Universal Equivariant Genus for Torus Actions

Victor M. Buchstaber

Steklov Mathematical Institute,
Russian Academy of Sciences,
Moscow

School of Mathematics,
University of Manchester,
Manchester

< *Victor.Buchstaber@manchester.ac.uk* >

Joint work with
Taras Panov and Nige Ray

(Manchester 13.02.2007)

Abstract

We consider the universal equivariant genus for stably complex $2n$-dimensional manifolds equipped with an action of the standard k-dimensional torus $\mathbb{T}^k$.

In the case of quasitoric manifolds M we explicitly calculate this genus in terms of the underlying combinatorial data (P, Λ) consisting of an oriented combinatorial n-dimensional simple polytope P with m facets and corresponding integer matrix $\Lambda = (I_n, \Lambda_*)$, where I_n is an identity $(n \times n)$-matrix and Λ_* is an $n \times (m-n)$-matrix.

By way of application, we obtain a formula evaluating the complex cobordism class of M in terms of (P, Λ).

The talk is based on:

1. Victor M Buchstaber and Nigel Ray, *Tangential structures on toric manifolds, and connected sums of polytopes.*, Internat Math Res Notices, 4, 2001, 193–219.

2. Victor M Buchstaber and Taras E Panov, *Torus Actions and Their Applications in Topology and Combinatorics.*, University Lecture Series, v. 24, Amer Math Soc, Providence RI, 2002.

3. Victor M Buchstaber, Taras E Panov and Nigel Ray, *Spaces of polytopes and cobordism of quasitoric manifolds.*, Moscow Math J, 7(2), 2007 (arXiv:math.AT/0609346).

4. Victor M Buchstaber and Nigel Ray, *The universal equivariant genus and Krichever's formula.*, Uspehi Mat.Nauk, 62(1), 2007, 195-196;
Russian Math. Surveys, 62(1), 2007 (English translation).

Notation and agreements

We denote by $\mathbb{R}^n$ the standard real n-dimensional Euclidean space with the **standard basis** consisting of vectors $e_i = (0, \ldots, 1, \ldots, 0)$ with 1 on the i-th place, for $1 \leqslant i \leqslant n$; and similarly for $\mathbb{Z}^n$ and $\mathbb{C}^n$.

The standard basis gives rise to the **canonical orientation** of $\mathbb{R}^n$.

We identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ by means of the real vector space isomorphism $\mathbb{C}^n \to \mathbb{R}^{2n}$ sending e_j to e_{2j-1} and $(\sqrt{-1})e_j$ to e_{2j} for $1 \leqslant j \leqslant n$. This provides the **canonical orientation** for $\mathbb{C}^n$.

Since $\mathbb{C}$-linear maps from $\mathbb{C}^n$ to $\mathbb{C}^n$ preserve the canonical orientation, we may also regard an arbitrary complex vector space as canonically oriented.

We denote by $\mathbb{T}^n$ the standard n-dimensional torus $\mathbb{R}^n/\mathbb{Z}^n$, which we identify with the product of n unit circles in $\mathbb{C}^n$:

$$\mathbb{T}^n = \{(e^{2\pi\sqrt{-1}\varphi_1}, \ldots, e^{2\pi\sqrt{-1}\varphi_n}) \in \mathbb{C}^n\},$$

where $(\varphi_1, \ldots, \varphi_n)$ runs over $\mathbb{R}^n$. The torus $\mathbb{T}^n$ is also **canonically oriented**.

Normally complex $\mathbb{T}^k$-manifolds

We consider smooth $2n$-dimensional manifolds M, equipped with a smooth action α of a k-dimensional torus $\mathbb{T}^k$.

We may choose an action of $\mathbb{T}^k$ on $\mathbb{C}^l$ and a $\mathbb{T}^k$-equivariant embedding $i\colon M \to \mathbb{C}^l$ for suitably large l.

Then we may choose a $\mathbb{T}^k$-equivariant unitary structure c_ν on the normal bundle $\nu(i)$ of i.

This is unique up to natural equivalence, and displays (M, α, c_ν) as a **normally complex $\mathbb{T}^k$-manifold**;
it represents a $2n$-dimensional element $[M, \alpha, c_\nu]$ of the geometric $\mathbb{T}^k$-equivariant complex cobordism ring $\Omega_*^{U,\mathbb{T}^k}$.

Construction of the universal equivariant genus

We use the standard (diagonal) action of $S^1 = \{z \in \mathbb{C} \colon |z| = 1\}$ on the unit sphere $S^{2m+1} \subset \mathbb{C}^{m+1}$.

We write the k-fold product $\prod_{i=1}^k S^{2m+1}$ as $\Pi S_m \subset \mathbb{C}^{k(m+1)}$, and note that it admits the standard action of $\mathbb{T}^k$. The quotient space ΠP_m is the corresponding product $\prod_{i=1}^k \mathbb{C}P^m$ of complex projective spaces.

So we may define a $2(km+n)$-dim smooth manifold $W_m = \Pi S_m \times_{\mathbb{T}^k} M$, and an l-dim complex vector bundle $q_m \colon E_m \to \Pi P_m$, where $E_m = \Pi S_m \times_{\mathbb{T}^k} \mathbb{C}^l$ for the above chosen action of $\mathbb{T}^k$ on $\mathbb{C}^l$.

Then i extends to an embedding $i' \colon W_m \to E_m$, and c_ν extends to a complex structure c' on the normal bundle $\nu(i')$.

The composition

$$p_m\colon W_m \xrightarrow{i'} E_m \xrightarrow{q_m} \varPi P_m$$

is complex oriented and determines a complex cobordism class $\varPhi_m(M,\alpha,c_\nu) \in U^{-2n}(\varPi P_m)$.

