1, \dots, \sigma_q)$ and $\tau = (\tau_1, \dots, \tau_q)$ such that $\sigma_i \tau_i = \alpha_i$, for $1 \leq i \leq q$. □

6. Products on Interval Cell-sets

We now turn to the study of product structures on interval cell-sets, for which we find the categorical framework of finite products especially useful.

A *Hopf interval cell-set* is a connected, closed interval cell-set which is equipped with a product that is compatible with the incidence coproduct. If $\mathcal{C}$ is a Hopf interval cell-set then it is a Hopf cell-set, and it follows from (7) that the antipode χ of the Hopf algebra $Z_*(\mathcal{C})$ is given by

$$\chi\langle P \rangle = \sum_{k \geq 0} \sum (-1)^k \prod_{i=1}^{k} \langle x_{i-1}, x_i \rangle, \tag{36}$$

where the inner sum is taken over all spanning chains $x_0 < x_1 < \cdots < x_k$ in P. The antipode formula (36) was first given, in more general form, in [24].

The next example highlights a particularly simple situation in which Hopf interval cell-sets naturally arise; we shall develop a more categorical approach to related phenomena in Theorem 50 below.

Example 37. If $\mathbf{C}$ is a full, closed interval category such that the object cell-set $\mathcal{C}$ is closed under formation of direct products, then the cell-map $\mu\colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ given by $\mu(P, Q) = P \times Q$ is a product. If $[x, y]$ and $[x', y']$ are subintervals of P and Q respectively, the direct product $[x, y] \times [x', y']$ is the subinterval $[(x, x'), (y, y')]$ of $P \times Q$, so that

$$(P \times Q)_{(x,y)} = P_x \times Q_y \quad \text{and} \quad (P \times Q)^{(x,y)} = P^x \times Q^y \tag{38}$$

for all $(x, y) \in P \times Q$. It follows immediately that the direct product μ is compatible with the incidence coproduct, and therefore that $\mathcal{C}$ is a Hopf interval cell-set. □

If **C** is not full in Example 37, then additional conditions are needed to ensure that the direct product is a cell-map, and that the class of one point intervals acts as the multiplicative identity. Specifically, for all $P, Q, R \in \mathcal{C}$, we require that

$$Q \sim R \quad \Longrightarrow \quad P \times Q \sim P \times R \quad \text{and} \quad Q \times P \sim R \times P, \tag{39}$$

and

$$|Q| = 1 \quad \Longrightarrow \quad P \times Q \sim Q \times P \sim Q. \tag{40}$$

These conditions hold automatically when **C** is full. Hopf interval cell-sets of this particular type were studied in [26], where they were referred to as hereditary families of posets.

Example 41. The category **B** of all finite Boolean algebras and Boolean algebra maps (that is, maps which preserve least upper bounds, greatest lower bounds, and complements) is a closed, graded interval category which is also closed under formation of direct products. The category **B** is not full, but satisfies (39) and (40); hence, according to Example 37, $\mathcal{B}$ is a Hopf interval cell-set.

We obtain a cell-map A from $\mathcal{B}$ to the cell-set of finite sets $\mathcal{S}$, introduced in Example 8, by setting $A(P)$ equal to the set of *atoms* $\{x \in P \colon \rho(x) = 1\}$, for any $P \in \mathcal{B}$; on the other hand, the correspondence $V \mapsto B(V)$ of Example 15 gives us a cell-map B in the reverse direction. The cell-maps $A \colon \mathcal{B} \to \mathcal{S}$ and $B \colon \mathcal{S} \to \mathcal{B}$ define an equivalence of Hopf cell-sets; hence $A_* \colon Z_*(\mathcal{B}) \to Z_*(\mathcal{S}) \cong \mathbb{Z}[x]$ and $B_* \colon Z_*(\mathcal{S}) \to Z_*(\mathcal{B})$ are inverse Hopf algebra isomorphisms. □

Example 42. Let $\mathbf{L}(\infty)$ denote the full, graded interval category having all finite direct products of finite nonempty chains as objects. The interval cell-set $\mathcal{L}(\infty)$ is closed under formation of subintervals and direct products and thus, according to Example 37, is a Hopf interval cell-set. Letting β_k denote the equivalence class of the chain $[k+1]$, we see that $Z_*(\mathcal{L}(\infty))$ is the polynomial Hopf algebra $\mathbb{Z}[\beta_1, \beta_2, \dots]$, where $\{\beta_k\}$ is a sequence of divided powers. In fact $\mathcal{L}(\infty)$ is the union $\bigcup_{q \geq 1} \mathcal{L}(q)$, and $Z_*(\mathcal{L}(\infty))$ is isomorphic as a coalgebra to the direct limit over q of the coalgebras $Z_*(\mathcal{L}(q))$. Also, $Z_*(\mathcal{L}(\infty))$ is isomorphic to the Hopf algebra of symmetric functions on infinitely many variables, where β_i maps to the ith complete homogeneous symmetric function h_i (see [6]). □

Suppose that $\mathcal{C}$ is a connected, closed interval cell-set. A *free Hopf interval cell-set* on $\mathcal{C}$ is a Hopf interval cell-set $\mathcal{D}$, together with a cell-map $i \colon \mathcal{C} \to \mathcal{D}$ that preserves coproducts up to equivalence and satisfies the following universal property: for any Hopf cell-set $\mathcal{P}$ and cell-map $f \colon \mathcal{C} \to \mathcal{P}$ which preserves coproducts up to equivalence, there exists a cell-map $\bar{f} \colon \mathcal{D} \to \mathcal{P}$, which is unique up to equivalence, preserves products and coproducts up to equivalence and has $f \sim \bar{f}i$. The free Hopf interval cell-set $\mathcal{D}$ is determined uniquely up to equivalence by $\mathcal{C}$, and the Hopf algebra $Z_*(\mathcal{D})$ satisfies the corresponding universal property, making it the free noncommutative Hopf algebra on the coalgebra $Z_*(\mathcal{C})$. Free commutative Hopf interval cell-sets are defined analogously.

The construction of free and free commutative hereditary families of posets given in [24] may readily be modified and translated into the language of cell-sets, yielding constructions of free and free commutative Hopf interval cell-sets. The next two propositions place these constructions into our categorical framework.

Proposition 43. *Suppose that* $\mathbf{C}$ *is a closed interval category whose object cell-set* $\mathcal{C}$ *is connected; then* $\mathbf{C}$ *may be embedded in an interval category* $\mathbf{C}_f$ *such that* $\mathcal{C}_f$ *is a Hopf interval cell-set under direct product and, together with the corresponding embedding of object cell-sets,* $\mathcal{C}_f$ *forms the free Hopf interval cell-set on* $\mathcal{C}$.

Proof. Let the objects of $\mathbf{C}_\mathrm{f}$ consist of all direct products $P_1 \times \cdots \times P_k$ of objects of $\mathbf{C}$, for $k \geq 1$. We distinguish the one-fold direct product of $P \in \mathcal{C}$, which we denote by (P), from P itself; in particular, if P, Q and the direct product $P \times Q$ all lie in $\mathcal{C}$, then $(P \times Q)$ and $P \times Q$ are distinct objects of $\mathbf{C}_\mathrm{f}$. The morphisms $P_1 \times \cdots \times P_k \to Q_1 \times \cdots \times Q_r$ in $\mathbf{C}_\mathrm{f}$ consist of maps of the form

$$(x_1, \ldots, x_k) \longmapsto (f_1(x_{\alpha(1)}), \ldots, f_r(x_{\alpha(r)})), \tag{44}$$

where $\alpha\colon [r] \to [k]$ is order-preserving and the maps $f_i\colon P_{\alpha(i)} \to Q_i$ are morphisms in $\mathbf{C}$. It is straightforward to verify that $\mathbf{C}_\mathrm{f}$ is a closed interval category. By construction, $\mathcal{C}_\mathrm{f}$ is closed under formation of direct products, and it is immediate from the definition of morphism (44) that $\mathcal{C}_\mathrm{f}$ satisfies (39) and (40); thus, according to Example 37, it is a Hopf interval cell-set.

By analogy with Example 32, intervals $P = P_1 \times \cdots \times P_k$ and $Q = Q_1 \times \cdots \times Q_r$ are equivalent in $\mathcal{C}_\mathrm{f}$ if and only if there exists an order-preserving bijection $\varphi\colon J_P \to J_Q$ such that $P_i \sim Q_{\varphi(i)}$ in $\mathcal{C}$, for all $i \in J_P$. The universal property required of $\mathcal{C}_\mathrm{f}$ then follows immediately. □

Proposition 45. *Suppose that* $\mathbf{C}$ *is a closed interval category whose corresponding interval cell-set* $\mathcal{C}$ *is connected; then* $\mathbf{C}$ *may be embedded in an interval category* $\mathbf{C}_{fc}$ *such that the object cell-set* $\mathcal{C}_{fc}$*, under direct product, is the free commutative Hopf interval cell-set on* $\mathcal{C}$.

Proof. The constuction on $\mathbf{C}_\mathrm{fc}$ is identical to that of $\mathbf{C}_\mathrm{f}$, except that we drop the requirement that α be order-preserving from the definition of morphism (44). □

Example 46. The free Hopf interval cell-set $\mathcal{L}_\mathrm{f}$ consists of all finite direct products of finite chains, with $P = L_1 \times \cdots \times L_k$ and $Q = M_1 \times \cdots \times M_r$ being equivalent in $\mathcal{L}_\mathrm{f}$ if and only if there exists an order-preserving bijection

$f\colon J_P \to J_Q$ such that the chains L_i and $M_{f(i)}$ have the same length for all $i \in J_P$. Letting θ_k denote the equivalence class of the chain $[k+1]$, the Hopf algebra $Z_*(\mathcal{L}_{\mathrm{f}})$ is isomorphic to the free associative algebra on the set $\{\theta_1, \theta_2, \ldots\}$, and $\theta_1, \theta_2, \ldots$ is a sequence of divided powers in $Z_*(\mathcal{L}_{\mathrm{f}})$. □

In fact $Z_*(\mathcal{L}_{\mathrm{f}})$ is isomorphic to the dual of the Hopf algebra QSym of quasi-symmetric functions, which was introduced by Gessel in [7], and has been recently studied by Ehrenborg in [5], and Malvenuto and Reutenauer in [19]. For all compositions $a = (a_1, \ldots, a_k)$, the basis element $\theta_a = \theta_{a_1} \cdots \theta_{a_k}$ of $Z_*(\mathcal{L}_{\mathrm{f}})$ is dual to the basis element

$$M_a = \sum_{x_1 < \cdots < x_k} x_1^{a_1} \cdots x_k^{a_k}$$

of QSym, defined in [7]. The elements M_a of QSym were given the name *monomial quasi-symmetric functions* in [5] because of their close relationship to monomial symmetric functions.

Example 47. The free commutative Hopf cell-set $\mathcal{L}_{\mathrm{fc}}$ is equal to the cell-set $\mathcal{L}(\infty)$ of Example 42, but the categories $\mathbf{L}_{\mathrm{fc}}$ and $\mathbf{L}(\infty)$ are different: for $\mathbf{L}(\infty)$ is a full, while $\mathbf{L}_{\mathrm{fc}}$ is not. Let $\mathbf{L}_{\mathrm{fc}}(q)$ denote the full subcategory of $\mathbf{L}_{\mathrm{fc}}$ having as objects all chain products P such that $|J_P| \leq q$. The interval category $\mathbf{L}_{\mathrm{fc}}(q)$ is closed (but of course is not closed under formation of direct products), and the cell-set $\mathcal{L}_{\mathrm{fc}}(q)$ is equal to the cell-set $\mathcal{L}(q)$ of Example 32, but the categories $\mathbf{L}_{\mathrm{fc}}(q)$ and $\mathbf{L}(q)$ are distinct. □

Many interval cell-sets are naturally equipped with products which are closely related to direct products of intervals. The categorical context is well-suited to exploiting these relationships, and we devote the remainder of this section to their development. We may work equally well in either of the categories $\mathbf{I}_s$ or $\mathbf{I}_g$ and we shall denote them both by $\mathbf{I}$ henceforth; the context will serve to tell them apart whenever the distinction is important.

The direct product of intervals invests the category $\mathbf{I}$ with finite products (in the sense of [18], for example), so that $\times$ induces a functor $\mathbf{I} \times \mathbf{I} \to \mathbf{I}$ which is coherent with respect to associativity. Given any two morphisms $f\colon P \to Q$ and $f'\colon P' \to Q'$, we write $(f \times f')\colon P \times P' \to Q \times Q'$ for their direct product.

Suppose $\mathbf{C}$ is a closed interval category whose object cell-set is connected. We call $\mathbf{C}$ *productive* if it admits an operation $\square$ that induces finite products, and the inclusion functor $\mathbf{C} \to \mathbf{I}$ is product-preserving. For all objects $P_1, \ldots, P_n$ in $\mathbf{C}$ and all $n \geq 0$, these conditions ensure the existence of an isomorphism

$$\eta_{P_1,\ldots,P_n}\colon P_1 \times \cdots \times P_n \longrightarrow P_1 \,\square \cdots \square\, P_n \tag{48}$$

in **I**, which preserves projections and is natural with respect to morphisms in $\mathbf{C} \times \cdots \times \mathbf{C}$. In particular, the square

$$\begin{array}{ccc} R \times S & \xrightarrow{i_{R,P} \times i_{S,Q}} & P \times Q \\ \eta_{R,S} \downarrow & & \downarrow \eta_{P,Q} \\ R \,\square\, S & \xrightarrow{i_{R,P} \square i_{S,Q}} & P \,\square\, Q \end{array} \tag{49}$$

commutes whenever R and S are subintervals of P and Q respectively.

