Chapter 2
Open and Closed Sets

Lecture 5

Given any metric space \((X,d)\), it is extremely important to generalise the notion of an open ball. So throughout this section, let \(U \subseteq X\) be an arbitrary subset.

**Definition 2.1.** An **interior point** \(u \in U\) is one for which there exists \(\epsilon > 0\) such that \(B_\epsilon(u) \subseteq U\); the **interior** of \(U\) is the subset \(U^o \subseteq U\) of all interior points. If \(U^o = U\), then \(U\) is **open** in \(X\).

So \(U\) is open precisely when every \(u \in U\) admits a \(B_\epsilon(u) \subseteq U\); in general, the choice of \(\epsilon\) depends on \(u\).

**Proposition 2.2.** Every open ball \(B_r(x)\) is open in \(X\).

**Proof.** Choose \(u \in B_r(x)\), let \(s = d(x,u) < r\), and consider \(y \in B_{r-s}(u)\); then
\[d(x,y) \leq d(x,u) + d(u,y) < s + (r-s) = r\]
by the triangle inequality, so \(y \in B_r(x)\). Hence \(B_{r-s}(u) \subseteq B_r(x)\), as required. \(\Box\)

Of course, Proposition 2.2 provides a host of examples; but there are others!

**Examples 2.3.**

1. Any union \(B_1 \cup B_2\) of two open balls is open; for \(u \in B_1\) implies that \(B_\epsilon(u) \subseteq B_1 \subseteq B_1 \cup B_2\), and similarly if \(u \in B_2\).

2. The complement \(U := X \setminus B_r(x)\) of a closed ball is given by \(\{y : d(x,y) > r\}\); so for any \(y \in U\) with \(d(x,y) = t\), it follows by an argument analogous to that of Proposition 2.2 that \(B_{t-r}(y) \subseteq U\). Hence \(U\) is open.

3. If \((X,d)\) is discrete, any subset \(U \subseteq X\) is open because \(B_{1/2}(u) = \{u\} \subseteq U\) by (1.10).
A particular case of Example 2.3.1 is provided by the union of two open intervals \((a, b) \cup (c, d)\) in the Euclidean line, where \(a < b < c < d\). It follows from Example 2.3.2 that \(\mathbb{R} \setminus [a, b] = (-\infty, a) \cup (b, +\infty)\) is also open.

On the other hand, many sets are not open in \(X\).

**Examples 2.4.**

1. The closed ball \(B_1(0)\) is not open in the Euclidean plane \(\mathbb{R}^2\). For consider \(u \in B_1(0)\) such that \(d(0, u) = 1\); then any \(B_\epsilon(u)\) contains points \(y\) for which \(d(x, y) = 1 + \epsilon/2\). Hence \(B_\epsilon(u) \not\subset B_1(0)\), as required. So Proposition 2.2 confirms that \(B_1(0)^\circ = B_1(0)\), by removing all \(u\) with \(|u| = 1\).

2. The subset \(\mathbb{Q}\) of the Euclidean line contains no open balls at all, and is therefore not open. For let \(q \in \mathbb{Q}\); then any open interval \((q - \epsilon, q + \epsilon)\) contains the irrationals \(q + \sqrt{2}/n\) for every integer \(n > \sqrt{2}/\epsilon\). So \(\mathbb{Q}^\circ = \emptyset\).

3. Similarly, the subset \(\mathbb{R}^x := \{(x, 0) : x \in \mathbb{R}\}\) of the Euclidean plane contains no open balls, because it contains no points \((x_1, x_2)\) with \(|x_2| > 0\). So \(\mathbb{R}_x^x = \emptyset\).

These examples confirm the importance of the metric for the openness of a set. By Examples 2.3.2 and 2.4.3, both \(\mathbb{Q} \subset \mathbb{R}\) and \(\mathbb{R}^x \subset \mathbb{R}^2\) are open with the discrete metric, but not with the Euclidean metric.

**Theorem 2.5.** Given any two subsets \(U, V \subseteq X\), the following hold:

1. \(U \subseteq V\) implies that \(U^\circ \subseteq V^\circ\)

2. \((U^\circ)^\circ = U^\circ\)

3. \(U^\circ\) is open in \(X\)

4. \(U^\circ\) is the largest subset of \(U\) that is open in \(X\).