The standard embedding $\iota_m\colon \varPi P_m \to \varPi P_{m+1}$ satisfies $\iota_m^*\varPhi_{m+1} = \varPhi_m$, so the inverse sequence $\{\varPhi_m(M,\alpha,c_\nu) : m \geqslant 0\}$ defines an element of $\lim U^{-2n}(\varPi P_m)$, which is isomorphic to $U^{-2n}(\varPi P_\infty)$.

We write this element as

$$\varPhi(M,\alpha,c_\nu) \in U^{-2n}(\varPi P_\infty),$$

and call it the **universal $\mathbb{T}^k$-equivariant genus** of (M,α,c_ν).

It is straightforward to check that it is well-defined on equivariant cobordism classes, and may therefore be displayed as a homomorphism

$$(1) \qquad \varPhi\colon \varOmega_{2n}^{U,\mathbb{T}^k} \longrightarrow U^{-2n}(\varPi P_\infty).$$

Tangentially stably complex $\mathbb{T}^k$-manifold

We interpret M as a **tangentially stably complex $\mathbb{T}^k$-manifold** $(M, \alpha, c_\mathcal{T})$ whenever an equivariant complex structure $c_\mathcal{T}$ is chosen for its stable tangent bundle. So

(2) $$c_\mathcal{T}\colon \tau(M) \oplus \mathbb{R}^{2(l-n)} \longrightarrow \xi$$

is a real isomorphism for some complex vector bundle ξ, and the composition

(3) $$r(t)\colon \xi \xrightarrow{c_\tau^{-1}} \tau(M) \oplus \mathbb{R}^{2(l-n)} \xrightarrow{d\alpha(t)\oplus I} \tau(M) \oplus \mathbb{R}^{2(l-n)} \xrightarrow{c_\tau} \xi$$

is a **complex transformation** for any $t \in \mathbb{T}^k$, where $d\alpha(t)$ is the differential of the action by $\alpha(t)$.

So (3) corresponds to a representation

$$\rho\colon \mathbb{T}^k \to \mathrm{Hom}_\mathbb{C}(\xi, \xi).$$

Up to natural equivalence an equivariant complex structure c_τ determines a unique $\mathbb{T}^k$-equivariant complex structure c_ν on the normal bundle $\nu(i)$ of i.

It is important to remember that this procedure is not generally reversible, because not all c_ν determine an equivariant c_τ.

We may define $\Phi(M, \alpha, c_\tau)$ to be the associated $\Phi(M, \alpha, c_\nu)$.

The genus Φ has been applied to deduce fundamental results on Hirzebruch genera of S^1-manifolds by Krichever.

The universal equivariant genus as a power series

The ring $U^*(\Pi P_\infty)$ is isomorphic to the algebra $\Omega^*[[u_1,\ldots,u_k]]$ of formal power series over $\Omega^* = U^*(pt)$. Here u_i is represented by the inverse sequence

$$\{\iota_m \colon \Pi_i P_m \to \Pi P_m : m \geqslant 0\}$$

of 2-codimensional submanifolds, obtained by substituting $\mathbb{C}P^{m-1}$ for the i-th copy of $\mathbb{C}P^m$ in ΠP_m.

An additive basis for $U^*(\Pi P_\infty)$ therefore consists of monomials of the form

$$u^s \colon = \prod_j u_j^{s(j)},$$

where s ranges over all maps $s : \{1,\ldots,k\} \to \mathbb{Z}_{\geqslant}$. Then u^s is represented by an appropriate inverse sequence

$$\{\iota_m \colon \Pi_s P_m \to \Pi P_m : m \geqslant 0\}$$

of $2|s|$-codimensional submanifolds, where $|s| = \sum_j s(j)$.

We want to give an explicit construction of stably complex manifolds $G_s(M)$ of dimension $2(n+|s|)$, such that

$$\Phi(M, \alpha, c_\nu) = \sum_s [G_s(M)]\, u^s.$$

Clearly $G_0(M) = M$, with stably complex structure c_ν.

At first consider the subspace $(S^3)^j$ of $\mathbb{C}^{2j}$

$$\{(y_1, z_1; \dots; y_j, z_j) : |y_q|^2 + |z_q|^2 = 1,\ q = 1, \dots, j\},$$

on which the torus $\mathbb{T}^j$ acts freely by

$$t \cdot (y_1, z_1; \dots; y_j, z_j) = (t_1 y_1, t_1^{-1} z_1; \dots; t_j y_j, t_{j-1}^{-1} t_j^{-1} z_j)$$

for all $t = (t_1, \dots, t_j)$.

The quotient manifold $B_j = (S^3)^j / \mathbb{T}^j$ is a j-fold iterated 2-sphere bundle over $B_0 = pt$ (the **Bott tower**), and admits complex line bundles η_1, ..., η_j such that $E(\eta_i) = (S^3)^j \times_{\mathbb{T}^j} \mathbb{C}$ via the action $t \cdot z = t_i^{-1} z$ for $z \in \mathbb{C}$.

Ray's basis

For any $j > 0$, the isomorphism

$$\tau(B_j) \oplus \mathbb{C}^j \cong (\overline{\eta}_1 \oplus \eta_1) \oplus \cdots \oplus (\overline{\eta_j} \oplus \eta_{j-1}\eta_j)$$

defines a stably complex structure c_j^∂ on B_j, which bounds the associated 3-disc bundle; so $[B_j] = 0$.

Proposition 1 (N.Ray, 1986) *The classifying maps $B_j \to \mathbb{C}P^\infty$ for the line bundles η_j represent a basis for $U_*(\mathbb{C}P^\infty)$, that is dual to the basis $\{u^k : k \geqslant 0\}$ of $U^*(\mathbb{C}P^\infty)$.*

We need the basis for $U_*(\Pi P_\infty)$ dual to $\{u^s,\, s : \{1, \ldots, k\} \to \mathbb{Z}_\geqslant\}$ in $U^*(\Pi P_\infty)$.