For all $(x, y) \in P \times Q$ we denote by $x \,\square\, y$ the element $\eta_{P,Q}(x, y) \in P \,\square\, Q$, which by commutativity of the square (49) is equal to the image of the one-point interval $\{x\} \,\square\, \{y\}$ under the map $i_{\{x\},P} \,\square\, i_{\{y\},Q}$. We remark that the element $x \,\square\, y$ is characterized by the property that its projections onto P and Q are x and y respectively. With this data, we may extend Example 37.

Theorem 50. *The object cell-set of a productive category is a Hopf interval cell-set.*

Proof. Given a productive category $\mathbf{C}$ and intervals $P, Q \in \mathcal{C}$, consider subintervals $R = [x, y] \subseteq P$ and $S = [x', y'] \subseteq Q$. We abbreviate the morphism $i_{R,P} \,\square\, i_{S,Q} \colon R \,\square\, S \to P \,\square\, Q$ to j, and write $\operatorname{im} j$ for its image. By commutativity of (49), the map j is injective and

$$\begin{aligned} \operatorname{im} j &= \eta_{P,Q}(R \times S) \\ &= \{z \,\square\, z' : z \in R \text{ and } z' \in S\} \\ &= [x \,\square\, x', y \,\square\, y']. \end{aligned}$$

The canonical factorization (27) ensures that j factorizes in $\mathbf{C}$ as a surjection $\widehat{\jmath} \colon R \,\square\, S \to \operatorname{im} j$ followed by the inclusion $i \colon \operatorname{im} j \to P \,\square\, Q$. Since j is injective, $\widehat{\jmath}$ is an isomorphism in **I**.

We denote the projections from $P \,\square\, Q$ onto P and Q by π_1 and π_2, respectively. The compositions $\pi_1 i \colon \operatorname{im} j \to P$ and $\pi_2 i \colon \operatorname{im} j \to Q$ are given by $x \,\square\, y \mapsto x$ and $x \,\square\, y \mapsto y$ and have images equal to R and S, respectively. Hence the canonical factorization (27) provides surjections $\widehat{\pi_1 i} \colon \operatorname{im} j \to R$ and $\widehat{\pi_2 i} \colon \operatorname{im} j \to S$ in $\mathbf{C}$, giving a morphism $\langle \widehat{\pi_1 i}, \widehat{\pi_2 i} \rangle \colon \operatorname{im} j \to R \,\square\, S$ which is inverse to $\widehat{\jmath}$. Therefore $\widehat{\jmath}$ is an isomorphism in $\mathbf{C}$. In particular, we have that

$$(P \,\square\, Q)_{x \square y} \sim P_x \,\square\, Q_y \quad \text{and} \quad (P \,\square\, Q)^{x \square y} \sim P^x \,\square\, Q^y \tag{51}$$

in $\mathcal{C}$, for all $(x, y) \in P \times Q$. Computation then yields

$$\begin{aligned} \delta(P \,\square\, Q) &= \{((P \,\square\, Q)_{x \square y}, (P \,\square\, Q)^{x \square y}) : (x, y) \in P \times Q\} \\ &\sim \big\{\big(P_x \,\square\, Q_y, P^x \,\square\, Q^y\big)\big) : (x, y) \in P \times Q\big\} \\ &= (\square \times \square)(1_{\mathcal{C}} \times \tau \times 1_{\mathcal{C}})\,\big(\big\{(P_x, P^x, Q_y, Q^y) \colon (x, y) \in P \times Q\big\}\big) \\ &= (\square \times \square)(1_{\mathcal{C}} \times \tau \times 1_{\mathcal{C}})(\delta \times \delta)(P, Q), \end{aligned}$$

so that δ and $\Box$ satisfy the compatibility condition (6). The proof is completed by observing that any of the equivalent one-point intervals may be taken as the empty product. $\Box$

Theorem 50 applies when $\Box$ is direct product, in which case the transformations $\eta_{P_1,\ldots,P_n}$ are all identities; of course, (39) and (40) are also true.

Example 52. Let $\mathbf{B_{set}}$ denote the full graded closed interval category having all intervals $[U,W]$ in the Boolean algebras $B(V)$ (introduced in Example 15) as objects. A product on $\mathbf{B_{set}}$ is defined by

$$[U,W] \Box [U',W'] = [U \sqcup U', W \sqcup W'],$$

so long as the disjoint union $\sqcup$ is chosen with sufficient care to ensure that $U \sqcup U'$ is a subset of $W \sqcup W'$. We may readily verify that $\mathbf{B_{set}}$ is productive, and hence is a Hopf interval cell-set. The Hopf algebra $Z_*(\mathcal{B}_{\mathrm{set}})$ is isomorphic to the Hopf algebra $Z_*(\mathcal{S}) \cong \mathbb{Z}[x]$ of Example 8. Cell-maps $g\colon \mathcal{B}_{\mathrm{set}} \to \mathcal{S}$ and $h\colon \mathcal{S} \to \mathcal{B}_{\mathrm{set}}$, given by $g[U,V] = V \setminus U$ and $h(V) = [\emptyset, V]$, define an equivalence of Hopf cell-sets; thus $g_*\colon Z_*(\mathcal{B}_{\mathrm{set}}) \to Z_*(\mathcal{S})$ and $h_*\colon Z_*(\mathcal{S}) \to Z_*(\mathcal{B}_{\mathrm{set}})$ are inverse Hopf algebra isomorphisms. $\Box$

The category $\mathbf{B_{set}}$ is closely related to $\mathbf{B}$ of Example 41, but differs by our insistence that objects are subintervals of actual lattices of subsets, rather than abstract Boolean algebras. On the other hand, the morphisms of $\mathbf{B_{set}}$ are more relaxed, although we might equally well restrict attention to maps of Boolean algebras without affecting the conclusions. In either case $\mathbf{B_{set}}$ does not admit *direct* products, and the main thrust of Example 52 is to illustrate how the concept of productivity can provide an effective alternative.

Example 53. Let $\mathbf{B_{par}}$ denote the full subgraded closed interval category whose objects consist of all intervals in the Boolean algebras $B(\sigma)$ (introduced in Example 16), with dimension function induced by the subgrading (17):

$$d([U,W]) \;=\; \sum_{B \in W \setminus U} |B|.$$

Thus $[U,W] \sim [U',W']$ in $\mathcal{B}_{\mathrm{par}}$ if and only if there is a bijection $W \setminus U \to W' \setminus U'$, preserving the cardinalities of blocks. Disjoint union invests $\mathbf{B_{par}}$ with finite products, exactly as in Example 52, and $\mathbf{B_{par}}$ is clearly productive; hence $\mathcal{B}_{\mathrm{par}}$ is a Hopf interval cell-set. If $|V| = k$ and $\sigma = \{V\}$ is the partition of V having one block, we denote the equivalence class of $B(\sigma)$ by x_k. It follows that the equivalence class of any interval $[U,W]$ in $\mathcal{B}_{\mathrm{par}}$ is given by

$$\langle U,W \rangle \;=\; \prod_{B \in W \setminus U} x_{|B|},$$

and that the Hopf algebra $Z_*(\mathcal{B}_{\mathrm{par}})$ is isomorphic to $\mathbb{Z}[x_1, x_2, \ldots]$ of Example 10. Cell-maps $g\colon \mathcal{B}_{\mathrm{par}} \to \mathcal{S}_\pi$ and $h\colon \mathcal{S}_\pi \to \mathcal{B}_{\mathrm{par}}$, given by $g[U,W] = W \setminus U$ and $h(\sigma) = [\emptyset, \sigma]$, define an equivalence of Hopf cell-sets; thus

$g_* \colon Z_*(\mathcal{B}_{\text{par}}) \to Z_*(\mathcal{S}_\pi)$ and $h_* \colon Z_*(\mathcal{S}_\pi) \to Z_*(\mathcal{B}_{\text{par}})$ are inverse Hopf algebra isomorphisms. □

Example 54. Let $\mathbf{F}$ denote the full graded closed interval category whose objects consist of all intervals $[\sigma, \tau]$ in the finite partition lattices $\Pi(V)$ of Example 18. For any $\sigma \le \tau$ in $\Pi(V)$ and $\sigma' \le \tau'$ in $\Pi(V')$ we define $[\sigma, \tau] \square [\sigma', \tau'] \in \Pi(V \sqcup V')$ by

$$[\sigma, \tau] \square [\sigma', \tau'] = [\sigma \sqcup \sigma', \tau \sqcup \tau']. \tag{55}$$

It is straightforward to verify that □ invests $\mathbf{F}$ with finite products, and that $\mathbf{F}$ is productive. Therefore $\mathcal{F}$ is a Hopf interval cell-set.

For any $\sigma \le \tau$ in $\Pi(V)$, the *quotient* partition τ/σ is the partition of the set σ induced by τ; it follows from the definition of □ that

$$[\sigma, \tau] \sim \mathop{\Large\square}_{B \in \tau/\sigma} \Pi(B).$$

Writing φ_k for the equivalence class of $\Pi(k+1)$, which has dimension k, the equivalence class of $[\sigma, \tau]$ factorizes as

$$\langle \sigma, \tau \rangle = \prod_{B \in \tau/\sigma} \varphi_{|B|-1}. \tag{56}$$

Hence $Z_*(\mathcal{F})$ is isomorphic to the polynomial Hopf algebra $\mathbb{Z}[\varphi_1, \varphi_2, \ldots]$, whose coproduct is given by

$$\delta_*(\varphi_k) = \sum_{i=0}^{k+1} B_{k+1,i+1}(1, \varphi_1, \varphi_2, \ldots) \otimes \varphi_i,$$

where $B_{n,k}$ denotes the partial Bell polynomial. The rational expression

$$\delta_*(\varphi_k) = (k+1)! \sum_{i+j=k} (1 + \frac{\varphi_1}{2!} + \frac{\varphi_2}{3!} + \cdots)_j^{i+1} \otimes \frac{\varphi_i}{(i+1)!} \tag{57}$$

is equivalent. □

The Hopf algebra $Z_*(\mathcal{F})$ was called the *Faà di Bruno Hopf algebra* by Joni and Rota in [13], and was first studied by Doubilet in [3]; the expression (57) shows that it represents the group scheme of formal Hurwitz series under substitution. Its antipode χ satisfies

$$\chi(\varphi_k) = \sum_{i \ge 1} (-1)^i B_{k+i,i}(0, \varphi_1, \varphi_2, \ldots),$$

for all $k \ge 1$. A combinatorial proof of this formula using the general antipode formula (36) was given by Haiman and Schmitt in [10], and the determinant formula

$$\chi(\varphi_k) = (-1)^k \det\left(B_{k-i+2,k-j+1}(1, \varphi_1, \varphi_2, \ldots)\right)_{1 \le i,j \le k},$$

appeared in [26].

These examples emphasize the utility of nondirect products in forming certain Hopf interval cell-sets. Up to equivalence, however, we may approach such cell-sets by a more roundabout route, in which direct products again play a dominant rôle.

Theorem 58. *Any productive category* $\mathbf{C}$ *may be embedded in a productive category* $\mathbf{C}^\times$ *whose product is direct, in such a way that the inclusion functor* $i'\colon \mathbf{C} \to \mathbf{C}^\times$ *is a product-preserving equivalence.*

Proof. Let $\mathbf{C}^\times$ have the same set of objects as $\mathbf{C}_{\mathrm{f}}$ of Proposition 43. The morphisms $P_1 \times \cdots \times P_n \to Q_1 \times \cdots \times Q_m$ consist of all compositions

$$f^\times = \eta^{-1}_{Q_1,\dots,Q_m} \cdot f \cdot \eta_{P_1,\dots,P_n}, \tag{59}$$

where $f\colon P_1 \square \cdots \square P_n \to Q_1 \square \cdots \square Q_m$ is a morphism in $\mathbf{C}$, and η_P or η_Q is the identity whenever n or m is 1. The dimension function satisfies $d(P_1 \times \cdots \times P_n) = \sum_{j=1}^n d(P_j)$, and $\mathbf{C}^\times$ has finite products induced by $\times$.

The inclusion of any subinterval in $Q_1 \times \cdots \times Q_m$ has the form $i_1 \times \cdots \times i_m$, where $i_j\colon R_j \to Q_j$ is the inclusion of a subinterval for all $1 \le j \le m$. Since each i_j is a morphism in $\mathbf{C}$, it follows that $i_1 \times \cdots \times i_m = \eta^{-1}_{Q_1,\dots,Q_m} \cdot (i_1 \square \cdots \square i_m) \cdot \eta_{R_1,\dots,R_m}$ is a morphism in $\mathbf{C}^\times$, and that $\mathbf{C}^\times$ contains inclusions of subintervals.