**Proof.**

1. If \(u \in U^\circ\), then there exists \(B_\epsilon(u) \subseteq U \subseteq V\); so \(u \in V^\circ\).

2. By definition, \((U^\circ)^\circ \subseteq U^\circ\). On the other hand, if \(u \in U^\circ\), then there exists \(B_\epsilon(u) \subseteq U\); thus \(B_\epsilon(u)^\circ \subseteq U^\circ\) by 1, and \(B_\epsilon(u) \subseteq U^\circ\) by Proposition 2.2. Thus \(u \in (U^\circ)^\circ\), so \(U^\circ \subseteq (U^\circ)^\circ\) as required.

3. This follows immediately from 2.

4. Suppose that \(U^\circ \subseteq W \subseteq U\), where \(W\) is open; then \(W = W^\circ \subseteq U^\circ\) by 1. So \(W \subseteq U^\circ\), and \(U^\circ = W\) as required. \(\square\)
Open sets satisfy certain criteria that ensure they define a topology on $X$. Topology is a more advanced form of geometry than metric spaces, but the relevant properties are crucial to both theories.

**Theorem 2.6.** The sets $X$ and $\emptyset$ are open in $X$; so are an arbitrary union $U = \bigcup_{i \in I} U_i$ of open sets $U_i$, and a finite intersection $U' = U'_1 \cap \cdots \cap U'_m$ of open sets $U'_j$.

**Proof.** The openness of $X$ is immediate. For $\emptyset$, consider the requirement that $u \in \emptyset$ implies $B_\epsilon(u) \subseteq \emptyset$; since its left- and right-hand sides are both false, the implication is true by virtue of the truth table.

For $U$, observe that any $u \in U$ must lie in some $U_i$; thus $B_\epsilon(u) \subseteq U_i \subseteq U$, and $U$ is open.

For $U'$, suppose that $u \in U'$. Then $u \in U'_j$ for each $j$, and there exists $\epsilon(j)$ such that $B_\epsilon(j)(u) \subseteq U'_j$ for $1 \leq j \leq m$. Hence $B_\epsilon(u) \subseteq U'_j$ for every $j$, so long as $\epsilon \leq \min(\epsilon(j) : 1 \leq j \leq m)$. Thus $B_\epsilon(u) \subseteq U'$, and $U'$ is open.

Note that the openness of $U$ generalises Examples 2.3.1. In particular, infinite unions of open intervals such as

$$U = \bigcup_{i=1}^\infty \left(1/(2i + 1), 1/2i\right) \subset \mathbb{R}$$

may be difficult to visualise, but nevertheless are open subsets of the Euclidean line. On the other hand, infinite intersections of open sets may not be open. For example, let $V_j$ be the open interval $(-1/j, 1/j)$ in the Euclidean line $\mathbb{R}$, for every $j = 1, 2, \ldots$. Then $\bigcap_j V_j = \{0\}$, which is not open in $\mathbb{R}$.

**Lecture 6**

Given any metric space $(X, d)$, it is also important to generalise the notion of a closed ball. Throughout this section, $U \subseteq X$ continues to denote an arbitrary subset.

**Definition 2.7.** A point $x \in X$ is a closure point of $U \subseteq X$ if $B_\epsilon(x) \cap U$ is non-empty for every $\epsilon > 0$; the closure of $U$ is the superset $\overline{U} \supseteq U$ of all closure points. If $\overline{U} = U$, then $U$ is closed in $X$.

So $U$ is closed precisely when $B_\epsilon(x) \cap U$ non-empty for every $\epsilon > 0$ implies that $x \in U$.

A simple criterion for recognising closed sets is the following.

**Proposition 2.8.** A set $V$ is closed in $X$ iff its complement $U := X \setminus V$ is open.
Proof. First suppose that $V$ is closed. Since $V = V$, no point of $U$ can have $B_\epsilon(x) \cap V$ non-empty for every $\epsilon > 0$; in other words, $B_\delta(x) \cap V = \emptyset$ for some $\delta$. Hence $B_\delta(x) \subset U$, and $U$ is open.