Proposition 2 *The bordism class dual to* u^s *is represented by the classifying map*

$$\eta_s = \prod \eta_{s(q)} \colon B_s = \prod_{q=1}^{k} B_q^{s(q)} \longrightarrow \mathit{\Pi}P_\infty.$$

Here $\eta_{s(q)} \colon B_q^{s(q)} \to \mathbb{C}P^\infty$ is the Ray basis element for the q-th copy of $\mathbb{C}P^\infty$ in $\mathit{\Pi}P_\infty$.

We wrote $\quad \mathit{\Phi}(M, \alpha, c_\nu) = \sum_s [G_s(M)]\, u^s.$

Theorem 3 *The manifold* $G_s(M)$ *is the total space of the bundle* $G_s(M) \to B_s$ *with* **fibre** M *determined by the pull-back diagram*

$$\begin{array}{ccc} G_s(M) & \longrightarrow & W_\infty \\ \big\downarrow & & \big\downarrow \\ B_s & \xrightarrow{\eta_s} & \mathit{\Pi}P_\infty \end{array}$$

Here $W_\infty = \mathit{\Pi}S_\infty \times_{\mathbb{T}^k} M$*.*

Rigid equivariant genera

Let $L\colon \Omega_U \to R$ be a Hirzebruch genus taking values in a ring R. Then the universal genus $\Phi\colon \Omega_{2n}^{U,\mathbb{T}^k} \to \Omega_U[[u_1,\dots,u_k]]$ gives rise to an equivariant genus

$$\Phi_L\colon \Omega_{2n}^{U,\mathbb{T}^k} \longrightarrow R[[u_1,\dots,u_k]],$$

$$\Phi_L(M,\alpha,c_\nu) = L[M] + \sum_{|s|>0} L[G_s(M)]\, u^s.$$

The genus Φ_L is **rigid** if
$\Phi_L(M,\alpha,c_\nu) = L[M] \in R \subset R[[u_1,\dots,u_k]]$
is a constant.

Let $A \subset \Omega_{2n}^{U,\mathbb{T}^k}$ be a subgroup. Then we say that the genus Φ_L is A-**rigid** if $\Phi_L(M,\alpha,c_\nu)$ is a constant for every $[M,\alpha,c_\nu] \in A$.

Multiplicativity with respect to fibre

Definition. A genus $L\colon \Omega_U \to R$ is said to be **multiplicative** with respect to M if

$$L[E] = (L[M])(L[B])$$

for every bundle $M \to E \xrightarrow{p} B$ of stably complex manifolds with a compact connected Lie group as structure group. We may write L as L^M to emphasise M.

Since the Ray basis $\{B_s\}$ consists of manifolds cobordant to zero, we get from Theorem 3:

Corollary. Let M be a normally complex $\mathbb{T}^k$-manifold. If the genus L is a multiplicative with respect to M then the equivariant genus $\Phi_L(M, \alpha, c_\nu)$ is rigid, that is, $\Phi_L(M, \alpha, c_\nu) = L[M]$.

Applications

Let I_U denote the ideal in Ω_U of differences $[E]-[N][B]$ for *all* fibre bundles $N \to E \to B$ of stably complex manifolds with a compact connected Lie group as structure group.

Consider the canonical projection

$$L^U : \Omega_U \longrightarrow R_U = \Omega_U / I_U,$$

which factors through L^M for any fixed M.

Definition. We refer to L^U as **the universal complex fibre multiplicative** genus.

Corollary. Let (M, α, c_ν) be a normally complex $\mathbb{T}^k$-manifold. Then

$$\Phi_{L^U}(M, \alpha, c_\nu) \ = \ L^U[M] = \text{const.}$$

It is known that $R_U = \mathbb{Z}[a, b]$, $\deg a = -2$, $\deg b = -4$ and L^U is the two-parameter Todd genus $\mathbf{T}_{z_1,z_2}$, where $z_1 + z_2 = a$, $z_1 z_2 = b$. The (classical, ungraded) Hirzebruch χ_y-genus is $\mathbf{T}_{-y,1}$.

Let I_{SU} denote the ideal in Ω_U defined in the same way as above, but with the additional assumption that N is an SU-manifold.

We have the canonical projection

$$L^{SU} : \Omega_U \longrightarrow R_{SU} = \Omega_U / I_{SU}.$$

Definition. We refer to L_{SU} as **the universal complex SU-fibre multiplicative** genus.

Corollary. Let (M, α, c_ν) be a normally complex $\mathbb{T}^k$-manifold with $c_1(M) = 0$ (an SU-manifold). Then

$$\Phi_{L^{SU}}(M, \alpha, c_\nu) \ = \ L^{SU}[M] = \text{const.}$$

It was shown by **Höhn (1991)** that $R_{SU} \otimes \mathbb{Q} = \mathbb{Q}[a_1, a_2, a_3, a_4]$, $\deg a_k = -2k$, and L^{SU} is **the Krichever genus**.

The sign of an isolated fixed point

Let $x \in M$ be an isolated fixed point of the $\mathbb{T}^k$-action α on a tangentially stably complex $\mathbb{T}^k$-manifold (M, α, c_τ). The representation

$$\rho_x \colon \mathbb{T}^k \longrightarrow GL(l, \mathbb{C})$$

associated with c_τ, decomposes the fibre $\xi_x \cong \mathbb{C}^l$ as $\mathbb{C}^n \oplus \mathbb{C}^{l-n}$, where ρ_x acts without trivial summands on $\mathbb{C}^n$, and trivially on $\mathbb{C}^{l-n}$. Moreover, the isomorphism $c_{\tau,x}$ of (2) induces an orientation of the tangent space $\tau_x(M)$.