Given f as in (59), we write f_j for the morphism $\pi_j \cdot f\colon P_1 \square \cdots \square P_n \to Q_j$ in $\mathbf{C}$. By (27), f_j factorizes as $i_j \cdot \widehat{f_j}$ in $\mathbf{C}$, where $i_j\colon \mathrm{im} f_j \to Q_j$ includes a subinterval and $\langle \widehat{f_1} \dots \widehat{f_m} \rangle\colon P_1 \square \cdots \square P_n \to \mathrm{im} f_1 \square \cdots \square \mathrm{im} f_m$ is the resulting surjection in $\mathbf{C}$. Then $f^\times$ factorizes as

$$(i_1 \times \cdots \times i_m) \cdot \left(\eta^{-1}_{\mathrm{im} f_1,\dots,\mathrm{im} f_m} \cdot \langle \widehat{f_1}, \dots, \widehat{f_m} \rangle \cdot \eta_{P_1,\dots,P_n}\right),$$

in $\mathbf{C}^\times$, which is equal to $i_{\,\mathrm{im} f^\times, Q_1 \times \cdots \times Q_m} \cdot \widehat{f^\times}$ and confirms that $\mathbf{C}^\times$ is a closed interval category. The inclusion functor $i^\times\colon \mathbf{C}^\times \to \mathbf{I}$ is product-preserving because the product in both categories is direct, so $\mathbf{C}^\times$ is productive.

It remains to show that the inclusion functor $i'\colon \mathbf{C} \to \mathbf{C}^\times$ is an equivalence, and that both i' and its inverse are product-preserving. To this end we define a functor $h\colon \mathbf{C}^\times \to \mathbf{C}$ by $h(P_1 \times \cdots \times P_n) = P_1 \square \cdots \square P_n$ on objects, and $h(f^\times) = f$ on morphisms. Clearly $h \cdot i' = 1$, and $i' \cdot h$ is naturally isomorphic to 1, so that i' is indeed an equivalence. Furthermore, i' is product-preserving because each isomorphism $\eta_{P_1,\dots,P_n}$ lies in $\mathbf{C}^\times$, and h is product-preserving because

$$\begin{aligned} \square(h \times h)(P_1 \times P_2, Q_1 \times Q_2) &= (P_1 \square P_2) \square (Q_1 \square Q_2) \\ &= P_1 \square P_2 \square Q_1 \square Q_2 \\ &= h\left((P_1 \times P_2 \times Q_1 \times Q_2)\right) \end{aligned}$$

for all objects P_1, P_2, Q_1, and Q_2 in $\mathbf{C}$. □

Corollary 60. *The cell-set of objects of any productive category is equivalent as a Hopf cell-set to a cell-set whose product is induced by the direct product of intervals.*

Proof. Considering the object cell-sets of Theorem 58, we need only confirm that both i' and h respect incidence coproducts. This is trivial for i'; for h, we obtain

$$(h \times h)\delta(P \times Q) = (h \times h)\Big\{\Big((P \times Q)_{(x,y)}, (P \times Q)^{(x,y)}\Big) : (x, y) \in P \times Q\Big\}$$

$$= \big\{\big(h(P_x \times Q_y), h(P^x \times Q^y)\big) : (x, y) \in P \times Q\big\}$$

$$= \big\{\big(P_x \,\square\, Q_y, P^x \,\square\, Q^y\big) : (x, y) \in P \times Q\big\} \tag{61}$$

in $\mathcal{C} \times \mathcal{C}$, for all P and Q in $\mathcal{C}$. On the other hand, we have that

$$\delta(P \,\square\, Q) = \{((P \,\square\, Q)_{x\square y}, (P \,\square\, Q)^{x\square y}) : (x, y) \in P \times Q\},$$

which is equivalent to (61) by virtue of (51). So h respects coproducts up to equivalence. □

7. Labelled interval cell-sets

We now generalize the notion of interval cell-set to allow intervals which are naturally equipped with order-preserving maps into other intervals, or *labellings*. Prompted by the previous two sections, we base our constructions on object cell-sets of certain categories of arrows; amongst other advantages, this helps us to crystallize a suitable notion of equivalence.

Given any category $\mathbf{C}$, we recall that the *category of arrows* $\mathbf{C}^2$ [18] has all morphisms f of $\mathbf{C}$ as objects; the morphisms $f \to g$ consist of all ordered pairs (α, β) of morphisms in $\mathbf{C}$ which make the square

$$\begin{array}{ccc} P & \xrightarrow{\;f\;} & Q \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta} \\ P' & \xrightarrow{\;g\;} & Q' \end{array} \tag{62}$$

commute. We occasionally find it helpful to write the source of f as $S(f)$, and the target as $T(f)$. At least for categories whose morphisms are functions, the commutativity of (62) is equivalent to the containment $(\alpha \times \beta)(f) \subseteq g$, where $\alpha \times \beta$ is the product map $P \times Q \to P' \times Q'$.

In the case of $\mathbf{I}^2$, we call (α, β) an *inclusion* if $\alpha \times \beta$ is an inclusion (or, equivalently, if both α and β are inclusions) in $\mathbf{I}$; we define *surjections* in $\mathbf{I}^2$ similarly. Just as (27) holds for $\mathbf{I}$, so the morphism (α, β) admits the canonical factorization

$$(\alpha, \beta) = (i_{\alpha(P),P'}, i_{\beta(Q),Q'}) \cdot (\hat{\alpha}, \hat{\beta}) \tag{63}$$

into an inclusion composed with a surjection in $\mathbf{I}^2$. We obtain the intermediate arrow $\alpha(P) \to \beta(Q)$ required for (63) by restricting the domain and codomain of g to $\alpha(P)$ and $\beta(P)$ respectively, and denote it by $\mathrm{im}(\alpha, \beta)$.

By a *labelled interval* we mean a morphism $f\colon P\to Q$ in $\mathbf{I}$ such that $[x,y]$ and $[f(x),f(y)]$ have the same dimension for all $x\le y$ in P. We define a *labelled interval category* $\mathbf{A}$ to be a subcategory of $\mathbf{I}^2$ (which may or may not be full) having labelled intervals as objects, in which every morphism admits the canonical factorization (63). We invest each such interval $f\colon P\to Q$ with the dimension common to P and $f(P)$, so that $\mathbf{A}$ is a cell category, and we refer to the object cell-set $\mathcal{A}$ as a *labelled interval cell-set*.

If $f\colon P\to Q$ is any morphism in $\mathbf{I}$ and $R=[x,y]$ is a subinterval of P, we write $\overline{f(R)}$ for the subinterval $[f(x),f(y)]$ of Q. The *birestriction* $f\|R$ is the map $R\to\overline{f(R)}$ obtained from f by restricting the source of f to R and the target to $\overline{f(R)}$. Note that $f\|R$ is a labelled interval whenever f is a labelled interval; we refer to $f\|R$ as a *labelled subinterval* of f. In particular, f_x and f^x denote the respective birestrictions $f\|P_x$ and $f\|P^x$, for all $x\in P$.

We decree that $\mathbf{A}$, and $\mathcal{A}$, are *closed* if, for all $f\colon P\to Q$ in $\mathcal{A}$ and subintervals $R\subseteq P$, the inclusion $(i_{R,P},i_{\overline{f(R)},Q})\colon f\|R\to f$ belongs to $\mathbf{A}$. For instance, a full labelled interval category $\mathbf{A}$ is closed if and only if $\mathcal{A}$ is closed under formation of labelled subintervals.

Proposition 64. *If $\mathcal{A}$ is a closed labelled interval cell-set, then the relation $\delta\colon\mathcal{A}\to\mathcal{A}\times\mathcal{A}$, given by*

$$\delta(f)=\big\{(f_x,f^x)\colon x\in S(f)\big\}\,, \tag{65}$$

is a coassociative cell-map and hence is a coproduct on $\mathcal{A}$.

Proof. Suppose that $f\colon P\to Q$ and $g\colon P'\to Q'$ are labelled intervals belonging to $\mathcal{A}$, that R is a subinterval of P and that $(\alpha,\beta)\colon f\to g$ is an $\mathcal{A}$-isomorphism. Let S denote the subinterval $\overline{f(R)}$ of Q. By closure, the composition

$$(\alpha|R,\beta|S)=(\alpha,\beta)\cdot(i_{R,P},i_{S,Q})\colon f\|R\to g$$

belongs to $\mathbf{A}$. The image of this morphism is the labelled subinterval $g\|R'$ of g, where $R'=\alpha(R)$. By the canonical factorization (63), the morphism

$$(\alpha\|R,\beta\|S)=(\widehat{\alpha|R},\widehat{\beta|S})\colon f\|R\to g\|R'$$

belongs to $\mathbf{A}$ and is in fact an $\mathcal{A}$-isomorphism. In particular, for all $x\in P$, the pairs $(\alpha\|P_x,\beta\|Q_{f(x)})$ and $(\alpha\|P^x,\beta\|Q^{f(x)})$ are $\mathcal{A}$-isomorphisms $f_x\to g_{\alpha(x)}$ and $f^x\to g^{\alpha(x)}$, respectively. Since $\delta(f)=\big\{(f_x,f^x)\colon x\in P\big\}$ and

$$\delta(g)=\big\{(g_y,g^y)\colon y\in P'\big\}=\Big\{(g_{\alpha(x)},g^{\alpha(x)})\colon x\in P\Big\}\,,$$

it follows that $\delta(f)\sim\delta(g)$ in $\mathcal{A}\times\mathcal{A}$. It is clear that δ preserves dimension and is coassociative. ☐

We remark that an interval cell-set can be regarded as a labelled interval cell-set consisting exclusively of identity arrows, and thus Proposition 64 generalizes Proposition 24. As Example 67 illustrates, a subgraded interval cell-set may also be considered as a labelled interval cell-set, whose arrows $f\colon P\to Q$ all have Q a graded chain and f strictly increasing.

As in the case of closed interval cell-sets, we refer to the cell-map δ as the *incidence coproduct*, and $Z_*(\mathcal{A})$ as the *incidence coalgebra* of $\mathcal{A}$. The induced coproduct δ_* on $Z_*(\mathcal{A})$ is determined by

$$\delta_*\langle f\rangle = \sum_{x\in S(f)} \langle f_x\rangle \otimes \langle f^x\rangle,$$

and a counit is given by

$$\epsilon_*\langle f\rangle = \begin{cases} 1 & \text{if } |S(f)| = 1, \\ 0 & \text{otherwise,} \end{cases}$$

which is induced by the augmentation ϵ on $\mathcal{A}$, if $\mathcal{A}$ is connected.

If all objects of a labelled interval category $\mathbf{A}$ are inclusion maps, we refer to $\mathbf{A}$ as an *interval pair category*, and $\mathcal{A}$ as an *interval pair cell-set*, and we denote the inclusion $i_{R,Q}\colon R \to Q$ by the pair (Q, R). For a closed interval pair cell-set the coproduct (65) takes the form

$$\delta(Q,R) = \left\{ \big((Q_x, R_x), (Q^x, R^x)\big) : x \in R \right\}. \tag{66}$$

Example 67. Consider the full, closed interval pair category $\mathbf{L}_+$ whose objects are all pairs (L, M) such that L is a graded chain and M is a spanning chain of L. There is a cell-map f from $\mathcal{L}_+$ to the cell-set $\mathcal{L}^+$ of all subgraded chains, introduced in Example 31, defined by $f(L, M) = M$, where M is understood to have the subgrading induced by the rank function of the graded chain L. On the other hand, there is a cell-map $g\colon \mathcal{L}^+ \to \mathcal{L}_+$ defined as follows. If M is the subgraded chain $\{x_1 < \cdots < x_k\}$, with $\rho(x_i) = n_i$, we let $g(M) = (L, M')$, where L is the subinterval $[n_1, n_k]$ of $\mathbb{N}$, and M' is the spanning chain $\{n_1, n_2, \ldots, n_k\}$ of L. The cell maps f and g respect coproducts and are mutually inverse up to equivalence. Thus $\mathcal{L}_+$ and $\mathcal{L}^+$ are equivalent cell-sets and so, in particular, the free abelian groups $Z_*(\mathcal{L}_+)$ and $Z_*(\mathcal{L}^+)$ are isomorphic coalgebras.

More generally, for any $T \subseteq \mathbb{N}$ we may define the subcell-set

$$\mathcal{L}_T = \{(L, M)\colon L \in \mathcal{L} \text{ and } M \text{ is a spanning } T\text{-chain in } L\}.$$

of $\mathcal{L}_+$. The restrictions of the cell-maps f and g to $\mathcal{L}_T \subseteq \mathcal{L}_+$ and $\mathcal{L}^T \subseteq \mathcal{L}^+$ are mutually inverse up to equivalence. Hence $\mathcal{L}_T$ and $\mathcal{L}^T$ are equivalent cell-sets. □

Suppose that $\mathcal{A}$ is a labelled interval cell-set, and that $\mathcal{C}$ is an interval cell-set which contains all targets of arrows in $\mathcal{A}$. For $Q \in \mathcal{C}$ we let $T^{-1}(Q)$ denote the set of labelled intervals $f \in \mathcal{A}$ such that $T(f) = Q$. We say that $\mathcal{C}$ is *targeted* by $\mathcal{A}$ whenever T^{-1} is a cell-map; that is, if $T^{-1}(Q)$ is finite for all $Q \in \mathcal{C}$, and $P \sim Q$ in $\mathcal{C}$ implies that $T^{-1}(P) \sim T^{-1}(Q)$ in $\mathcal{A}$.