Now suppose that $U$ is open. Choose $u \in V$, and assume that $u \notin V$; then $u \in U$, so there exists $B_\epsilon(u) \subseteq U$. Hence $B_\epsilon(u) \cap V = \emptyset$, a contradiction. So $u \in V$, and $\overline{V} \subseteq V$. Hence $V$ is closed.

Corollary 2.9. Every closed ball $B_r(x)$ is closed in $X$.

Proof. By Example 2.3.2, $X \setminus B_r(x)$ is open in $X$. So $B_r(x)$ is closed.

Note that Proposition 2.8 also states that $U$ is open in $X$ iff its complement $X \setminus U$ is closed.

There are therefore two ways to proceed with many properties and problems concerning closed sets. The first is to use the definitions directly; and the second is to take complements, and use the properties of open sets. Both approaches will be much in evidence below.

Examples 2.10.

1. Any intersection $B_1 \cap B_2$ of two closed balls is closed; for $B_\epsilon(x) \cap (B_1 \cap B_2)$ non-empty implies that $B_\epsilon(x) \cap B_1$ and $B_\epsilon(x) \cap B_2$ are non-empty, and hence that $x \in B_1$ and $x \in B_2$. Thus $x \in B_1 \cap B_2$, which is therefore closed.

2. The complement $V := X \setminus B_r(x)$ of an open ball is given by $\{y : d(x, y) \geq r\}$; so $V$ is closed by Proposition 2.8.

3. If $(X, d)$ is discrete, any subset $V \subseteq X$ is closed, because $X \setminus V$ is open by Example 2.3.3.

A particular case of Example 2.10.1 is provided by the intersection of two closed balls $\overline{B}_1(0, 0) \cap \overline{B}_1(1, 0)$ in the Euclidean plane. This is a segment-shaped region, with the point $(1/2, 0)$ in its interior. It follows from Example 2.10.2 that the set $\{(x_1, x_2) : x_1^2 + x_2^2 \geq 1\}$ is also closed in $\mathbb{R}^2$.

On the other hand, many sets are neither open nor closed in $X$. Many more are open but not closed; so their complements are closed but not open!

Definition 2.11. A partially open ball in $X$ is a set $P_r(x) := B_r(x) \cup P$, where $P$ is a proper subset of $\{p : d(x, p) = r\}$.

A partially open ball $P_1(0)$ in $(\mathbb{R}^n, d_2)$ is an open disc, with some (but not all) unit vectors adjoined; $P_1(0)$ in $(\mathbb{R}^2, d_1)$ is an open diamond, with some (but not all) vectors satisfying $x_1 \pm x_2 = \pm 1$ adjoined.

Examples 2.12.

1. The open ball $B_1(0)$ is not closed in the Euclidean plane $\mathbb{R}^2$, because every point $x$ with $|x| = 1$ is a closure point.
2. Example 2.3.2 confirms that no point \( x \in \mathbb{R}^2 \) with \( |x| > 1 \) is a closure point of \( B_1(0) \), so \( \overline{B_1(0)} = \overline{B_1(0)} \). This explains the notation for closed balls. But beware (see Problem 20); \( \overline{B_r(x)} \neq B_r(x) \) in some metric spaces!

3. A partially open ball \( P_r(x) \subset \mathbb{R}^2 \) is neither open nor closed, with either metric \( d_1 \) or \( d_2 \); for the points \( p \in P \) are not in the interior of \( P_r(x) \), and the points \( p \notin P \) distant \( r \) from \( x \) are closure points of \( P_r(x) \).

**Theorem 2.13.** Given any two subsets \( U, V \subseteq X \), the following hold:

1. \( U \subseteq V \) implies that \( \overline{U} \subseteq \overline{V} \)
2. \( \overline{\overline{V}} = \overline{V} \)
3. \( \overline{V} \) is closed in \( X \)
4. \( \overline{V} \) is the smallest set containing \( V \) that is closed in \( X \).