Definition. The **sign** $\sigma(x)$ is $+1$ if the map

$$\tau_x(M) \xrightarrow{I \oplus 0} \tau_x(M) \oplus \mathbb{R}^{2(l-n)} \xrightarrow{c_{\tau,x}} \xi_x \cong \mathbb{C}^n \oplus \mathbb{C}^{l-n} \xrightarrow{\pi} \mathbb{C}^n,$$

preserves the orientation, and -1 otherwise, where π is projection onto the first summand.

The weights of the tangential representation at an isolated fixed point

The representation $\rho_x\colon \mathbb{T}^k \to GL(n,\mathbb{C})$ decomposes as

$$\rho_{x,1} \oplus \cdots \oplus \rho_{x,n},$$

where $\rho_{x,j}$ is a non-trivial one-dimensional representation of $\mathbb{T}^k$ given by

$$\rho_{x,j}(e^{2\pi\sqrt{-1}\varphi_1}, \ldots, e^{2\pi\sqrt{-1}\varphi_k})v = e^{2\pi\sqrt{-1}\langle \mathbf{w}_j(x),\, \phi\rangle}v$$

where $\phi = (\varphi_1, \ldots, \varphi_k) \in \mathbb{R}^k$,
$\mathbf{w}_j(x) = (w_{1j}(x), \ldots, w_{kj}(x)) \in \mathbb{Z}^k$ and

$$\langle \mathbf{w}_j(x),\, \phi\rangle = \sum_{q=1}^{k} w_{qj}(x)\varphi_q.$$

To each isolated fixed point x, we may therefore assign a sequence

$$\{\mathbf{w}_1(x), \ldots, \mathbf{w}_n(x)\}$$

of **weight vectors**.

Localization formula for the universal equivariant genus

Let $F(u,v)$ be the formal group for complex cobordism. The corresponding power system $\{[w](u) \in \Omega^*[[u]] : w \in \mathbb{Z}\}$ is uniquely defined by $[0](u) = 0$, and $[w](u) = F(u, [w-1](u))$ for $w \in \mathbb{Z}$.
Note that $[w](u) = wu$ modulo (u^2).

Let $\mathbf{w} = (w_1, \ldots, w_k) \in \mathbb{Z}^k$ and $\mathbf{u} = (u_1, \ldots, u_k)$. Define inductively $[\mathbf{w}](\mathbf{u}) = [w](u)$ for $k = 1$;

$[\mathbf{w}](\mathbf{u}) = F_{q=1}^{k}[w_q](u_q) = F\Big(F_{q=1}^{k-1}[w_q](u_q), [w_k](u_k)\Big)$

for $k \geqslant 2$.

Note that

$$[\mathbf{w}](t\mathbf{u}) = \langle \mathbf{w}, \mathbf{u} \rangle t + \varphi(\mathbf{w}, \mathbf{u}; t)t^2,$$

where $\varphi(\mathbf{w}, \mathbf{u}; t) \in \Omega_U[[u_1, \ldots, u_k, t]]$.

Theorem 4 *If the action α has a finite set X of isolated fixed points, then*

$$(4) \quad \Phi(M,\alpha,c_{\mathcal{T}}) = \sum_{x\in X}\sigma(x)\prod_{j=1}^{n}\frac{1}{[\mathbf{w}_j(x)](\mathbf{u})}$$

is equal $[M]+\mathcal{L}(\mathbf{u})$, where $\mathcal{L}(\mathbf{u})\in\Omega^*[[u_1,\dots,u_k]]$ and $\mathcal{L}(\mathbf{0})=0$.

Corollary. For $0\leqslant l\leqslant n$

$$\frac{1}{l!}\frac{d^l}{dt^l}\left(\sum_{x\in X}\sigma(x)\prod_{j=1}^{n}\frac{1}{\langle\mathbf{w}_j(x),\mathbf{u}\rangle+\varphi(\mathbf{w}_j(x),\mathbf{u};t)t}\right)\Big|_{t=0}$$

is equal $\delta_n^l[M^{2n}]$, where $\delta_n^l=\begin{cases}0, & l\neq n,\\ 1, & l=n.\end{cases}$

This is a $\mathbb{T}^k$-analogues of the famous Conner-Floid relations for the $\mathbb{Z}/p$-actions with isolated fixed points.

Thus we have for $G=\mathbb{T}^k$ an answer to **the generalized P.Smith problem**:
Let be given a G-manifold with isolated fixed points. How are the isotropy representations of G related, at distinct fixed points?

Applications

Hirzebruch genus as a **ring homomorphism** $L_h : \Omega_U \longrightarrow R$ is given by the series $t/h(t)$, where $h(t) \in R \otimes \mathbb{Q}[[t]]$. Let $L_h : \Omega_U \longrightarrow R$ classify the formal group $f(u,v)$. Then $g_f(h(t)) = t$, where

$$g_f(u) = u + \sum L_h[\mathbb{C}P^n] \frac{u^{n+1}}{n+1}$$

is the **logarithm** of f(u,v), i.e. $h(t)$ is the **exponential** of this group.

In the notation of Theorem 4 we have

Theorem 5 *Let $h(t) \in R \otimes \mathbb{Q}[[t]]$ be a series such that $h(0) = 0$ and $h'(0) = 1$. Then the expression*

$$\sum_{x \in X} \sigma(x) \prod_{j=1}^{n} \frac{1}{h\Big(\sum_{q=1}^{k} w_{qj}(x) t_q \Big)}$$

is equal $L_h[M] + \mathcal{L}_h(t_1, \ldots, t_k)$, where $\mathcal{L}_h(t_1, \ldots, t_k) \in R \otimes \mathbb{Q}[[t_1, \ldots, t_k]]$ and $\mathcal{L}_h(0, \ldots, 0) = 0$.