Proposition 68. *Suppose that $\mathcal{A}$ is a closed labelled interval cell-set and that $\mathcal{C}$ is a closed interval cell-set which is targeted by $\mathcal{A}$; then the cell-map $T^{-1}\colon \mathcal{C} \to \mathcal{A}$ preserves coproducts if and only if, for all $x \in Q \in \mathcal{C}$ and $(g, h) \in T^{-1}(Q_x) \times T^{-1}(Q^x)$, there exists $f \in T^{-1}(Q)$ such that*

$$f_y = g \qquad \textit{and} \qquad f^y = h,$$

for some $y \in S(f)$.

Proof. For all $Q \in \mathcal{C}$ we have

$$(T^{-1} \times T^{-1})\delta(Q) = \left\{T^{-1}(Q_x) \times T^{-1}(Q^x) \colon x \in Q\right\}$$

and

$$\delta_{\mathcal{A}} T^{-1}(Q) = \left\{(f_y, f^y) \colon f \in T^{-1}(Q) \text{ and } y \in S(f)\right\}.$$

For any $f \in T^{-1}(Q)$ and $y \in S(f)$, the maps f_y and f^y have respective targets $Q_{f(y)}$ and $Q^{f(y)}$, so that $(f_y, f^y) \in T^{-1}(Q_{f(y)}) \times T^{-1}(Q^{f(y)})$; therefore

$$\delta_{\mathcal{A}} T^{-1}(Q) \subseteq (T^{-1} \times T^{-1})\delta(Q). \tag{69}$$

Thus T^{-1} preserves coproducts if and only the reverse containment holds, which is precisely our hypothesis. □

The reverse containment to (69) also holds when the sets

$$\delta_{\mathcal{A}} T^{-1}(Q) \cap \left(T^{-1}(Q_x) \times T^{-1}(Q^x)\right) \qquad \text{and} \qquad T^{-1}(Q_x) \times T^{-1}(Q^x)$$

have the same cardinality for all $x \in Q$, allowing the following restatement.

Corollary 70. *Suppose that $\mathcal{A}$ is a closed labelled interval cell-set and that $\mathcal{C}$ is a closed interval cell-set which is targeted by $\mathcal{A}$; then the cell-map $T^{-1} \colon \mathcal{C} \to \mathcal{A}$ preserves coproducts if and only if*

$$\left|T^{-1}(Q_x)\right| \cdot \left|T^{-1}(Q^x)\right| = \left|\{f \in T^{-1}(Q) \colon x \in \operatorname{im} f\}\right|, \tag{71}$$

for all $Q \in \mathcal{C}$ and $x \in Q$.

We say that a labelled interval category $\mathbf{A}$, and corresponding cell-set $\mathcal{A}$, are *uniquely labelled* if it is always the case that distinct labelled intervals belonging to $\mathcal{A}$ have distinct sources. If $\mathbf{A}$ is uniquely labelled then we lose no information by dropping the labellings from our notation and denoting the labelled interval $f \colon P \to Q$ simply as P. In the closed case, the incidence coproduct then takes the familiar form $\delta(P) = \left\{(P_x, P^x) \colon x \in P\right\}$.

We now turn to product structures, and base our generalizations of Section 6 on the observation that the direct product of arrows invests the category $\mathbf{I}^2$ with finite products. We therefore define a labelled interval category $\mathbf{A}$ to be *productive* if it is closed, has finite products induced by an operation $\square$, and the inclusion functor $\mathbf{A} \to \mathbf{I}^2$ is product-preserving. The proof of the following result is then completely analogous to that of Theorem 50.

Theorem 72. *The object cell-set of a productive labelled interval category is a Hopf cell-set.*

Theorem 58 and Corollary 60 generalize just as easily to labelled interval categories.

By way of illustration, we refer back to Example 19. For any finite set V, recall the natural map θ_V which takes a delineation J to the partition whose blocks are the underlying sets of the chains which constitute J.

Example 73. Let $\mathbf{N}$ denote the full labelled interval category having as objects all labelled subintervals $\theta_V \| [J, K]$ such that $J \leq K$ in some poset of delineations $\Delta(V)$. The category $\mathbf{N}$ is full and $\mathcal{N}$ is closed under formation of labelled subintervals, so that $\mathbf{N}$ is a closed labelled interval category. Since $\mathbf{N}$ is uniquely labelled, we may abbreviate each labelled interval $\theta_V \| [J, K]$ to $[J, K]$. The operation

$$[J, K] \,\Box\, [J', K'] = [J \sqcup J', K \sqcup K'] \tag{74}$$

equips $\mathbf{N}$ with finite products, and makes it a productive category. For any interval $[J, K]$ in $\Delta(V)$, we have

$$[J, K] \;\sim \bigsqcap_{M \in K/J} \Omega(M)$$

in $\mathcal{N}$. Writing b_k for the equivalence class of $\Omega([k+1])$, which has dimension k, the equivalence class of $[J, K]$ factorizes as

$$\langle J, K \rangle \;=\; \prod_{M \in K/J} b_{|M|-1}.$$

It follows that $Z_*(\mathcal{N})$ is equal to the polynomial Hopf algebra $\mathbb{Z}[b_1, b_2, \ldots]$, and that the expression

$$\delta_*(b_k) = \sum_{i+j=k} (1 + b_1 + b_2 + \cdots)_j^{i+1} \otimes b_i \tag{75}$$

determines the coproduct. □

These formulae confirm that the Hopf algebra $Z_*(\mathcal{N})$ is isomorphic to the dual of the *Landweber-Novikov algebra* [2], and illustrate the connection with Example 54. The expression (75) shows that $Z_*(\mathcal{N})$ represents the group scheme of formal power series under substitution, and should be compared with the formula (57) for the coproduct of the Faà di Bruno Hopf algebra. We may now formalize this relationship in combinatorial terms.

The Hopf interval cell-set $\mathcal{F}$ of all intervals in full partition lattices is targeted by $\mathcal{N}$ and hence the inverse target relation $T^{-1}\colon \mathcal{F} \to \mathcal{N}$, which satisfies

$$T^{-1}(\Pi(V)) = \{\Omega(L)\colon L \text{ is a linear ordering of } V\} \tag{76}$$

for all finite sets V, is a cell-map. It is easy to check that T^{-1} satisfies the hypotheses of either Proposition 68 or Corollary 70, and thus preserves coproducts. It is clear from the formulae (55) and (74) for the products on $\mathcal{F}$ and $\mathcal{N}$ that T^{-1} also preserves products, and is thus determined by (76). We conclude that the induced Hopf algebra map $(T^{-1})_*$ from $Z_*(\mathcal{F}) = \mathbb{Z}[\varphi_1, \varphi_2, \ldots]$ to $Z_*(\mathcal{N}) = \mathbb{Z}[b_1, b_2, \ldots]$ is given by

$$(T^{-1})_*(\varphi_i) = (i+1)! b_i,$$

for all $i \geq 0$; this encodes the passage from divided to ordinary power series.

8. Quotients of closed interval cell-sets

In this section (and the next), we work exclusively with interval cell-sets and interval pair cell-sets, without making the corresponding cell-categories explicit. This convention allows us to concentrate on the constructions at hand, but may easily be circumvented when the details of the categories are of greater importance. Here we apply the notion of interval pair cell-set to form certain *quotient cell-sets*, and thereby realize the corresponding quotients of the incidence coalgebras.

Suppose that $\mathcal{C}$ is a closed interval cell-set with incidence coproduct δ. A *coideal* of $\mathcal{C}$ consists of a subset $\mathcal{K} \subseteq \mathcal{C}$ which is a union of equivalence classes, contains no one-point intervals, and is such that

$$\delta(x) \subseteq (\mathcal{C} \times \mathcal{K}) \cup (\mathcal{K} \times \mathcal{C}),$$

for all $x \in \mathcal{K}$; these conditions ensure that $Z_*(\mathcal{K})$ is a coideal of the coalgebra $Z_*(\mathcal{C})$. Given a coideal $\mathcal{K} \subseteq \mathcal{C}$ and any interval $P \in \mathcal{C}$, we let P' denote the subset of P given by

$$P' = \{x \colon P_x,\ P^x \notin \mathcal{K}\}.$$

In particular, since $\mathcal{K}$ is a coideal, it follows that $P' = \emptyset$ for all $P \in \mathcal{K}$.

Proposition 77. *For each P in $\mathcal{C}$, the set P' is partially ordered by setting $u \leq v$ in P' if and only $u \leq v$ in P and $[u, v] \notin \mathcal{K}$.*

Proof. The relation $\leq$ on P' is clearly reflexive and antisymmetric. Suppose that $u \leq v \leq w$ in P'. Then we have $u \leq v \leq w$ in P, so that $u \leq w$ in P; and also, $[u, v], [v, w] \notin \mathcal{K}$, which implies that $[u, w] \notin \mathcal{K}$, because $\mathcal{K}$ is a coideal. Hence $u \leq w$ in P'. □

We define the *quotient cell-set* $\mathcal{C}/\mathcal{K}$ to be the interval pair cell-set given by

$$\mathcal{C}/\mathcal{K} = \{(P, P') \colon P \in \mathcal{C} \setminus \mathcal{K}\},$$

with $\mathcal{C}/\mathcal{K}$-isomorphisms $(P, P') \to (Q, Q')$ given by pairs (f, f'), where $f \colon P \to Q$ is a $\mathcal{C}$-isomorphism satisfying $f(P') = Q'$, and f' is obtained from f by restricting its domain and range to P' and Q', respectively. We may simplify the description of equivalence in $\mathcal{C}/\mathcal{K}$ by using the fact that $\mathcal{K}$ is a union of equivalence classes.

Proposition 78. *The pairs (P, P') and (Q, Q') are equivalent in $\mathcal{C}/\mathcal{K}$ if and only if $P \sim Q$ in $\mathcal{C}$.*

Proof. Suppose $(P, P') \sim (Q, Q')$ in $\mathcal{C}/\mathcal{K}$. Then there exists a $\mathcal{C}$-isomorphism $f \colon P \to Q$ with $f(P') = Q'$, so in particular, $P \sim Q$ in $\mathcal{C}$. On the other hand, if $P \sim Q$ in $\mathcal{C}$ and $f \colon P \to Q$ is a $\mathcal{C}$-isomorphism, then by order compatibility we have $P_x \sim Q_{f(x)}$ and $P^x \sim Q^{f(x)}$, for all $x \in P$. Since $\mathcal{K}$ is a union of equivalence classes, we have $P_x, P^x \notin \mathcal{K}$ if and only if $Q_{f(x)}, Q^{f(x)} \notin \mathcal{K}$, and hence $x \in P'$ if and only if $f(x) \in Q'$. Therefore $(P, P') \sim (Q, Q')$ in $\mathcal{C}/\mathcal{K}$. □

Proposition 78 tells us that the cell-set $\mathcal{C}$ is targeted by $\mathcal{C}/\mathcal{K}$, so that the inverse target map T^{-1}, given by

$$T^{-1}(P) = \begin{cases} (P, P') & \text{if } P \notin \mathcal{K} \\ \emptyset & \text{otherwise,} \end{cases} \tag{79}$$

is a cell-map $\mathcal{C} \to \mathcal{C}/\mathcal{K}$.

Proposition 80. *The interval pair cell-set $\mathcal{C}/\mathcal{K}$ is closed, and the cell-map $T^{-1}\colon \mathcal{C} \to \mathcal{C}/\mathcal{K}$ preserves coproducts.*

Proof. Let (P, P') be an element of $\mathcal{C}/\mathcal{K}$. Suppose that R is a subinterval of P' having minimal element x and maximal element y, and that $\overline{R}$ is the corresponding subinterval $[x, y]$ of P. Then we have

$$R = \{z\colon x \leq z \leq y \text{ in } P, \text{ and } P_z,\ P^z,\ [x,z],\ [z,y] \notin \mathcal{K}\}$$

and

$$(\overline{R})' = \{z\colon x \leq z \leq y \text{ in } P, \text{ and } [x,z],\ [z,y] \notin \mathcal{K}\}.$$

Since x and y belong to P', we deduce that $P_x, P^y \notin \mathcal{K}$. Hence, because $\mathcal{K}$ is a coideal,

$$[x,z] \notin \mathcal{K} \quad \Longrightarrow \quad P_z \notin \mathcal{K} \qquad \text{and} \qquad [z,y] \notin \mathcal{K} \quad \Longrightarrow \quad P^z \notin \mathcal{K},$$

for all $z \in R$. It follows that $(\overline{R})' = R$, and thus the pair $(\overline{R}, R)$ belongs to $\mathcal{C}/\mathcal{K}$. Given any $\mathcal{C}/\mathcal{K}$-isomorphism $(f, f')\colon (P, P') \to (Q, Q')$, the pair of birestrictions $(f\|\overline{R}, f\|R)$ is therefore also a $\mathcal{C}/\mathcal{K}$-isomorphism, as required.