**Proof.**

1. If \( B_\epsilon(x) \cap U \) is non-empty for every \( \epsilon > 0 \), so is \( B_\epsilon(x) \cap V \).

2. By definition, \( \overline{V} \subseteq \overline{V} \). Conversely, if \( x \in \overline{V} \), then \( B_{\epsilon/2}(x) \cap \overline{V} \) is non-empty for every \( \epsilon > 0 \). So there exists \( w \) for which \( d(x, w) < \epsilon/2 \) and \( w \in \overline{V} \), whence \( B_{\epsilon/2}(w) \cap V \) is non-empty; thus \( B_\epsilon(x) \cap V \) is also non-empty, by the triangle inequality. Hence \( x \in \overline{V} \), and \( \overline{\overline{V}} \subseteq \overline{V} \) as required.

3. This follows immediately from 2.

4. Suppose that \( V \subseteq W \subseteq \overline{V} \), where \( W \) is closed; then \( \overline{V} \subseteq \overline{W} = W \) by 1. So \( \overline{V} \subseteq W \), and \( W = \overline{V} \) as required. \( \square \)

Crucial properties of closed sets follow from Proposition 2.8 and Theorem 2.6.

**Theorem 2.14.** The sets \( X \) and \( \emptyset \) are closed in \( X \); so are an arbitrary intersection \( V = \bigcap_{i \in I} V_i \) of closed sets \( V_i \) and a finite union \( V' = V'_1 \cup \cdots \cup V' \) of closed sets \( V'_j \).

**Proof.** Since \( X \setminus \emptyset = X \) and \( X \setminus X = \emptyset \), both \( X \) and \( \emptyset \) are closed.

For \( V \), observe that \( X \setminus V = \bigcup_{i \in I} (X \setminus V_i) \) is an arbitrary union of open sets, and is therefore open. Similarly, \( X \setminus V' = (X \setminus V'_1) \cap \cdots \cap (X \setminus V'_m) \) is a finite intersection of open sets, and is also open. \( \square \)

Note that the closedness of \( V \) generalises Examples 2.10.1.
Lecture 7

For many applications it is useful to introduce the idea of a sequence into the theory of metric spaces, and to link the concept of limit to that of closure.

For any metric space $X = (X, d)$, a sequence in $X$ is a function $s : \mathbb{N} \to X$. It is standard practise to write $s(n)$ as $x_n$, and display the sequence as $(x_n : n \geq 1)$.

**Definition 2.15.** A sequence $(x_n)$ **converges** to the point $x \in X$ whenever

$$\forall \epsilon > 0, \exists N \in \mathbb{N} \text{ such that } n \geq N \Rightarrow d(x, x_n) < \epsilon;$$

in this situation, $x$ is known as the **limit** of $(x_n)$.

Convergence of $(x_n)$ to $x$ may be written as $\lim_{n \to \infty} x_n = x$, or $x_n \to x$ as $n \to \infty$. Of course, Definition 2.15 requires that the sequence of real numbers $d(x, x_n)$ tends to 0 in the standard sense. In terms of open balls, $x_n \to x$ whenever\(^{16}\)

$$\forall \epsilon > 0, \exists N \in \mathbb{N} \text{ such that } n \geq N \Rightarrow x_n \in B_\epsilon(x).$$

**Examples 2.17.**

1. In Euclidean $m$-space, $d_2(x_n, x) = |x_n - x|$; so Definition 2.15 coincides with the standard notion that $x_n \to x$ iff $|x_n - x| \to 0$.

2. In the discrete metric space $(X, d)$, a ball $B_\epsilon(x)$ consists only of $\{x\}$ for any $\epsilon < 1$, by (1.10); so Definition 2.15 reduces to the condition that $x_n \to x$ iff $x_n = x$ for all $n \geq$ some $N$.

3. The same considerations apply to the vertices of a graph $\Gamma$ with edge metric $e$; again, $v_n \to v$ iff $v_n$ is eventually constant at $v$.

4. In the function space $C[a,b]$ of (1.20), $d_{sup}(f_n, f) = \sup_{x \in [a,b]} |f_n(x) - f(x)|$; Definition 2.15 means that $f_n \to f$ iff $f_n(x) \to f(x)$ uniformly on $[a,b]$.