Corollaries

Applying the augmentation (the trivial Hirzebruch genus) $L_\varepsilon\colon \Omega_U \to \mathbb{Q}$, we obtain a **strong condition** on the set of signs $\{\sigma(x)\}$ and weight vectors $\{\mathbf{w}(x)\}$:

Corollary.

$$\sum_{x\in X} \sigma(x) \prod_{j=1}^{n} \frac{1}{\left(\sum_{q=1}^{k} w_{qj}(x)t_q\right)} \equiv 0.$$

Proof. Hirzebruch genus L_ε corresponds to $\varepsilon(t) = t$.

In the more general case we obtain:

Corollary. If the genus L_h is multiplicative with respect to M then $\mathcal{L}_h(t_1, \dots, t_k) \equiv 0$.

The universal complex fibre multiplicative genus

Consider the formal group law

$$f_{z_1,z_2}(u,v) = \frac{u+v-auv}{1-buv}$$

with $a = z_1 + z_2$, $b = z_1 z_2$ and $\deg a = -2$, $\deg b = -4$. Then

$$\begin{aligned} g_f'(u) =& \frac{1}{z_1 - z_2}\Big(\frac{z_1}{1-z_1 u} - \frac{z_2}{1-z_2 u}\Big) \\ =& 1 + \sum \mathbf{T}_{z_1,z_2}[CP^n]u^n, \end{aligned}$$

where $\mathbf{T}_{z_1,z_2}$ is the **two-parameter Todd genus**.

In this case, the exponential series is given by

$$h_{z_1,z_2}(x) = \frac{\exp\frac{z_1-z_2}{2}x - \exp\frac{z_2-z_1}{2}x}{z_1\exp\frac{z_1-z_2}{2}x - z_2\exp\frac{z_2-z_1}{2}x}.$$

Corollary. Let (M, α, c_ν) be a tangentially stably complex $\mathbb{T}^k$-manifold. Then

$$\mathbf{T}_{z_1,z_2}[M] = \sum_{x\in X}\sigma(x)\prod_{j=1}^{n}\frac{1}{h_{z_1,z_2}\Big(\sum_{q=1}^{k}\omega_{qj}(x)t_q\Big)}$$

The Krichever genus

Krichever's complex elliptic genus $\mathbf{T}^K$ is the Hirzebruch genus L_h with $h(x) = \frac{1}{\Phi(x)} \exp ax$ where

$$\Phi(x) = \Phi(x, z) = \frac{\sigma(z-x)}{\sigma(x)\sigma(z)} \exp \zeta(z)x.$$

Here $\sigma(z)$ is the classical **Weierstrass function** and $\zeta(z) = (\ln \sigma(z))'$.

The function $\Phi(x, z)$ is the solution of the **Lame equation**

$$\left(\frac{d^2}{dx^2} - 2\wp(x)\right)\Phi(x) = \wp(z)\Phi(x)$$

of the form

$$\Phi(x) = \frac{1}{x} + \Phi_{reg}(x)$$

in a neighborhood of $x = 0$, where $\Phi_{reg}(x)$ is a power series such that $\Phi_{reg}(0) = 0$. Here $\wp(z) = -\zeta(z)'$.

The formal group of the Krichever genus

Over the graded ring $\mathbf{R} = \sum_{i\geqslant 0}\mathbf{R}^{-2i}$, let

$$c_i(u) = \sum_{j\geqslant 0} c_{i,j}u^j, \quad i = 1,2,$$

where $\deg c_{1,j} = -2j$, $\deg c_{2,j} = -2j-4$ and $c_{1,0} = 1$, $c_{1,1} = 0$, $c_{2,0} = 0$.

Theorem 6 (V.M.Buchstaber, 1990) *The universal formal group $f^K(u,v)$ of the form*

$$uc_1(v) + vc_1(u) - auv - \frac{c_2(u) - c_2(v)}{uc_1(v) - vc_1(u)}u^2v^2$$

over $\mathbf{R} = \mathbb{Z}[a, c_{1,j},\ j \geqslant 2,\ c_{2,k},\ k \geqslant 1]/J$ has the exponential $h(x)$ of the form

$$\frac{\sigma(x)\sigma(z)}{\sigma(z-x)}\exp[(a-\zeta(z))x] \in \mathbb{Q}[a,\xi,\gamma,g_2][[x]],$$

where J is the ideal generated by polynomial given by associativity condition and

$$\mathbf{R}\otimes\mathbb{Q} = \mathbb{Q}[a, c_{1,2}, c_{1,3}, c_{1,4}].$$

Here $\xi = \wp(z)$, $\gamma = \wp'(z)$.

The universal complex SU-fibre multiplicative genus

Corollary. Let $(M, \alpha, c_\mathcal{T})$ be a tangentially stably complex $\mathbb{T}^k$-manifold with $c_1(M) = 0$. Then

$$\mathbf{T}^K[M] = \sum_{x \in X} \sigma(x) \prod_{j=1}^{n} \Phi\Big(\sum_{q=1}^{k} \omega_{qj}(x) t_q, z \Big).$$

Important examples:

1. The T_{z_1,z_2}-genus.

The formal group law

$$f_{z_1,z_2}(u,v) = \frac{u + v - auv}{1 - buv}$$

is given according to Theorem 6 by

$$c_1(u) = 1 + bu^2$$
$$c_2(u) = bu(a - bu).$$

2. The elliptic genus $\mathbf{T}_{ell}$.
The formal group law

$$f_{ell}(u,v) = \frac{u\sqrt{R(v)} + v\sqrt{R(u)}}{1-\varepsilon u^2 v^2}$$

with $R(u) = 1 - 2\delta u^2 + \varepsilon u^4$, is given according to Theorem 6 by

$$c_1(u) = \sqrt{R(u)}$$
$$c_2(u) = -\varepsilon u^2$$
$$a = 0$$

Here $\deg \delta = -4$, $\deg \varepsilon = -8$. Then

$$g'_{ell}(u) = \frac{1}{\sqrt{R(u)}} = 1 + \sum \mathbf{T}_{ell}[CP^n]u^n.$$

So $\mathbf{T}_{ell}[\mathbb{C}P^{2n-1}] = 0$ and

$$\mathbf{T}_{ell}[\mathbb{C}P^{2n}] = (\sqrt{\varepsilon})^n P_n\left(\frac{\delta}{\sqrt{\varepsilon}}\right),$$

where $P_n(z)$ is the **Legendre polynomial**.