Proposition 68 and the fact that $\mathcal{K}$ is a coideal together imply that T^{-1} preserves coproducts. □

Corollary 81. *Whenever $\mathcal{K} \subseteq \mathcal{C}$ is a coideal, there is a coalgebra isomorphism*

$$Z_*(\mathcal{C}/\mathcal{K}) \cong Z_*(\mathcal{C})/Z_*(\mathcal{K}).$$

Proof. By (79), the cell-map T^{-1} sends precisely the elements of $\mathcal{K}$ to the empty set; by Proposition 80 the map $(T^{-1})_*$ is therefore an epimorphism of coalgebras with kernel $Z_*(\mathcal{K})$. □

Example 82. Let $\mathcal{R}(q)$ denote the quotient cell-set $\mathcal{L}(q)/\mathcal{L}(q-1)_-$, where $\mathcal{L}(q)$ is the cell-set of q-fold chain products, defined in Example 32, and $\mathcal{L}(q-1)_-$ denotes the cell-set $\mathcal{L}(q-1)$ with all one-point intervals removed (which is a coideal). Any interval $P = [x, y]$ in $\mathcal{L}(q) \setminus \mathcal{L}(q-1)_-$ is equal to a direct product of chains $[n_1] \times \cdots \times [n_q]$, where $(n_1, \ldots, n_q)$ is a nonincreasing q-tuple of integers which is uniquely determined by P and satisfies $n_i \geq 2$, for $1 \leq i \leq q$. The spanning subposet P' of P is given by

$$P' = \{x\} \cup \{y\} \cup \{(s_1, \ldots, s_q)\colon 1 < s_i < n_i, \text{ for } 1 \leq i \leq q\},$$

with partial ordering determined by $(u_1, \ldots, u_q) < (v_1, \ldots, v_q)$ in P' if and only $u_i < v_i$, for $1 \leq i \leq q$. The free abelian group $Z_*(\mathcal{R}(q))$ thus has a single generator β_u of dimension $|u|$ for each nonincreasing sequence of positive integers $u = (u_1, \dots, u_q)$, together with the generator $1 = \beta_{(0,\dots,0)}$ of dimension zero. By Proposition 80, $\mathcal{R}(q)$ is a closed interval pair cell-set and hence has coproduct δ given by Equation 66. The induced coproduct on $Z_*(\mathcal{R}(q))$ is given by

$$\delta_*(\beta_u) = \sum \beta_{s'} \otimes \beta_{t'},$$

where the summation ranges over all ordered pairs of sequences of integers s and t , either all positive or all zero, such that $s_i + t_i = u_i$, for $1 \leq i \leq q$. □

By Corollary 81, the sequence of cell-maps $\mathcal{L}(q-1) \to \mathcal{L}(q) \xrightarrow{T^{-1}} \mathcal{R}(q)$ induces the quotient exact sequence

$$0 \to \widetilde{Z}_*(\mathcal{L}(q-1)) \to Z_*(\mathcal{L}(q)) \to Z_*(\mathcal{R}(q)) \to 0.$$

9. Chains and cell-maps

We now turn our attention to cell-*maps*. We investigate a class of closed interval pair cell-sets which target a given closed interval cell-set $\mathcal{C}$, by considering various families of spanning subchains of the constituent intervals. These cell-sets satisfy the hypotheses of Proposition 68, and we therefore obtain coproduct-preserving cell-maps T^{-1} defined on $\mathcal{C}$.

If $\mathcal{D}$ is a subset of $\mathcal{C}$, we define a *$\mathcal{D}$-chain* in an interval $P \in \mathcal{C}$ to be a spanning chain $x_0 < \cdots < x_k$ of P such that $[x_{i-1}, x_i]$ is equivalent to some element of $\mathcal{D}$, for $1 \leq i \leq k$. We define the *cell-set of labelled $\mathcal{D}$-chains in $\mathcal{C}$* to be the interval pair cell-set

$$\mathcal{C}_{\mathcal{D}} = \{(P, C) \colon P \in \mathcal{C} \text{ and } C \text{ is a } \mathcal{D}\text{-chain in } P\},$$

with $\mathcal{C}_{\mathcal{D}}$-isomorphisms $(P, C) \to (Q, D)$ given by pairs (f, f_C), where $f \colon P \to Q$ is a $\mathcal{C}$-isomorphism such that $f(C) = D$, and f_C is obtained from f by restricting its domain and range to C and D, respectively. If $(P, C) \in \mathcal{C}_{\mathcal{D}}$ and $f \colon P \to Q$ is a $\mathcal{C}$-isomorphism, we deduce that $f(C)$ is a $\mathcal{D}$-chain from the fact that $\mathcal{C}$ is closed. Since any interval in C is a $\mathcal{D}$-chain, it follows that $\mathcal{C}_{\mathcal{D}}$ is a closed interval pair cell-set. It is also immediate that the inverse target relation $T^{-1} \colon \mathcal{C} \to \mathcal{C}_{\mathcal{D}}$, given in this case by

$$T^{-1}(P) = \{(P, C) \colon C \text{ is a } \mathcal{D}\text{-chain in } P\},$$

is a cell-map, so that $\mathcal{C}$ is targeted by $\mathcal{C}_{\mathcal{D}}$. If $x \in P$, and C and D are $\mathcal{D}$-chains in P_x and P^x, respectively, then the union $C \cup D$ is a $\mathcal{D}$-chain in P satisfying $(C \cup D)_x = C$ and $(C \cup D)^x = D$; Proposition 68 therefore implies that T^{-1} preserves coproducts.

If (P, C) is any labelled $\mathcal{D}$-chain then C inherits the induced subgrading, and we may regard the source map S, given by $S(P, C) = C$, as taking values in the cell-set of subgraded chains $\mathcal{L}^+$. More specifically, if $T \subseteq \mathbb{N}$ contains $d(P)$ for all $P \in \mathcal{D}$, then S takes values in the subcell-set of T-chains $\mathcal{L}^T \subseteq \mathcal{L}^+$.

Proposition 83. *If $\mathcal{D}$ is a subset of the closed interval cell-set $\mathcal{C}$ and $T \subseteq \mathbb{N}$ contains $d(P)$ for all $P \in \mathcal{D}$, then the relation*

$$\lambda_{\mathcal{D}}(P) = \{C \colon C \text{ is a } \mathcal{D}\text{-chain in } P\},$$

defines a coproduct-preserving cell-map $\mathcal{C} \to \mathcal{L}^T$.

Proof. We observe that $S \colon \mathcal{C}_{\mathcal{D}} \to \mathcal{L}^T$ is a coproduct-preserving cell-map and that $\lambda_{\mathcal{D}}$ is equal to the composition ST^{-1}. The result thus follows from the fact that T^{-1} preserves coproducts. □

For example, letting $T = \mathbb{N}$ and $\mathcal{D} = \mathcal{C}$, we obtain the *spanning chain cell-map* $\lambda_s \colon \mathcal{C} \to \mathcal{L}^+$, which maps an interval to the set of all of its spanning chains; and letting $\mathcal{D}$ be the set of all intervals of length (not necessarily dimension) 1 in $\mathcal{C}$, we obtain the *maximal chain cell-map* $\lambda_m \colon \mathcal{C} \to \mathcal{L}^+$, which maps an interval to the set of all of its maximal chains. If we let $\mathcal{D}$ be the set of all intervals of dimension one, we obtain the *maximal graded chain cell-map* $\lambda_{mg} \colon \mathcal{C} \to \mathcal{L}$, which maps an interval to the set of all of its graded maximal chains. If $\mathcal{C}$ consists of graded intervals, then λ_m and λ_{mg} take the same values on $\mathcal{C}$. Each of the cell-maps λ_s, λ_m and λ_{mg} preserves coproducts by virtue of Proposition 83.

Example 84. Recall from Examples 30 and 31 that the free abelian group $Z_*(\mathcal{L})$ has a single generator β_k for each nonnegative integer k, and that $Z_*(\mathcal{L}^+)$ has one generator β_e for each composition e of every nonnegative integer. The maximal graded chain cell-map $\lambda_{mg} \colon \mathcal{L} \to \mathcal{L}$ is the identity cell-map $1_{\mathcal{L}}$, and the maximal chain cell-map $\lambda_m \colon \mathcal{L} \to \mathcal{L}^+$ is the inclusion, which induces the map $\beta_n \mapsto \beta_{(1,\dots,1)}$ on free abelian groups. The maximal graded chain cell-map $\lambda_{mg} \colon \mathcal{L}^+ \to \mathcal{L}$ is the projection, which induces the map

$$\beta_e \mapsto \begin{cases} \beta_n & \text{if } e = (1, \dots, 1) \\ 0 & \text{otherwise} \end{cases}$$

on free abelian groups. Recalling Example 41, we see that $\lambda_{mg} \colon \mathcal{B} \to \mathcal{L}$ maps a Boolean algebra of rank k to the set of all $k!$ of its maximal chains, and hence the induced coalgebra injection $(\lambda_{mg})_*$ from $Z_*(\mathcal{B}) = \mathbb{Z}[x]$ to $Z_*(\mathcal{L})$ is given by $x^k \mapsto k!\beta_k$. □

In our next example we consider maps of cell-sets which fail to be cell-maps only because they do not preserve dimension. Following conventions in K-theory (to which this example is related, and about which we shall say more in the sequel) we may avoid this problem by redefining all dimensions to be zero; alternatively, we may simply accept that the induced homomorphisms of the free abelian groups are ungraded.

Example 85. Consider the map $\gamma \colon \mathcal{L}^+ \to \mathcal{L}$, which maps a subgraded chain to a graded chain by substituting the canonical rank function ρ for its subgrading. Since γ preserves coproducts, so does the composition $\gamma\lambda_s \colon \mathcal{C} \to$

$\mathcal{L}$, for any closed interval cell-set $\mathcal{C}$. In particular $\gamma\lambda_s\colon \mathcal{L} \to \mathcal{L}$ induces the (ungraded) coalgebra endomorphism of $Z_*(\mathcal{L})$ given by $\beta_0 \mapsto \beta_0$ and

$$\beta_n \longmapsto \sum_{k=1}^{n} \binom{n-1}{k-1} \beta_k,$$

for $n \geq 1$; and the induced coalgebra map $(\gamma\lambda_s)_*\colon Z_*(\mathcal{B}) \to Z_*(\mathcal{L})$ is given by $x^0 \mapsto \beta_0$ and

$$x^n \longmapsto \sum_{k=1}^{n} k! S(n,k) \beta_k,$$

for $n \geq 1$, where the $S(n,k)$ denote Stirling numbers of the second kind. □

We now recall from Example 34 the closed interval cell-set $\mathcal{L}^T(2)$ of all direct products of two T-chains, defined for any $T \subseteq \mathbb{N}$. If $\mathcal{D}$ is a subset of $\mathcal{L}^T(2)$ such that $d(L) + d(M)$ lies in T for all $L \times M$ in $\mathcal{D}$, then the cell-map $\lambda_{\mathcal{D}}\colon \mathcal{L}^T(2) \to \mathcal{L}^T$ commutes up to equivalence with coproducts by virtue of Proposition 83. Furthermore, the direct product map $\pi\colon \mathcal{L}^T \times \mathcal{L}^T \to \mathcal{L}^T(2)$, given by $\pi(L, M) = L \times M$, is a cell-map which commutes with coproducts. The composition $\mu_{\mathcal{D}} = \lambda_{\mathcal{D}}\pi$ is therefore a cell-map $\mathcal{L}^T \times \mathcal{L}^T \to \mathcal{L}^T$, which commutes up to equivalence with the incidence coproduct.

For many choices of $\mathcal{D}$ the cell-map $\mu_{\mathcal{D}}$ is not associative, even up to equivalence, and $\mathcal{L}^T$ fails to be a Hopf interval cell-set. Characterizing the general associativity properties of $\mu_{\mathcal{D}}$ is a problem of considerable subtlety, which is related to the study of one-dimensional formal group laws [11]. We pursue these and related questions elsewhere.

In the following two examples, $\mathcal{D}$ is of an appropriate form to verify associativity without much difficulty. We thereby obtain examples of Hopf interval cell-sets which are not the object cell-sets of productive interval categories.

Example 86. For any T, the cell-map $\mu_m = \lambda_m\pi\colon \mathcal{L}^T \times \mathcal{L}^T \to \mathcal{L}^T$ is given by

$$\mu_m(L, M) = \{C\colon C \text{ is a maximal chain in } L \times M\},$$

for all pairs of T-chains (L, M). It is easy to verify by counting maximal chains that μ_m is associative, and hence $\mathcal{L}^T$ is a Hopf interval cell-set.

In particular, if $T = \{1\}$ then $\mathcal{L}^T = \mathcal{L}$, and we denote the product μ_m simply as μ. The free abelian group $Z_*(\mathcal{L}) = \mathbb{Z}\{\beta_1, \beta_2, \dots\}$ is a Hopf algebra, with the induced product satisfying

$$\beta_j \beta_k = \binom{j+k}{k} \beta_{j+k},$$

for all $j, k \geq 0$. □

We remark that $Z_*(\mathcal{L})$ is the dual of the binomial Hopf algebra, for which we provided various constructions in Examples 8, 41 and 52.

