On the algebraic topology of weighted projective spaces

Nigel Ray, University of Manchester: 31.10.09

Proposal for talk

From the viewpoint of algebraic topology, a *quasitoric orbifold* is a singular space that is constructed out of a simple polytope and a related integral matrix, and admits a well-behaved torus action. *Weighted projective spaces* $\mathbb{P}(\chi)$ provide a particularly illuminating class of examples (where χ denotes a weight vector of natural numbers), but their topological literature is remarkably sparse. Our aim would be to introduce an audience of geometers to some of the more fundamental topological properties of $\mathbb{P}(\chi)$, based on recent and ongoing work with Tony Bahri and Matthias Franz. We would expect to describe their integral cohomology rings, both ordinary and equivariant, in terms of piecewise polynomials, and to sketch relationships between $\mathbb{P}(\chi)$ and the Borel construction, homotopy colimits, and iterated Thom complexes.

Research Report

Until the appearance of Davis and Januszkiewicz's seminal paper [5] in 1991, the subject of toric geometry lay mainly within the realms of algebraic, combinatorial, and symplectic geometry. Since 2000, however, their work has led to a rapidly increasing number of publications concerning the algebraic topology of toric objects. These have demonstrated that many properties depend less on the geometric superstructure than was originally imagined, and therefore lie within the realms of an alternative version of the subject, namely *toric topology*. A detailed discussion of this viewpoint can be found in [4], which views the four contributing disciplines as the vertices of a *Toric Tetrahedron*.

Davis and Januszkiewicz introduced a class of well-behaved actions a of the n–dimensional torus T^n on smooth $2n$–dimensional manifolds M. These are more immediately suited to topological analysis than the examples of toric geometry, and the pairs (M, a) are now known as *quasitoric manifolds*. For any such M, their construction requires a simple n–dimensional polytope P^n with m facets, and a *dicharacteristic homomorphism* $\lambda\colon \mathbb{Z}^m \to \mathbb{Z}^n$; this must obey a certain *condition* $(*)$, which describes its behaviour on subgroups $\mathbb{Z}^k < \mathbb{Z}^m$ corresponding to sets of k intersecting facets, for $1 \leq k \leq n$. Then M is obtained as an identification space $(T^n \times P^n)/\sim$, in which the equivalence relation is determined by λ. The torus action a is induced by multiplication on the first coordinate, and the projection $M \to P^n$ encodes its isotropy subgroups.

The construction may be rephrased in terms of the $(m+n)$-dimensional *moment-angle complex* $\mathcal{Z}_P$, defined as an identification space $(T^m \times P^n)/\simeq$, where $\simeq$ uses only the relations amongst the faces of P. The resulting action by T^m also has quotient P, but λ determines a freely acting $(m-n)$–dimensional subtorus $T(\lambda)$; then M is given by $\mathcal{Z}_P/T(\lambda)$, and the quotient action of T^m/T^{m-n} yields a.

Some simple polytopes are *unsupportive*, in the sense that no λ can exist satisfying condition $(*)$; examples include the duals of those cyclic polytopes

C_k^n [8] for which $k \geq 2^n$. But we may immediately describe a weaker *condition* $(**)$, such that appropriate functions $\lambda\colon \mathbb{Z}^m \to \mathbb{Z}^n$ exist for any P. The price we pay is that the corresponding identification spaces $Q(\lambda)$ have singularities, although they are never more complicated locally than the quotient of $\mathbb{C}^n$ by the standard action of some finite group. A significant torus action a survives, and the resulting pairs $(Q(\lambda), a)$ are the *quasitoric orbifolds* of toric topology.

For any *weight vector* $(\chi_0, \ldots, \chi_n)$ of natural numbers, the *weighted projective spaces* $\mathbb{P}(\chi)$ provide an important family of examples that illustrate several phenomena associated with the general case. Moreover, they arise quite naturally in other areas of geometry and theoretical physics, including mirror symmetry. From the perspective of toric topology, the underlying simple polytope is the n-simplex Δ^n (for which $m = n+1$), and λ is determined by the weights. In this case, the moment angle complex is the sphere $S^{2n+1} \subset \mathbb{C}^{n+1}$, and $T(\lambda) < T^{n+1}$ is the weighted circle of points $(t^{\chi_0}, \ldots, t^{\chi_n})$, where $|t| = 1$. So $\mathbb{P}(\chi)$ is the orbit space $S^{2n+1}/T\langle\chi\rangle$; yet it can be surprisingly difficult to make the quotient action by the n-torus $T^{n+1}/T\langle\chi\rangle$ explicit. Observe that $T\langle\chi\rangle$ only acts freely when all the weights are 1, in which case $\mathbb{P}(\chi)$ is $\mathbb{C}P^n$.

Algebraic and symplectic geometers currently appear to focus on invariants of $\mathbb{P}(\chi)$ that are defined specifically for orbifolds, such as *orbifold cohomology*. Algebraic topologists, on the other hand, have long been successful in evaluating homotopy invariants on spaces such as CW-complexes, whose singularities may be much more complicated than those of orbifolds. So it was natural that Kawasaki [6] should initiate topological activity by computing the integral cohomology ring $H^*(\mathbb{P}(\chi); \mathbb{Z})$. His results are tantalising; these rings display considerable multiplicative complexity, but are zero in odd dimensions and free of additive torsion.

After reworking Kawasaki's calculations [1], Al Amrani computed the complex K-theory ring $K^*(\mathbb{P}(\chi))$ by means of the Chern character [2]. Subsequently, Nishimura and Yoshimura [7] determined the additive structure of the real K-theory ring $KO^*(\mathbb{P}(\chi))$. Surprisingly, no further developments appear to have taken place until 2008 — in particular, topologists did not take advantage of the toric structure, and therefore were unable to bring the apparatus of modern equivariant homotopy theory to bear. This situation has been rectified in [3], where we compute generators and relations for the equivariant integral cohomology ring $H_T^*(\mathbb{P}(\chi); \mathbb{Z})$ with respect to the action of $T = T^{n+1}/T\langle\chi\rangle$. The answer is expressed as a ring of piecewise polynomials on the cones of the fan associated to χ, and is highly dependent on the number theoretic relationships amongst the weights. This language also illuminates Kawasaki's original multiplicative structure.

Our current programme is to apply additional techniques of classical homotopy theory to study the real and complex K-theory and complex cobordism of $\mathbb{P}(\chi)$, both standard and equivariant. In particular, we interpret $\mathbb{P}(\chi)$ as a homotopy colimit, to be compared and contrasted with certain of Kawasaki's *weighted lens spaces*, from which $\mathbb{P}(\chi)$ is obtained by quotienting the action of a free circle.

We have made most progress by placing number theoretic restrictions on the weights. For example, χ may be made *p-specific* by extracting the powers of a single prime p from the weights χ_k, which ensures that they may be presented as $(1, 1, p^{k_1}, \ldots, p^{k_{n-1}})$ for some increasing sequence (k_j) of non-negative integers. We prove that $\mathbb{P}(\chi)$ is then an *iterated Thom complex*, and that Kawasaki's calculations may be formulated in terms of iterated Thom isomorphisms; extensions to complex cobordism (and even $BP^*(-)$) follow immediately, using the p-series of the appropriate formal group law. Our approach is motivated by the localisation and completion processes that are so influential in modern homotopy theory. Equivariant calculations may also be pursued from this viewpoint.

So it is extremely important to understand the procedures by which p-specific weights may be recombined, as p ranges over the primes that appear in the weights. This problem has both algebraic and geometric aspects, and is one of the most intriguing components of our current research.

References

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