5. Thus $x^n \not\to 0$ in $C[0,1]$, because $\sup_{x \in [0,1]} |x^n| = 1 \not\to 0$. On the other hand, $x^n \to 0$ in $L_1[0,1]$, because

$$d_1(x^n, 0) = \int_0^1 |t^n| \, dt = 1/(n+1) \to 0 \text{ as } n \to \infty.$$

**Theorem 2.18.** *In any a metric space $(X, d)$, the limit of a convergent sequence is unique.*

**Proof.** Suppose that $x_n \to x$ and $x_n \to x'$, where $x \neq x'$; then $d := d(x, x') > 0$, and $\epsilon$ may be chosen as $d/2$. So $B_\epsilon(x)$ and $B_\epsilon(x')$ are disjoint, by the triangle inequality.

On the other hand, (2.16) implies that $x_n \in B_\epsilon(x)$ for $n \geq N$, and $x_n \in B_\epsilon(x')$ for $n \geq N'$. So $x_n \in B_\epsilon(x) \cap B_\epsilon(x')$ for $n \geq \max(N, N')$, a contradiction.

Hence $x = x'$. \[\square\]
The connection of limits with closure arises as follows.

**Theorem 2.19.** Suppose that $Y \subseteq X$ and $y \in X$; then $y$ lies in $\overline{Y}$ iff there exists a sequence $(y_n)$ in $Y$ such that $y_n \to y$ as $n \to \infty$.

**Proof.** Suppose that $y_n \to y$ as $n \to \infty$. Then $y_n \in B_\epsilon(y)$ for all sufficiently large $n$, and any $\epsilon > 0$. But $y_n \in Y$, so $B_\epsilon(y) \cap Y$ is non-empty and $y$ lies in $\overline{Y}$.

Conversely, let $y$ lie in $\overline{Y}$. Then $B_{1/n}(y) \cap Y$ is non-empty for any integer $n \geq 1$, and contains at least one point $y_n$. Moreover, any $\epsilon > 0$ admits an $n$ for which $1/n < \epsilon$; so $y_n \in B_\epsilon(y)$. Hence $y_n \to y$ as $n \to \infty$.

**Examples 2.20.**

1. Let $Y$ be the subset $\{1/n : n = 1, 2, \ldots\}$ of the Euclidean line; so $0 \notin Y$. But $(y_n = 1/n)$ lies in $Y$, and has limit 0; so $0 \notin \overline{Y}$. Moreover, any convergent sequence in $Y$ must tend to some $1/m$, or 0. So $\overline{Y} = Y \cup \{0\}$.

2. Let $Y \subseteq V$ be any subset of the vertices of a graph $\Gamma$, with edge metric $e$, and let $(v_n)$ be a sequence in $Y$. Since $v_n \to v$ only if $v_n$ is eventually constant, any such $v$ must also lie in $Y$. Thus $Y$ is closed.

A more substantial example involves the function spaces of (1.20).

**Example 2.21.** Let $Y$ be the subset $\mathcal{P}[0,1] \subseteq C[0,1]$, and consider the sequence

$$f_n(x) = 1 + x/2 + x^2/4 + \cdots + x^n/2^n$$

of polynomials $f_n$ in $Y$. Let $f(x) = (1 - x/2)^{-1}$ in $C[0,1]$; then the Binomial Theorem implies that $f(x) - f_n(x) = \sum_{k \geq n+1} x^k/2^k \leq 1/2^n$ for $x \in [0,1]$. So

$$d_{\text{sup}}(f_n, f) = \sup_{x \in [0,1]} |f_n(x) - f(x)| = 1/2^n \to 0 \text{ as } n \to \infty,$$

and $f_n \to f$ in $C[0,1]$. But $f$ is not a polynomial, so $\mathcal{P}[0,1]$ is not closed.

As in real analysis, sequences present themselves in several alternative forms.