Example 87. Suppose that $T = \mathbb{N}$, so that $\mathcal{L}^T = \mathcal{L}^+$, and let $\mathcal{D} \subseteq \mathcal{L}^+(2)$ be the set of all direct products $L \times M$ such that each of the chains L

and M has length 1 or 0. In other words, $\mathcal{D}$ is the set of all intervals in $\mathcal{L}^+(2)$ which are Boolean algebras. We may confirm that the cell-map $\mu_+ = \lambda_{\mathcal{D}}\pi\colon \mathcal{L}^+ \times \mathcal{L}^+ \to \mathcal{L}^+$ is associative by constructing, for each triple of subgraded chains (L, M, N), subranking-preserving bijections from the sets

$$\mu_+(\mu_+(L, M) \times N) \quad \text{and} \quad \mu_+(L \times \mu_+(M, N))$$

onto the set of all spanning chains $\{x_0 < \cdots < x_k\}$ in $L \times M \times N$ in which each subinterval $[x_{i-1}, x_i]$ is a Boolean algebra. It follows that $Z_*(\mathcal{L}^+)$ is a Hopf algebra, and we may write its product as

$$\beta_c\beta_d = \sum_C \beta_{a(C)}, \tag{88}$$

for all compositions c and d, where the sum is over all $\mathcal{D}$-chains C in a direct product of subgraded chains $L \times M$ with $a(L) = c$ and $a(M) = d$. □

It follows from the product formula (88) that $Z_*(\mathcal{L}^+)$ is the dual of the Hopf algebra $Z_*(\mathcal{L}_\mathrm{f})$ of Example 46, where the basis element β_e of $Z_*(\mathcal{L}^+)$ is dual to the basis element $\theta_e = \theta_{e_1} \cdots \theta_{e_k}$ of $Z_*(\mathcal{L}_\mathrm{f})$. Thus there is an isomorphism from $Z_*(\mathcal{L}^+)$ onto the Hopf algebra of quasi-symmetric functions QSym, which takes β_e to the monomial quasi-symmetric function M_e, for all compositions e. It is not difficult to check that the product formula (88) corresponds to the formula for the product of M_e's given by Ehrenborg in [5]. The product μ_m of Example 86 furnishes $\mathcal{L}^+$ with an alternative Hopf cell-set structure and therefore induces a second Hopf algebra structure on $Z_*(\mathcal{L}^+)$.

We now turn to the multiplicative behaviour of λ_s, λ_m and λ_{mg} when $\mathcal{C}$ has a product structure.

Proposition 89. *If* $\mathbf{C}$ *is a productive interval category then, up to equivalence, the spanning chain cell-map* $\lambda_s\colon \mathcal{C} \to \mathcal{L}^+$ *commutes with* μ_+*, the maximal chain cell-map* $\lambda_m\colon \mathcal{C} \to \mathcal{L}^+$ *commutes with* μ_m*, and the maximal graded chain cell-map* $\lambda_{mg}\colon \mathcal{C} \to \mathcal{L}$ *commutes with* μ*.*

Proof. For all intervals P_1 and P_2 in $\mathcal{C}$, the product $P_1 \square P_2$ and direct product $P_1 \times P_2$ are isomorphic subgraded intervals and hence the sets of spanning chains in $P_1 \square P_2$ and in $P_1 \times P_2$ are equivalent in $\mathcal{L}^+$. Suppose that $C_1 \subseteq P_1$ and $C_2 \subseteq P_2$ are spanning chains. The set $\mu_+(C_1, C_2)$ consists of precisely those spanning chains C in $P_1 \times P_2$ such that $\pi_1(C) = C_1$ and $\pi_2(C) = C_2$, where π_1 and π_2 denote the projections $P_1 \times P_2 \to P_1$ and $P_1 \times P_2 \to P_2$. It follows that $\mu_+(\lambda_s(P_1) \times \lambda_s(P_2))$ is equivalent in $\mathcal{L}^+$ to the set of all spanning chains in $P_1 \times P_2$, and so λ_s preserves products up to equivalence. The proofs for λ_m and λ_{mg} are similar. □

With the hypotheses of Proposition 89, we deduce that the cell-maps λ_s, λ_m and λ_{mg} commute both with products and coproducts up to equivalence, and hence the corresponding maps of free abelian groups are Hopf algebra maps. In particular, after identifying $Z(\mathcal{L}^+)$ with the Hopf algebra of quasi-symmetric functions, the induced map $(\lambda_s)_*\colon Z_*(\mathcal{C}) \to \mathrm{QSym}$ corresponds to

Ehrenborg's *F-quasi-symmetric function map*, shown to be a Hopf algebra map in [5].

10. Cell-set coactions

In our final section we discuss coactions by cell-sets with coproducts, and study the induced comodule structures on the corresponding free abelian groups. These aims have motivated many of our previous constructions.

Suppose $\mathcal{C}$ and $\mathcal{M}$ are cell-sets and that $\mathcal{C}$ has coproduct δ. A *left $\mathcal{C}$-coaction* on $\mathcal{M}$ is a cell-map $\psi\colon \mathcal{M} \to \mathcal{C} \times \mathcal{M}$ such that

$$(\delta \times 1_{\mathcal{M}})\psi \sim (1_{\mathcal{C}} \times \psi)\psi, \tag{90}$$

and a *right $\mathcal{C}$-coaction* is a cell-map $\gamma\colon \mathcal{M} \to \mathcal{M} \times \mathcal{C}$ satisfying

$$(1_{\mathcal{M}} \times \delta)\gamma \sim (\gamma \times 1_{\mathcal{C}})\gamma. \tag{91}$$

By a *coaction*, we shall always mean a left coaction. Whenever ψ is a $\mathcal{C}$-coaction on the cell-set $\mathcal{M}$, the free abelian group $Z_*(\mathcal{M})$ becomes $Z_*(\mathcal{C})$-comodule, with coaction given by the induced map ψ_*, and similarly for right coactions.

Given a cell-map $f\colon \mathcal{C} \to \mathcal{D}$ which commutes with coproducts up to equivalence, any $\mathcal{C}$-coaction ψ on $\mathcal{M}$ may be extended to a $\mathcal{D}$-coaction ψ_f by the formula

$$\psi_f = (f \times 1_{\mathcal{M}})\psi. \tag{92}$$

This is the cell-set version of a classical device for the construction of new coactions from old, which is dual to the notion of change of rings for modules.

Now suppose that $\mathcal{C}$ has a product $\mu_{\mathcal{C}}$ which is compatible with its coproduct. If $\psi_{\mathcal{M}}\colon \mathcal{M} \to \mathcal{C} \times \mathcal{M}$ and $\psi_{\mathcal{P}}\colon \mathcal{P} \to \mathcal{C} \times \mathcal{P}$ are $\mathcal{C}$-coactions then the cell-map $\psi_{\mathcal{M}\times\mathcal{P}}$, defined by

$$\psi_{\mathcal{M}\times\mathcal{P}} = (\mu_{\mathcal{C}} \times 1_{\mathcal{M}} \times 1_{\mathcal{P}})(1_{\mathcal{C}} \times \tau \times 1_{\mathcal{P}})(\psi_{\mathcal{M}} \times \psi_{\mathcal{P}}), \tag{93}$$

is a $\mathcal{C}$-coaction on $\mathcal{M} \times \mathcal{P}$, and the isomorphism (5) of free abelian groups

$$Z_*(\mathcal{M} \times \mathcal{P}) \cong Z_*(\mathcal{M}) \otimes Z_*(\mathcal{P}),$$

is in fact an isomorphism of $Z_*(\mathcal{C})$-comodules, where the right-hand side is the usual tensor product of comodules over a bialgebra (see [16]). If $\mathcal{M} = \mathcal{P}$, and $\mathcal{M}$ has a coproduct $\delta_{\mathcal{M}}$ which respects coactions up to equivalence, in the sense that

$$\psi_{\mathcal{M}\times\mathcal{M}} \cdot \delta_{\mathcal{M}} \sim (1_{\mathcal{C}} \times \delta_{\mathcal{M}}) \cdot \psi_{\mathcal{M}}, \tag{94}$$

then $Z_*(\mathcal{M})$ is a $Z_*(\mathcal{C})$-comodule-coalgebra.

We now assume given an object cell-set $\mathcal{M}$, and establish our general framework for constructing coactions by object cell-sets of various subcategories of the category of arrows $\mathbf{M}^2$. By a *triangle* in a category $\mathbf{M}$, we mean an ordered pair (g, h) of composable morphisms (or, equivalently, a commutative diagram of the form $f = gh$) in $\mathbf{M}$. The set of all triangles in $\mathbf{M}$ is the object set of a category, denoted by $\mathbf{M}^3$, with morphisms of triangles $(g, h) \to (g', h')$ given by triples of morphisms (α, β, γ) in $\mathbf{M}$ such that the diagram

$$\begin{array}{ccccc} x & \xrightarrow{h} & y & \xrightarrow{g} & z \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} \\ x' & \xrightarrow{h'} & y' & \xrightarrow{g'} & z' \end{array}$$

commutes. We say that $\mathbf{M}$ is *decomposition-finite* if, for every morphism f in $\mathbf{M}$ there exist only finitely many triangles (g, h) in $\mathbf{M}$ such that $f = gh$.

Suppose that $\mathbf{M}$ is a cell-category and that $\mathbf{C}$ is a subcategory of $\mathbf{M}$. We denote by $\mathbf{MC}$ the full subcategory of $\mathbf{M}^2$ having the same object set as $\mathbf{C}^2$. Hence the objects of $\mathbf{MC}$ are arrows in $\mathbf{C}$, a morphism $f \to g$ in $\mathbf{MC}$ consists of an ordered pair (α, β) of morphisms in $\mathbf{M}$ such that $g\alpha = \beta f$, and we have inclusions of categories $\mathbf{C}^2 \to \mathbf{MC} \to \mathbf{M}^2$. We make $\mathbf{MC}$ into a cell-category by defining the dimension of an arrow $f \colon x \to y$ to be $d(y) - d(x)$, where d is the dimension function on $\mathbf{M}$.

Suppose that $\mathbf{M}$ and $\mathbf{C}$ are cell-categories having the same object set, with $\mathbf{C}$ a decomposition-finite subcategory of $\mathbf{M}$. We say that the pair $(\mathbf{M}, \mathbf{C})$ is *nice* if, whenever $f \sim f'$ in $\mathcal{MC}$, there exists a bijection between the sets of triangles

$$\{(g, h)\colon\ gh = f \text{ in } \mathbf{C}\} \quad \text{and} \quad \{(g', h')\colon\ g'h' = f' \text{ in } \mathbf{C}\}$$

such that corresponding triangles are isomorphic in $\mathbf{M}^3$. A composable pair of arrows (g, h) in $\mathbf{C}$ may be regarded as a special type of object in either $\mathbf{M}^3$ or $\mathbf{MC} \times \mathbf{MC}$. Observe that isomorphism of pairs in $\mathbf{M}^3$ implies isomorphism in $\mathbf{MC} \times \mathbf{MC}$, but not conversely.

Proposition 95. *If* $(\mathbf{M}, \mathbf{C})$ *is a nice pair of cell-categories then the relation* $\delta \colon \mathcal{MC} \to \mathcal{MC} \times \mathcal{MC}$ *given by*

$$\delta(f) = \{(g, h)\colon\ gh = f \text{ in } \mathbf{C}\} \tag{96}$$

is a coassociative coproduct on the cell-set $\mathcal{MC}$.

Proof. The fact that δ is a cell-map follows immediately from the definition of nice, and coassociativity is a direct consequence of the associativity of composition of morphisms. It is clear the δ preserves dimension. □

Given a nice pair $(\mathbf{M}, \mathbf{C})$ we consider the source and target maps S and T as having their domains restricted to $\mathcal{MC}$, so that in particular the inverse target relation $T^{-1} \colon \mathcal{M} \to \mathcal{MC}$ is given by

$$T^{-1}(x) = \{f \colon f \in \mathcal{MC} \text{ and } T(f) = x\}.$$

We say that the nice pair $(\mathbf{M}, \mathbf{C})$ is *target-nice* if $T^{-1}(x)$ is finite for all $x \in \mathcal{M}$, and $T^{-1}(x) \sim T^{-1}(y)$ in $\mathcal{MC}$ whenever $x \sim y$ in $\mathcal{M}$.

Proposition 97. *If* $(\mathbf{M}, \mathbf{C})$ *is a target-nice pair of cell-categories then the relation* $\psi \colon \mathcal{M} \to \mathcal{MC} \times \mathcal{M}$ *given by*

$$\psi(x) = \{(f, S(f))\colon f \in T^{-1}(x)\} \tag{98}$$

is an $\mathcal{MC}$-coaction on $\mathcal{M}$.