The exponential $h(x)$ of $f_{ell}(u,v)$ is the **Jacobi elliptic function** sn(x).

Combinatorial quasitoric data

I. We consider a simple n-dimensional polytope P given as a bounded intersection of m closed half-spaces in $\mathbb{R}^n$:

(5) $$P = \{x \in \mathbb{R}^n \colon \langle a_i, x\rangle + b_i \geqslant 0 \text{ for } 1 \leqslant i \leqslant m\},$$

where a_i lies in $\mathbb{R}^n$ and b_i is a real scalar.

We assume that there are no redundant inequalities in (5), that is, every hyperplane bounding a half-space in (5) intersects P at an $(n-1)$-dimensional **facet**.

It follows that there are m facets $F_1, \ldots, F_m$ in total; and we further assume that they are **finely ordered**, in the sense that $F_1 \cap \cdots \cap F_n$ defines the **initial vertex** v_1 of P.

II. We associate to a simple n-dim polytope P an integral $(n \times m)$-matrix of the form

$$\Lambda = \begin{pmatrix} 1 & 0 & \dots & 0 & \lambda_{1,n+1} & \cdots & \lambda_{1,m} \\ 0 & 1 & \dots & 0 & \lambda_{2,n+1} & \cdots & \lambda_{2,m} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 & \lambda_{n,n+1} & \dots & \lambda_{n,m} \end{pmatrix},$$

in which the column $\lambda_j = (\lambda_{1j}, \dots, \lambda_{nj})$ corresponds to the facet F_j, $j = 1, \dots, m$, and the columns λ_{j_1}, ..., λ_{j_n} corresponding to any vertex $F_{j_1} \cap \dots \cap F_{j_n}$ are required to form a basis for $\mathbb{Z}^n$.

In other words, the associated $(n \times n)$-submatrices have determinant ± 1. We partition Λ as $(I_n \mid \Lambda_\star)$, where I_n is identity $(n \times n)$-matrix.
So that $\Lambda_\star$ is $n \times (m-n)$-matrix, and refer to it as the **refined submatrix**.

Definition. The **combinatorial quasitoric data** (P, Λ) consists of an oriented combinatorial simple polytope P and integer $(n \times m)$-matrix Λ with the properties above.

The Canonical embedding of a polytope in the positive cone

We may specify P by a matrix inequality $A_P x + b_P \geqslant 0$, where A_P is the $(m \times n)$-matrix of row vectors a_i, and b_P is the column vector of scalars b_i in $\mathbb{R}^m$.

We may interpret the matrix A_P as a linear transformation $\mathbb{R}^n \to \mathbb{R}^m$. Since the points of P are specified by the constraint $A_P x + b_P \geqslant 0$, the intersection of the affine subspace $A_P(\mathbb{R}^n) + b_P$ with the positive cone $\mathbb{R}^m_{\geqslant}$ is a copy of P in $\mathbb{R}^m$. The formula $i_P(x) = A_P x + b_P$ defines an affine injection

$$i_P \colon \mathbb{R}^n \longrightarrow \mathbb{R}^m,$$

which embeds P as a submanifold with corners of the positive cone.

Moment-angle manifolds

The **moment-angle manifold** $\mathcal{Z}_P$ is defined by the pull-back diagram

$$\begin{array}{ccc} \mathcal{Z}_P & \stackrel{i_{\mathcal{Z}}}{\longrightarrow} & \mathbb{C}^m \\ \varrho_P \Big\downarrow & & \Big\downarrow \varrho \\ P & \stackrel{i_P}{\longrightarrow} & \mathbb{R}^m_{\geqslant} \end{array}$$

Here $\varrho(z_1,\ldots,z_m)$ is given by $(|z_1|^2,\ldots,|z_m|^2)$, the vertical maps are projections onto the quotients by the $\mathbb{T}^m$-actions, and $i_{\mathcal{Z}}$ is a $\mathbb{T}^m$-equivariant embedding.

The manifold $\mathcal{Z}_P$ is a **complete intersection** of $(m-n)$ real quadratic hypersurfaces in $\mathbb{C}^m$ and is therefore smooth.

There is a $\mathbb{T}^m$-equivariant decomposition

$$\tau(\mathcal{Z}_P) \oplus \nu(i_{\mathcal{Z}}) \cong \mathcal{Z}_P \times \mathbb{C}^m,$$

where $\tau(\mathcal{Z}_P)$ is the tangent bundle of $\mathcal{Z}_P$ and $\nu(i_{\mathcal{Z}})$ is the normal bundle of the embedding $i_{\mathcal{Z}}$.

Quasitoric manifolds

Matrix Λ defines a surjective homomorphism $\ell\colon \mathbb{T}^m \to \mathbb{T}^n$. The kernel of ℓ (which we denote $K(\Lambda)$) is isomorphic to $\mathbb{T}^{m-n}$.

The action of $K(\Lambda)$ on $\mathcal{Z}_P$ is free due to the condition on the minors of Λ. So its quotient $M = \mathcal{Z}_P/K(\Lambda)$ is a $2n$-dimensional smooth manifold with an action of the n-dimensional torus $\mathbb{T}^m/K(\Lambda)$. We denote this action by α.