**Definition 2.22.** In any metric space $(X, d)$, a [Cauchy sequence] $(x_n)$ satisfies

$$\forall \epsilon > 0, \exists N \in \mathbb{N} \text{ such that } m, n \geq N \Rightarrow d(x_m, x_n) < \epsilon.$$

**Proposition 2.23.** If $x_n \to x$ in $(X, d)$, then $(x_n)$ is a Cauchy sequence.

**Proof.** Given any $\epsilon > 0$, suppose that $d(x_n, x) < \epsilon/2$ for all $n \geq N$. Then

$$d(x_m, x_n) \leq d(x_m, x) + d(x, x_n) < \epsilon/2 + \epsilon/2 = \epsilon$$

for $m, n \geq N$, by the triangle inequality. So $(x_n)$ is Cauchy.
The converse is false. For example, the sequence 1, 1.4, 1.41, 1.414, ... lies in the subset \( \mathbb{Q} \) of the Euclidean line, but converges to \( \sqrt{2} \in \mathbb{R} \). It is therefore Cauchy in \( \mathbb{Q} \), but not convergent in \( \mathbb{Q} \). This example also proves that \( \mathbb{Q} \) is not closed in \( \mathbb{R} \), by Theorem 2.19; on the other hand \( \overline{\mathbb{Q}} = \mathbb{R} \), essentially because every real number has an infinite decimal expansion.

**Definition 2.24.** A subset \( Y \) is dense in \((X,d)\) whenever \( \overline{Y} = X \).

Thus \( \mathbb{Q} \) is dense in \( \mathbb{R} \).

### Lecture 8

**Definition 2.25.** A subset \( A \) of a metric space \((X,d)\) is bounded whenever there exists \( x_0 \in X \) and \( M \in \mathbb{R} \) such that \( d(x,x_0) \leq M \) for every \( x \in A \). A function \( f: S \to X \) is bounded whenever its image \( f(S) \subset X \) is a bounded, for any set \( S \).

So \( A \) is bounded whenever \( A \subseteq \overline{B}_M(x_0) \).

If \( A \) satisfies Definition 2.25, it is necessary to check that the same criterion holds for any other point \( x_1 \in X \). This is true because

\[
d(x,x_1) \leq d(x,x_0) + d(x_0,x_1) \leq M + d(x_0,x_1) = M'
\]

by the triangle inequality. Any subset of a bounded subset is also automatically bounded.

If \( x \) and \( y \) lie in a bounded set \( A \), then \( d(x,y) \leq d(x,x_0) + d(x_0,y) \leq 2M \), so \( d(x,y) \) is bounded in \( \mathbb{R} \). The following definition is therefore meaningful.

**Definition 2.26.** The diameter \( \text{diam}(A) \) of a bounded non-empty subset \( A \subseteq X \) is the real number

\[
\sup\{d(x,y) : x,y \in A\}.
\]

It is sometimes convenient to interpret the diameter of the empty set as 0.

**Examples 2.27.**

1. The open ball \( B_r(0) \) in Euclidean \( n \)-space is bounded, because \( d(x,0) < r \) for any \( x \in B_r(0) \); so its diameter is \( \leq 2r \). On the other hand, points such as \( x = (r-\epsilon,0,\ldots,0) \) and \( -x = (\epsilon-r,0,\ldots,0) \) satisfy \( d(x,-x) = 2(r-\epsilon) \) for any \( \epsilon > 0 \); so its diameter is \( \geq 2(r-\epsilon) \). Hence \( \text{diam}(B_r(0)) = 2r \). It is even easier to deduce that \( \text{diam}(\overline{B}_r(0)) = 2r \).

2. Let \( e: \mathbb{R} \to \mathbb{R}^2 \) denote the function \( e(x) = (r \cos x,r \sin x) \) into the Euclidean plane; its image \( e(\mathbb{R}) \) is a circle of radius \( r \), which is a subset of \( \overline{B}_r(0) \), and therefore bounded. So \( e \) is a bounded function.
Calculations such as those of Example 2.27.1 suggest the need to codify the properties of sets such as the Euclidean \((n-1)\)-sphere \(\{x : |x| = r \}\) of radius \(r\), which is the difference \(B_r(0) \setminus B_r(0)\) between the closed and open balls. So let \(A \subseteq X\) be any subset of \((X, d)\).

**Definition 2.28.** A **boundary point** \(x \in X\) of \(A\) is one for which every open ball \(B_\epsilon(x)\) meets both \(A\) and \(X \setminus A\); the **boundary** \(\partial A\) of \(A\) is the set of all such boundary points.