Proof. Let us denote the diagonal map $f \mapsto (f, f)$ on $\mathcal{MC}$ by Δ. Each of the three maps $T^{-1}: \mathcal{M} \to \mathcal{MC}$, $\Delta: \mathcal{MC} \to \mathcal{MC} \times \mathcal{MC}$ and $1_{\mathcal{MC}} \times S: \mathcal{MC} \times \mathcal{MC} \to \mathcal{MC} \times \mathcal{M}$ preserves equivalence and therefore so does the composition $(1_{\mathcal{MC}} \times S)\Delta T^{-1}$, which is equal to ψ. It is clear that ψ also preserves dimension, and so it is a cell-map. The fact that ψ is a coaction follows from the observation that the common value of $(\delta \times 1_{\mathcal{M}})\psi(x)$ and $(1_{\mathcal{MC}} \times \psi)\psi(x)$ is

$$\{(f, g, S(g))\colon fg \in T^{-1}(x)\}.$$

□

Every poset may be realized as a category $\mathbf{P}$ in which there is at most one morphism between any two objects x and y. We refer to such categories as *categorical posets*, noting that the original poset is recovered from $\mathbf{P}$ by stipulating that $x \leq y$ precisely when there exists a morphism $y \to x$ in $\mathbf{P}$.

Example 99. Suppose that $\mathbf{P}$ is a categorical poset such that $T^{-1}(x) = \{y\colon y \geq x\}$ is finite, for all objects x of $\mathbf{P}$, and regard $\mathbf{P}$ as a cell-category whose dimension function is identically zero. The pair $(\mathbf{P}, \mathbf{P})$ is obviously target-nice, since the finiteness conditions hold and the equivalence relations on the cell-sets $\mathcal{P}$ and $\mathcal{P}^2 = \mathcal{PP}$ are trivial. For all $x \leq z \in \mathcal{P}$ we denote the unique morphism from z to x by (x, z). We may then write the coproduct on $\mathcal{P}^2$ as

$$\delta(x, z) = \{((x, y), (y, z)) : \ x \leq y \text{ and } y \leq z \text{ in } \mathcal{P}\},$$

and the $\mathcal{P}^2$-coaction on $\mathcal{P}$ as

$$\psi(x) = \{((x, y), y)\colon y \geq x \text{ in } \mathcal{P}\}.$$

Mapping each morphism (x, z) in $\mathbf{P}$ to the interval $[x, z]$ in $\mathcal{P}$ defines a natural isomorphism from the coalgebra $Z_*(\mathcal{P}^2)$ onto the incidence coalgebra of the poset $\mathcal{P}$. The composition of this isomorphism with the induced coaction ψ_* gives a coaction of the incidence coalgebra of $\mathcal{P}$ on $Z_*(\mathcal{P})$, which we also denote by ψ_*, satisfying

$$\psi_*(x) = \sum_{y \geq x} [x, y] \otimes y.$$

This coaction (along with a dual, right coaction) was introduced by Graves in [9]; so equipped, we refer to $Z_*(\mathcal{P})$ as the *incidence comodule* of $\mathcal{P}$. □

Experience with the theory of incidence coalgebras confirms that their full potential is realized only after they have been *reduced* by an appropriate equivalence relation. Such reduction is essential, for example, in the construction of incidence Hopf algebras. We now apply the same principle to coactions, and enrich the applications of Example 99 by imposing various types of equivalence relations on the basic underlying model.

Given an interval category $\mathbf{M}$ we denote by $\mathbf{M}_{\text{inc}}$ the subcategory of all inclusion maps in $\mathbf{M}$. By an *inclusion subcategory* of $\mathbf{M}$, we mean a subcategory of the form $\mathbf{D}_{\text{inc}}$, where $\mathbf{D}$ is a subcategory of $\mathbf{M}$ having the same isomorphisms as $\mathbf{M}$. If $\mathbf{C} = \mathbf{D}_{\text{inc}}$ is an inclusion subcategory of $\mathbf{M}$ it follows that $\mathbf{C}$ and $\mathbf{M}$ have the same set of objects, and that the arrow category $\mathbf{MC}$ is an isomorphism-closed subcategory of $\mathbf{MM}_{\text{inc}}$; in other words, the cell-set $\mathcal{MC}$ is a union of equivalence classes of the cell-set of all inclusions $\mathcal{MM}_{\text{inc}}$.

Proposition 100. *If* $\mathbf{C}$ *is an inclusion subcategory of an interval category* $\mathbf{M}$*, then* $(\mathbf{M}, \mathbf{C})$ *is a target-nice pair.*

Proof. Since all morphisms in $\mathbf{C}$ are inclusions, it is immediate that $\mathbf{C}$ is decomposition-finite and that $T^{-1}(x)$ is finite for all $x \in \mathcal{M}$.

Suppose that $(\alpha, \beta)\colon i_{R,P} \to i_{R',P'}$ is an isomorphism in $\mathbf{C}$, and that Q is an interval such that $(i_{Q,P}, i_{R,Q})$ belongs to $\delta(i_{R,P})$. Since $\mathbf{M}$ is an interval category, and thus retains all factorizations of the form (27), the image $Q' = \beta(Q)$ of the map $\beta|Q = \beta \cdot i_{Q,P}\colon Q \to P'$ belongs to $\mathcal{M}$, and we have the factorization $\beta|Q = i_{Q',P'} \cdot \beta\|Q$ in $\mathbf{M}$, where $\beta\|Q = \widehat{\beta|Q}\colon Q \to Q'$ is an isomorphism. Therefore the inclusion $i_{R',Q'} = (\beta\|Q) \cdot i_{R,Q} \cdot \alpha^{-1}$ belongs to $\mathbf{M}$ and the triple $(\alpha, \beta\|Q, \beta)$ is an isomorphism between the triangles $(i_{Q,P}, i_{R,Q})$ and $(i_{Q',P'}, i_{R',Q'})$ in $\mathbf{M}^3$. Since $\mathbf{MC}$ is an isomorphism-closed subcategory of $\mathbf{MM}_{\text{inc}}$, it follows that $(i_{Q',P'}, i_{R',Q'})$ belongs to $\delta(i_{R',P'})$, and hence the correspondence $(i_{Q,P}, i_{R,Q}) \mapsto (i_{Q',P'}, i_{R',Q'})$ defines a bijection from $\delta(i_{R,P})$ onto $\delta(i_{R',P'})$, such that corresponding terms are isomorphic in $\mathbf{M}^3$.

The verification that $x \sim y$ in $\mathcal{M}$ implies $T^{-1}(x) \sim T^{-1}(y)$ in $\mathcal{MC}$ is similar. $\square$

Any inclusion subcategory $\mathbf{C}$ of an interval category $\mathbf{M}$ is a categorical poset, and hence the cell-set of objects $\mathcal{M}$ is partially ordered, with $P \leq Q$ in $\mathcal{M}$ if and only if the inclusion $i_{Q,P}$ belongs to $\mathcal{MC}$. For any $P \in \mathcal{M}$ we therefore denote the set $\{Q\colon Q \geq P \text{ in } \mathcal{M}\}$ by $\mathcal{M}^P$. Note that $\mathcal{M}^P$ is finite and has minimal element P, but in general has many maximal elements, and hence is not a subinterval of $\mathcal{P}$. Using the pair notation (P, Q) for the inclusion map $i_{Q,P}$, we can thus write the coproduct δ on $\mathcal{MC}$ as

$$\delta(P, R) = \{((P, Q), (Q, R))\colon Q \in [P, R]\}, \tag{101}$$

and the $\mathcal{MC}$-coaction ψ on $\mathcal{M}$ as

$$\psi(P) = \{((P, Q), Q) : Q \in \mathcal{M}^P\}. \tag{102}$$

The set of all subintervals of the poset $\mathcal{M}$, together with the obvious dimension function, and equivalence relation given by $[P, Q] \sim [P', Q']$ if and only if $(P, Q) \sim (P', Q')$ in $\mathcal{MC}$, is a closed interval cell-set, which we shall denote by $\mathcal{MC}'$. The correspondence $(P, Q) \mapsto [P, Q]$ is a cell-set equivalence which respects coproducts.

Given an interval category $\mathbf{M}$, we denote by $\mathbf{M}_{\text{int}}$ the subcategory of $\mathbf{M}$ consisting of all inclusion maps $i_{Q,P}$ such that Q is a subinterval of P in $\mathcal{M}$. When $\mathbf{M}$ is closed, $\mathcal{MM}_{\text{int}}$ is intimately related to $\mathcal{M}$ and to $\mathcal{M}^{\text{op}}$, the cell-set $\mathcal{M}$ equipped with the opposite coproduct. We have cell-maps $\eta\colon \mathcal{M} \to \mathcal{M}\mathcal{M}_{\text{int}}$ and $\nu\colon \mathcal{M}^{\text{op}} \to \mathcal{M}\mathcal{M}_{\text{int}}$, defined by

$$\eta(P) = (P, \{x\}) \qquad \text{and} \qquad \nu(P) = (P, \{y\}),$$

for all $P = [x, y]$ in $\mathcal{M}$ and $\mathcal{M}^{\text{op}}$ respectively, which preserve coproducts up to equivalence. We also have cell-maps $\gamma\colon \mathcal{M}\mathcal{M}_{\text{int}} \to \mathcal{M}$ and $\pi\colon \mathcal{M}\mathcal{M}_{\text{int}} \to \mathcal{M}^{\text{op}}$, given by

$$\gamma(P, R) = \begin{cases} P_x & \text{if } R = P^x, \text{ for some } x \in P \\ \emptyset & \text{otherwise} \end{cases}$$

and

$$\pi(P, R) = \begin{cases} P^x & \text{if } R = P_x, \text{ for some } x \in P \\ \emptyset & \text{otherwise,} \end{cases}$$

which preserve coproducts up to equivalence. The induced coaction $\psi_\gamma = (\gamma \times 1_{\mathcal{M}})\psi$ of $\mathcal{M}$ on itself is equal to the incidence coproduct of $\mathcal{M}$.

Example 103. Consider the pair $(\mathbf{L}, \mathbf{L}_{\text{int}})$, where $\mathbf{L}$ is the graded interval category of chains introduced in Example 30, and suppose (L, M) belongs to $\mathcal{L}\mathcal{L}_{\text{int}}$, where M is the subinterval $[x, y]$ of the chain L. The equivalence class of (L, M) in $\mathcal{L}\mathcal{L}_{\text{int}}$ is determined by the triple of integers $(d(L_x), d(M), d(L^y))$. Hence the free abelian group $Z_*(\mathcal{L}\mathcal{L}_{\text{int}})$ has one generator $\beta_{n,k,m}$ of dimension $n + m$, for each triple of nonnegative integers (n, k, m). The induced coproduct on $Z_*(\mathcal{L}\mathcal{L}_{\text{int}})$ is given by

$$\delta_*(\beta_{n,k,m}) = \sum_{i=1}^{n} \sum_{j=1}^{m} \beta_{n-i,i+k+j,m-j} \otimes \beta_{i,k,j},$$

and the induced coaction of $Z_*(\mathcal{L}\mathcal{L}_{\text{int}})$ on $Z_*(\mathcal{L})$ is given by

$$\psi_*(\beta_n) = \sum_{k=0}^{n} \left(\sum_{i+j=n-k} \beta_{i,k,j} \right) \otimes \beta_k.$$

□

Suppose that $\mathbf{C}$ is an inclusion subcategory of a closed interval category $\mathbf{M}$. We define $\mathbf{MC}$ to be *closed* by analogy with labelled interval categories; since $\mathbf{MC}$ is a full subcategory of $\mathbf{M}^2$, this amounts to the requirement that

$$(P_x, Q_x), (P^x, Q^x) \in \mathcal{M}\mathcal{C} \tag{104}$$

for all $(P, Q) \in \mathcal{M}\mathcal{C}$ and $x \in Q$. If $\mathcal{M}\mathcal{C}$ admits a product μ which is compatible with the coproduct, the following result establishes conditions which ensure that the incidence coproduct $\delta_{\mathcal{M}}$ respects coactions, so that $Z_*(\mathcal{M})$ is a $Z_*(\mathcal{M}\mathcal{C})$-comodule-coalgebra.

Proposition 105. *Suppose that a product μ on $\mathcal{M}\mathcal{C}$ satisfies*

$$\mu\left((P_x, Q_x), (P^x, Q^x)\right) \sim (P, Q) \tag{106}$$

for all $(P,Q) \in \mathcal{MC}$ *and* $x \in Q$*, and that, for all* $x \in P \in \mathcal{M}$*,*

$$R \in \mathcal{M}^{P_x} \text{ and } S \in \mathcal{M}^{P^x} \quad \Longrightarrow \quad R = Q_x \text{ and } S = Q^x \tag{107}$$

for some $Q \in \mathcal{M}^P$*; then the incidence coproduct on* $\mathcal{M}$ *respects coactions.*

Proof. For any interval $P \in \mathcal{M}$, (93) and (102) imply that $\psi_{\mathcal{M}\times\mathcal{M}} \cdot \delta_{\mathcal{M}}(P)$ and $(1_{\mathcal{M}C} \times \delta_{\mathcal{M}}) \cdot \psi_{\mathcal{M}}(P)$ are equal to

$$\left\{(\mu\left((P_x,R),(P^x,S)\right),R,S) : x \in P,\ R \in \mathcal{M}^{P_x} \text{ and } S \in \mathcal{M}^{P^x}\right\} \tag{108}$$

and

$$\{((P,Q),Q_x,Q^x) : Q \in \mathcal{M}^P \text{ and } x \in Q\} \tag{109}$$

respectively. Since **MC** is closed, it follows from (106) that the set (109) is equivalent to a subset of (108) in $\mathcal{MC} \times \mathcal{M} \times \mathcal{M}$, and (107) implies the reverse inclusion up to equivalence. Hence $\delta_{\mathcal{M}}$ satisfies (94), and therefore respects coactions. □

We remark that the (107) is closely related to the condition on the inverse target relation T^{-1} given in the context of labelled interval cell-sets in Proposition 68.