It satisfies the **Davis–Januszkiewicz' conditions**:

(a) α is locally isomorphic to the standard coordinatewise representation of $\mathbb{T}^n$ in $\mathbb{C}^n$.
(b) there is a projection $\pi\colon M \to P$ whose fibres are orbits of α.

We refer to $M = M(P,\Lambda)$ as the **quasitoric manifold associated with the combinatorial data** (P,Λ).

Facial submanifold structure

Additional structure on quasitoric manifold M is associated to the **facial submanifolds** M_i, defined as the inverse images of the facet F_i under π, for $1 \leqslant i \leqslant m$. Every M_i has codimension 2, and its isotropy subgroup is the subcircle $\ell(\mathbf{T}_i) \subset \mathbb{T}^n$, where $\mathbf{T}_i \subset \mathbb{T}^m$ is the i-th coordinate subcircle.
The quotient map

$$\mathcal{Z}_P \times_K \mathbb{C}_i \longrightarrow M$$

defines a canonical **complex line bundle** ρ_i, whose restriction to M_i is isomorphic to the **normal bundle** ν_i of its embedding in M.

The submanifolds M_i are mutually **transverse**.

Definition. An **omniorientation** of a quasitoric manifold M consists of a choice of an **orientation** for M and for **every** facial normal bundle ν_i, $i = 1, \ldots, m$.

Omnioriented quasitoric manifold

Theorem 7 *Every pair* (P, Λ) *determines a* $2n$*-dim* **omnioriented quasitoric manifold***.*

Proof. An interior point of the quotient polytope P admits an open neighborhood U, whose inverse image under the projection π is canonically diffeomorphic to $U \times \mathbb{T}^n$ as a subspace of M.

Since $\mathbb{T}^n$ is oriented by the standard choice of basis, orientations of M correspond bijectively to orientations of P.
Since the homomorphism $\ell\colon \mathbb{T}^m \to \mathbb{T}^n$ determines a complex structure on each ρ_i, it encodes equivalent information.

Theorem 8 *Any omnioriented quasitoric manifold admits a* **canonical stably complex** $\mathbb{T}^n$**-invariant structure***.*

Sign of a fixed point for $M(P,\Lambda)$

Every $\mathbb{T}^n$-fixed point $x \in M = M(P,\Lambda)$ can be obtained as the intersection $M_{j_1} \cap \cdots \cap M_{j_n}$ of n facial submanifolds. The tangent space to M at x therefore decomposes into the sum of normal subspaces to M_{j_k} for $1 \leqslant k \leqslant n$:

$$\tau_x(M) = \nu_{j_1}|_x \oplus \ldots \oplus \nu_{j_n}|_x. \tag{6}$$

Lemma 9 *Let $x = M_{j_1} \cap \ldots \cap M_{j_n}$ be a fixed point.*
1. *We have $\sigma(x) = 1$ if in (6) the orientation of $\tau_x(M)$ determined by the orientation of M coincides with the orientation of $\nu_{j_1}|_x \oplus \ldots \oplus \nu_{j_n}|_x$ determined by the orientations of ν_{j_k} for $1 \leqslant k \leqslant n$, and $\sigma(x) = -1$ otherwise.*

2. *In terms of combinatorial data (P,Λ), we have*

$$\sigma(x) = \operatorname{sign}\Big(\det(\lambda_{j_1}, \ldots, \lambda_{j_n})\det(a_{j_1}, \ldots, a_{j_n})\Big).$$

Weights for $M(P, \Lambda)$

Let $\mathbb{T}^n$-fixed point $x \in M = M(P, \Lambda)$ be the intersection $M_{j_1} \cap \cdots \cap M_{j_n}$ of n facial submanifolds.

Denote by Λ_x the $(n \times n)$-submatrix of Λ formed by the columns $\lambda_{j_1}, \ldots, \lambda_{j_n}$ (note $\det \Lambda_x = \pm 1$).

Lemma 10 *The weight vectors* $\{\mathbf{w}_1(x), \ldots, \mathbf{w}_n(x)\}$ *of the tangential* $\mathbb{T}^n$*-representation in* $\tau_x(M)$ *are given by the column vectors* $\mu_1, \ldots, \mu_n$ *of the matrix* $\mathcal{M}_x$ *satisfying*

$$\mathcal{M}_x^t \Lambda_x = I_n.$$

In other words, $\{\mathbf{w}_1(x), \ldots, \mathbf{w}_n(x)\}$ *is the basis of* $\mathbb{R}^n$ *conjugated to* $\{\lambda_{j_1}, \ldots, \lambda_{j_n}\}$*.*

Toric manifolds

Every nonsingular projective toric variety (**toric manifold**) M is determined by the normal fan of a simple polytope $Q \subset \mathbb{R}^n$; it is **integral**, insofar as its vertices lie in the lattice $\mathbb{Z}^n$. We may assume that the origin is a distinguished vertex, that its incident facets lie in the respective coordinate hyperplanes and the remaining facets $F_{n+1}, \ldots, F_m$ are ordered.

For any such M we let P be the oriented combinatorial type of Q and the columns of Λ be the primitive integral inward pointing normal vectors to $F_1, \ldots, F_m$ respectively. So $\Lambda = A_Q^t$.

We can identify the stably complex structure associated to the combinatorial data (P, Λ) with the canonical complex structure on M.

Corollary. $\sigma(x) = 1$ for **any fixed point** of the canonical $\mathbb{T}^n$-action on toric manifold M.

Corollary.

$$\sum_{x \in X} \prod_{j=1}^{n} \frac{1}{\left(\sum_{q=1}^{n} \mu_{qj}(x) t_q \right)} \equiv 0.$$

where $\mu_1, \ldots, \mu_n$ are the column vectors of the matrix A_x^{-1}.