**Proposition 2.29.** The boundary of any subset \(A\) is the set \(\overline{A} \setminus A^\circ\).

**Proof.** Suppose that \(x \in \partial A\). Then every \(B_\epsilon(x)\) meets \(A\), so \(x \in \overline{A}\); but \(B_\epsilon(x)\) meets \(X \setminus A\), so \(x \notin A^\circ\). Hence \(\partial A \subseteq \overline{A} \setminus A^\circ\).

Conversely, suppose that \(x \in \overline{A} \setminus A^\circ\). Then \(x \in \overline{A}\), so that every \(B_\epsilon(x)\) meets \(A\); and \(x \notin A^\circ\), so \(B_\epsilon(x)\) meets \(X \setminus A\). Thus \(B_\epsilon(x)\) meets both \(A\) and \(X \setminus A\), and \(x \in \partial A\). Hence \(\overline{A} \setminus A^\circ \subseteq \partial A\), as required. \(\Box\)

If \(A\) fails to be closed, then it has closure points not in \(A\), so \(\partial A \not\subseteq A\). If \(A^\circ\) is empty, then \(\partial A = \overline{A}\).

**Examples 2.30.**

1. The closed ball \(\overline{B}_r(0)\) in Euclidean \(n\)-space is closed by Corollary 2.9, and its interior is the open ball \(B_r(0)\); thus \(\partial \overline{B}_r(0)\) is the sphere \(\{x : |x| = r \}\).

The open ball \(B_r(0)\) is open by Proposition 2.2, but its closure is the closed ball \(\overline{B}_r(0)\); thus \(\partial B_r(0)\) is also the sphere \(\{x : |x| = r \}\)!

2. The subset \(Q\) of the Euclidean line is dense, but \(Q^\circ\) is empty by Examples 2.4.2. Thus \(\partial Q = \mathbb{R}\).

3. The subset \(A := \mathbb{R}^n \setminus \{0\}\) of Euclidean \(n\)-space is open, because \(\{0\}\) is closed. Moreover, \(\overline{A} = \mathbb{R}^n\), because the sequence \((1/n : n \geq 1)\) in \(A\) tends to 0 in \(\mathbb{R}^n\). Thus \(\partial A = \mathbb{R}^n \setminus A = \{0\}\).

A particular case of Example 2.30.1 is provided by the closed interval \([a, b]\) and the open interval \((a, b)\) in the Euclidean line, for any \(a < b\). Both have boundary \(\{a, b\}\).

**Theorem 2.31.** Any subset \(A\) of \((X, d)\) satisfies

1. \(A \setminus \partial A = \overline{A} \setminus \partial A = A^\circ\)
2. \(\partial A = \partial(X \setminus A)\)
3. \(\partial A\) is closed in \(X\)

**Proof.**
1. It suffices to show that
\[ A \setminus \partial A \subseteq \overline{A} \setminus \partial A \subseteq A^\circ \subseteq A \setminus \partial A, \]
as follows. If \( x \in A \setminus \partial A \), then \( x \in \overline{A} \setminus \partial A \) because \( A \subseteq \overline{A} \). If \( x \in \overline{A} \setminus \partial A \), then every \( B_\epsilon(x) \) meets \( A \), but not \( X \setminus A \); thus \( B_\epsilon(x) \subseteq A \), and \( x \in A^\circ \). If \( x \in A^\circ \), then \( B_\epsilon(x) \) meets \( A \), but not \( X \setminus A \) for small \( \epsilon \); thus \( x \in A \setminus \partial A \).

2. This is an immediate consequence of Definition 2.28, since \( X \setminus (X \setminus A) = A \).

3. If \( x \in \overline{\partial A} \), then every \( B_\epsilon(x) \) meets \( \partial A \). So there exists \( y \in B_\epsilon(x) \) such that \( B_\delta(y) \) meets both \( A \) and \( X \setminus A \), and \( B_\delta(y) \subseteq B_\epsilon(x) \). Thus \( B_\epsilon(x) \) also meets them both, whence \( x \in \partial A \). Thus \( \overline{\partial A} \subseteq \partial A \), as required. \( \square \)