In each of our final examples we consider an interval category **M**, and introduce the subcategory $\mathbf{M}_{\rm sp}$ of all inclusions (P,Q) for which Q is a spanning subposet of P; it is important to note that this is an inclusion subcategory.

Example 110. Consider the pair $(\mathbf{L},\mathbf{L}_{\rm sp})$, where **L** is the graded interval category of chains introduced in Example 30, and for each $L, C \in \mathcal{L}$ define $L \circ C$ to be the chain obtained by identifying the maximal element of L with the minimal element of C. The cell-set $\mathcal{LL}_{\rm sp}$ has product μ given by

$$\mu((L,M),(C,D)) = (L \circ C, M \circ D),$$

which is compatible with the coproduct (101). The free abelian group $Z_*(\mathcal{LL}_{\rm sp})$ is therefore a noncommutative, noncocommutative bialgebra; it is isomorphic to the free associative algebra $\mathbb{Z}\langle c_0, c_1, \ldots\rangle$, where the generator c_n has dimension n and is represented by the pair $([n+2],\{1,n+2\})$, and the multiplicative identity $1 \neq c_0$ is represented by the pair $(\{1\},\{1\})$. The coproduct is determined by $\delta_*(1) = 1 \otimes 1$ and

$$\delta_*(c_k) = \sum_{i+j=k} (c_0 + c_1 + c_2 + \cdots)^{i+1}_j \otimes c_i, \tag{111}$$

for $k \geq 0$. Note that $Z_*(\mathcal{LL}_{\rm sp})$ is not a Hopf algebra, for the element c_0 is group-like but not invertible. The coaction $\psi\colon \mathcal{L} \to \mathcal{LL}_{\rm sp} \times \mathcal{L}$ satisfies

$$\psi(D) = \{((D,C),C) : C \text{ is a spanning chain of } D\},$$

so that the induced coaction of $Z_*(\mathcal{L}\mathcal{L}_{\text{sp}})$ on $Z_*(\mathcal{L})$ is given by $\psi_*(\beta_0) = 1 \otimes \beta_0$, and

$$\psi_*(\beta_k) = \sum_{i=1}^{k} (c_0 + c_1 + c_2 + \cdots)^i_{k-i} \otimes \beta_i, \tag{112}$$

for $k \geq 0$. We may easily verify that the pair $(\mathbf{L}, \mathbf{L}_{\text{sp}})$ satisfies conditions (104), (106) and (107), ensuring that $Z_*(\mathcal{L})$ is a $Z_*(\mathcal{L}\mathcal{L}_{\text{sp}})$-comodule-coalgebra by virtue of Proposition 105. □

Let D be a finite chain, and recall from §4 the canonical map ω of the poset $\text{Sp}(D) = \mathcal{L}^D$ onto the poset of delineations $\Delta(D_{\text{cut}})$, satisfying $\omega(C) = \{J_1, \ldots, J_k\}$ for each spanning chain C, where J_i is determined by (20). Any subinterval $[D, C]$ of $\mathcal{L}^D$ is mapped isomorphically onto the subinterval $\Delta(D_{\text{cut}})_{\omega(C)}$ of $\Delta(D_{\text{cut}})$. We thus obtain a coproduct-preserving cell-map, also written ω, from $\mathcal{L}\mathcal{L}_{\text{sp}}$ to the cell-set $\mathcal{N}$ of all labelled subintervals in delineation posets (see Example 73), given by $(D, C) \mapsto \Delta(D_{\text{cut}})_{\omega(C)}$. It is apparent from the definition of the products on $\mathcal{L}\mathcal{L}_{\text{sp}}$ and $\mathcal{N}$ that ω preserves products up to equivalence. The induced map ω_* from $Z_*(\mathcal{L}\mathcal{L}_{\text{sp}})$ onto $Z_*(\mathcal{N}) = \mathbb{Z}[b_1, b_2, \ldots]$, determined by $c_k \mapsto b_k$ for all $k \geq 0$ (where $b_0 = 1$), is therefore a bialgebra map. According to (92) we obtain a coaction $\psi_\omega \colon \mathcal{L} \to \mathcal{N} \times \mathcal{L}$, which satisfies

$$\psi_\omega(D) = \left\{ \left(\Delta(D_{cut})_{\omega(C)}, C \right) : C \in \text{Sp}(D) \right\}.$$

The induced map $(\psi_\omega)_*$ is determined by

$$\psi_*(\beta_k) = \sum_{i+j=k} (1 + b_1 + b_2 + \cdots)^i_j \otimes \beta_i \tag{113}$$

for $k \geq 0$. Since ω_* preserves products as well as coproducts, $Z_*(\mathcal{L})$ is a $Z_*(\mathcal{N})$-comodule-coalgebra.

Suppose that $\mathbf{M}$ is one of the categories $\mathbf{L}_{\text{f}}$, $\mathbf{L}_{\text{fc}}$ or $\mathbf{L}(q)_{\text{fc}}$ from Examples 46 and 47. An inclusion map in $\mathbf{M}$ has the form $i_1 \times \cdots \times i_k$ for some $k \geq 1$ (if $\mathbf{M} = \mathbf{L}_{\text{fc}}(q)$, then $k = q$), where each i_j is an inclusion of chains; hence the cell-set $\mathcal{M}\mathcal{M}_{\text{sp}}$ consists of pairs $(D_1 \times \cdots \times D_k, C_1 \times \cdots \times C_k)$, where $C_i \in \text{Sp}(D_i)$. The cell-map $\widehat{\omega} \colon \mathcal{M}\mathcal{M}_{\text{sp}} \to \mathcal{N}$ given by

$$(D_1 \times \cdots \times D_k, C_1 \times \cdots \times C_k) \mapsto \prod_{i=1}^{k} \omega(D_i, C_i)$$

preserves coproducts up to equivalence, so that the $\mathcal{M}\mathcal{M}_{\text{sp}}$-coaction ψ induces an $\mathcal{N}$-coaction $\psi_{\widehat{\omega}}$ on $\mathcal{M}$. The inclusion map $\mathcal{L}(q-1) \to \mathcal{L}(q)$ respects these coactions, giving rise to a coaction of $\mathcal{N}$ on the quotient cell-set $\mathcal{R}(q)$ of Example 82.

Such structures are interrelated in many ways, as the formulae suggest. For example, given any $q \geq 1$ we define $f_q \colon \mathcal{L}(q) \setminus \mathcal{L}(q-1) \to \mathcal{L}(q)\mathcal{L}(q)_{\text{sp}}$ by

$$f_q(D_1 \times \cdots \times D_q) = (D_1 \times \cdots \times D_q, \{x_1, y_1\} \times \cdots \times \{x_q, y_q\})$$

on each product of chains $D_i = [x_i, y_i]$; the domain of f_q may be interpreted as $\mathcal{R}(q)$, divested of its zero-dimensional intervals. If we redefine the dimension of $D_1 \times \cdots \times D_q$ as $\sum(d(D_i) - 1)$, then f_q becomes a cell-map which commutes with $\mathcal{L}(q)\mathcal{L}(q)_{\mathrm{sp}}$-coactions up to equivalence (where $\mathcal{L}(q)\mathcal{L}(q)_{\mathrm{sp}}$ coacts on itself by its coproduct). When $q = 1$, this procedure reconciles the coproduct (111) with the coaction (112), under the appropriate shift of dimension; for higher values of q, we take products.

Example 114. Consider the pair $(\mathbf{B}, \mathbf{B}_{\mathrm{sp}})$, where $\mathbf{B}$ is the category of finite Boolean algebras introduced in Example 41; the inclusion subcategory $\mathbf{B}_{\mathrm{sp}}$ consists of all pairs (P, Q) such that $P, Q \in \mathcal{B}$ and Q is a subBoolean algebra of P. The cell-set $\mathcal{B}\mathcal{B}_{\mathrm{sp}}$ has product μ defined by

$$\mu((P, Q), (R, S)) = (P \times R, Q \times S),$$

which is compatible with its coproduct and commutative up to equivalence.

Given $(P, Q) \in \mathcal{B}\mathcal{B}_{\mathrm{sp}}$, and an atom $y \in A(Q)$, we write U_y for the set $\{x \in A(P) \colon x \leq y\}$. We obtain an isomorphism σ from the poset $\mathcal{B}^P$ of subBoolean algebras of P (ordered by reverse inclusion) onto the partition lattice $\Pi(A(P))$ by setting

$$\sigma(Q) = \{U_y \colon y \in A(Q)\}$$

(see Examples 16 and 41). Pairs (P, Q) and (R, S) are equivalent in $\mathcal{B}\mathcal{B}_{\mathrm{sp}}$ if and only if the partitions $\sigma(Q)$ and $\sigma(S)$ have the same number of blocks of size i, for all $i \geq 1$. Thus $(P, Q) \sim (R, S)$ implies that the intervals $[\sigma(P), \sigma(Q)] = \Pi(A(P))_{\sigma(Q)}$ and $[\sigma(R), \sigma(S)] = \Pi(A(R))_{\sigma(S)}$ are isomorphic posets, but not conversely. It follows that $Z_*(\mathcal{B}\mathcal{B}_{\mathrm{sp}})$ is the polynomial bialgebra $\mathbb{Z}[\nu_0, \nu_1, \ldots]$, where ν_k has dimension k and is represented by the pair $(B([k+1]), \{\emptyset, [k+1]\})$, and the identity element $1 \neq \nu_0$ is represented by $(B(\emptyset), B(\emptyset))$. The induced coproduct on $Z_*(\mathcal{B}\mathcal{B}_{\mathrm{sp}})$ is determined by $\delta_*(1) = 1 \otimes 1$ and

$$\begin{aligned}\delta_*(\nu_k) &= \sum_{i=0}^{k+1} B_{k+1,i+1}(\nu_0, \nu_1, \nu_2, \ldots) \otimes \nu_i \\ &= (k+1)! \sum_{i+j=k} \left(\nu_0 + \frac{\nu_1}{2!} + \frac{\nu_2}{3!} + \cdots\right)_j^{i+1} \otimes \frac{\nu_i}{(i+1)!},\end{aligned}$$

and the coaction on $Z_*(\mathcal{B})$ by

$$\begin{aligned}\psi_*(x^k) &= \sum_{i=0}^{k} B_{k,i}(\nu_0, \nu_1, \nu_2, \ldots) \otimes x^i \\ &= k! \sum_{i+j=k} \left(\nu_0 + \frac{\nu_1}{2!} + \frac{\nu_2}{3!} + \cdots\right)_j^i \otimes \frac{x^i}{i!}.\end{aligned}$$

The pair $(\mathbf{B}, \mathbf{B}_{\mathrm{sp}})$ satisfies conditions (104), (106) and (107), ensuring that $Z_*(\mathcal{B})$ is a $Z_*(\mathcal{B}\mathcal{B}_{\mathrm{sp}})$-comodule-coalgebra by Proposition 105. $\square$

The cell-map $\sigma\colon \mathcal{B}\mathcal{B}_{\mathrm{sp}} \to \mathcal{F}$ given by $\sigma(P,Q) = \Pi(A(P))_{\sigma(Q)}$ preserves products and coproducts up to equivalence, and therefore the induced map σ_* from $Z_*(\mathcal{B}\mathcal{C})$ onto the Faà di Bruno Hopf algebra $Z_*(\mathcal{F}) = \mathbb{Z}[\varphi_1, \varphi_2 \ldots]$, which maps ν_k to φ_k, is a bialgebra map. The induced $Z_*(\mathcal{F})$-coaction on $Z_*(\mathcal{B})$ is determined by the rational expression

$$(\psi_\sigma)_*(x^k) = k!\sum_{i=1}^{k}(1 + \frac{\varphi_1}{2!} + \frac{\varphi_2}{3!} + \cdots)^i_{k-i} \otimes \frac{x^i}{i!} \tag{115}$$

for $k \geq 0$. Since σ_* is a bialgebra map, $Z_*(\mathcal{B})$ is a $Z_*(\mathcal{F})$-comodule-coalgebra.

The coaction (113) is well known to algebraic topologists, and is linked to the coaction (115) by means of the commutative diagrams of cell maps

$$\begin{array}{ccccc} \mathcal{B} & \xrightarrow{\psi} & \mathcal{F} \times \mathcal{B} & \xrightarrow{T^{-1}\times 1_{\mathcal{B}}} & \mathcal{N} \times \mathcal{B} \\ {\scriptstyle \lambda_m}\downarrow & & & & \downarrow{\scriptstyle 1_{\mathcal{N}}\times\lambda_m} \\ \mathcal{L} & & \xrightarrow{\qquad\psi\qquad} & & \mathcal{N} \times \mathcal{L} \end{array} .$$

Here λ_m and T^{-1} are given in Example 84 and (76) respectively, and the composition of the upper horizontal arrows is the induced $\mathcal{N}$-coaction on $\mathcal{B}$. The coaction (115) has no immediate topological counterpart.

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