Corollary. Let $h(t) \in R \otimes \mathbb{Q}[[t]]$ be a series such that $h(0) = 0$ and $h'(0) = 1$. Then for any toric manifold M the expression

$$\sum_{x \in X} \prod_{j=1}^{n} \frac{1}{h\left(\sum_{q=1}^{k} \mu_{qj}(x) t_q \right)}$$

is equal $L_h[M] + \mathcal{L}_h(t_1, \ldots, t_k)$, where $\mathcal{L}_h(t_1, \ldots, t_k) \in R \otimes \mathbb{Q}[[t_1, \ldots, t_k]]$ and $\mathcal{L}_h(0, \ldots, 0) = 0$.

Applications

Theorem 11

$$\Phi(\mathbb{C}P(n)) = \frac{1}{u_1 \cdots u_n} + \sum_{i=1}^{n} \frac{1}{\bar{u}_i} \prod_{j \neq i} \frac{1}{F(u_j, \bar{u}_i)}$$

where $\bar{u} = [-1](u)$.

Set $u_i = F(v_{n+1}, \bar{v}_i)$ then $F(u_j, \bar{u}_i) = F(v_i, \bar{v}_j)$.

Corollary.

$$\Phi_v(\mathbb{C}P(n)) = \sum_{i=1}^{n+1} \prod_{j \neq i} \frac{1}{F(v_i, \bar{v}_j)} \in \Omega_U[[v_1, \ldots, v_{n+1}]].$$

Example.

$$\Phi_v(\mathbb{C}P(1)) = \frac{1}{F(v_1, \bar{v}_2)} + \frac{1}{F(v_2, \bar{v}_1)} = [\mathbb{C}P^1] + \cdots$$

Corollary. Let $h(t) \in R\otimes\mathbb{Q}[[t]]$ be a series such that $h(0) = 0$ and $h'(0) = 1$. Then the Hirzebuch genus $L_h = L^{\mathbb{C}P^n}$ is multiplicative with respect to $\mathbb{C}P^n$ if and only if

$$L_h[\mathbb{C}P^n] = \sum_{i=1}^{n+1} \prod_{j\neq i} \frac{1}{h(t_i - t_j)} = \text{const.}$$

This corollary is a classical result (see Hirzebruch's book, 1992, p. 52). Theorem 11 provides a form of it in the case of universal genus, and therefore for arbitrary genus.

In particular, in the expression $\sum_{i=1}^{n+1} \prod_{j\neq i} \frac{1}{h(t_i-t_j)}$ all the singularities cancel;
so $\sum_{i=1}^{n+1} \prod_{j\neq i} \frac{1}{h(t_i-t_j)}$ is a power series in $(n+1)$ variables, whose constant term is $L_h[\mathbb{C}P^n]$.

The two-parameter Todd genus T_{z_1,z_2}

The formal group law is

$$f_{z_1,z_2}(u,v) = \frac{u+v-auv}{1-buv}$$

where $a = z_1 + z_2,\ b = z_1 z_2$.

We have:

$$\bar{u} = \frac{u}{au-1} \quad \text{and} \quad f_{z_1,z_2}(u,\bar{v}) = \frac{u-v}{1-(a-bu)v}$$

Thus,

$$f^{-1}_{z_1,z_2}(u,\bar{v}) = \frac{1-av+bv^2+bv(u-v)}{u-v} =$$

$$= bv + \frac{(1-z_1v)(1-z_2v)}{u-v}.$$

$\mathbf{T}_{z_1,z_2}$ is the universal complex fibre multiplicative genus, and $\mathbf{T}_{z_1,z_2}[\mathbb{C}P^n] = \frac{z_1^{n+1}-z_2^{n+1}}{z_1-z_2}$.

Therefore we obtain

$$\frac{z_1^{n+1}-z_2^{n+1}}{z_1-z_2} = \sum_{i=1}^{n+1} \prod_{j\neq i} \left(bv_j + \frac{(1-z_1v_j)(1-z_2v_j)}{v_i-v_j} \right).$$

Main examples

I. The genus $\mathbf{T}_{a,0}$ gives the **Todd genus** $\mathbf{Td}$. Thus, we obtain a **partition of** a^n for every $n > 0$

$$a^n = \sum_{i=1}^{n+1} \prod_{j \neq i} \frac{1 - a v_j}{v_i - v_j}.$$

In the case $a \neq 0$ set $s_j = 1 - a v_j$.
Then we obtain a **partition of unity**

$$1 = \sum_{i=1}^{n+1} \prod_{j \neq i} \frac{s_j}{s_i - s_j}.$$

Similar formulae can be retrieved from the combinatorial data of other quasitoric manifolds.
For **any toric manifold** M^{2n} we have $\mathbf{Td}(M) = a^n$, so we again obtain a partition of a^n or of unity.

II. The genus $\mathbf{T}_{z,-z}$ gives the **Signature**.

In this case we obtain the partition

$$\left(\frac{1-(-1)^{n+1}}{2}\right) z^n = \sum_{i=1}^{n+1} \prod_{j \neq i} \frac{1 - z^2 v_i v_j}{v_i - v_j}$$

and therefore

$$(-b)^n = \sum_{i=1}^{2n+1} \prod_{j \neq i} \frac{1 + b v_i v_j}{v_i - v_j}.$$

III. The genus $\mathbf{T}_{z,z}$ gives the **top Chern number**, i.e. $\mathbf{T}_{z,z}[M^{2n}] = \Big(c_n(\tau)(M^{2n})\Big) z^n$.

In this case we obtain the partition

$$(n+1) z^n = \sum_{i=1}^{n+1} \prod_{j \neq i} \frac{1 - 2z v_j + z^2 v_i v_j}{v_i - v_j}.$$