Inverse Spectral Problems for Semi-simple Damped Vibrating Systems

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Abstract

Computational schemes are investigated for the solution of inverse spectral problems for $n \times n$ real systems of the form $L(\lambda) = M\lambda^2 + D\lambda + K$. Thus, admissible sets of data concerning systems of eigenvalues and eigenvectors are examined and procedures for generating associated (isospectral) families of systems are developed. The analysis includes symmetric systems, systems with mixed real/non-real spectrum, systems with positive definite coefficients, and hyperbolic systems (with real spectrum). A one-to-one correspondence between Jordan pairs and structure preserving similarities is clarified. An examination of complex symmetric matrices is included.

1 Introduction

Inverse eigenvalue problems are addressed here in the context of vibrating systems which, for our present purposes, are defined as follows:

Definition 1. A *(vibrating) system* is a triple of $n \times n$ **real** matrices $\{M, D, K\}$ for which M is nonsingular.

Many problems of physical interest also require that some or all of the coefficients M, D, K be symmetric and positive definite (or semidefinite).

In this paper, an idea introduced in [10] is extended from the restriction to systems with purely non-real spectrum, to the full range of real and non-real spectrum, but with the continuing limitation (seen as unrestrictive by many in the field) to semisimple eigenvalues, i.e.

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an eigenvalue of multiplicity $m \geq 1$ has m associated linearly independent eigenvectors. This hypothesis has the added advantage that analysis is simplified considerably. The generation of real systems, real symmetric systems, and systems with positive semidefinite or positive definite coefficients will be considered, in this order.

First, it is necessary to summarise the spectral properties admitted in this analysis. Since the systems of interest have real coefficients, the eigenvalues may be real or may appear in complex conjugate pairs. All eigenvalues are required to be semisimple for both real and non-real eigenvalues. The set of all the eigenvalues, both real and complex, is denoted by σ.

The central problem considered here is, given an admissible set of spectral data (real and complex eigenvalues and a sign characteristic), construct a family of systems consistent with this data. Another closely related problem is: given a real system (with spectral data implicitly defined), construct a family of systems consistent with *this* data. In both cases the objective is the construction of *isospectral* families of systems, i.e. each member of the family has spectrum σ and it is semisimple.

For the second formulation, in particular, it is natural to re-formulate the problem in terms of the well-known *companion matrices*, in which case all eigenvalues, at least, are preserved by similarity transformations. This, in turn, leads to the notion of *structure preserving similarities*, which have been discussed elsewhere (in [10] and [15], for example), and which are developed further in Sections 2 and 3. In Section 4 these ideas are re-examined in terms of Jordan pairs (see Theorem 3), and this leads to constructions for families of real systems in Section 5.

The study of *symmetric* systems (in which $M,\ D,\ K$ are real and symmetric) is taken up in Sections 6 and 7. A strategy is adopted in which the real eigenvectors are assigned (subject to some necessary constraints) and then the eigenvectors for *non-real* eigenvalues are determined from them. This requires the symmetric factorization of a complex symmetric matrix and is accomplished with the aid of *Takagi's factorization* (Section 7 and Appendix A). This also requires some detailed knowledge of the rank of complex symmetric matrices which is presented in Appendix C.

Hypotheses that ensure the positivity conditions of $M,\ D,\ K$ are the subject of Section 8. Systems with all eigenvalues real (quasi-hyperbolic or over-damped, for instance) are the subject of Section 9.

2 Massaging the spectrum

If the system is $n \times n$ then $2r$ real eigenvalues are admitted $(0 \leq r \leq n)$. The non-real eigenvalues in the upper half of the complex plane are determined by a complex diagonal matrix $\Lambda = U_1 + iW$ of size $(n-r) \times (n-r)$ with $W > 0$. Their complex conjugates are also eigenvalues, and make up the diagonal entries of $\bar{\Lambda}$. Then there are $2r$ real eigenvalues which are distributed between the diagonal entries of two $r \times r$ real diagonal matrices U_2 and U_3. The way in which these two matrices are formed will be discussed in what follows.

A complex (canonical) diagonal $2n \times 2n$ matrix including all the eigenvalues is now

$$J = \begin{bmatrix} \Lambda & 0 & 0 & 0 \\ 0 & U_2 & 0 & 0 \\ 0 & 0 & U_3 & 0 \\ 0 & 0 & 0 & \bar{\Lambda} \end{bmatrix} = \begin{bmatrix} U_1 + iW & 0 & 0 & 0 \\ 0 & U_2 & 0 & 0 \\ 0 & 0 & U_3 & 0 \\ 0 & 0 & 0 & U_1 - iW \end{bmatrix}. \tag{1}$$

Defining $\Omega_1^2 = U_1^2 + W^2$ it is easily seen that there is an associated (diagonal, real symmetric) vibrating system:

$$L_0(\lambda) := \lambda^2 I_n - 2\lambda \begin{bmatrix} U_1 & 0 \\ 0 & \frac{1}{2}(U_2 + U_3) \end{bmatrix} + \begin{bmatrix} \Omega_1^2 & 0 \\ 0 & U_2 U_3 \end{bmatrix}. \tag{2}$$

It is simply a direct sum of the two diagonal systems

$$\lambda^2 I_{n-r} - 2\lambda U_1 + \Omega_1^2 = (\lambda I_{n-r} - \Lambda)(\lambda I_{n-r} - \overline{\Lambda})$$

and

$$\lambda^2 I_{2r} - \lambda(U_2 + U_3) + U_2 U_3 = (\lambda I_r - U_2)(\lambda I_r - U_3)$$

with non-real and real eigenvalues, respectively.

Make the abbreviations

$$U = \begin{bmatrix} U_1 & 0 \\ 0 & \frac{1}{2}(U_2 + U_3) \end{bmatrix}, \quad \text{and} \quad \Omega^2 = \begin{bmatrix} \Omega_1^2 & 0 \\ 0 & U_2 U_3 \end{bmatrix} \tag{3}$$

so that equation (2) takes the form

$$L_0(\lambda) = \lambda^2 I - 2\lambda U + \Omega^2.$$

Now a particular linearisation of $L_0(\lambda)$ is $\lambda I_{2n} - C_0$, where C_0 is the associated *companion matrix*:

$$C_0 := \begin{bmatrix} 0 & I_n \\ -\Omega^2 & 2U \end{bmatrix}. \tag{4}$$

Our objective is to generate vibrating systems whose companion matrices are similar to C_0, and which, consequently, are isospectral.

A first step in the analysis is to show that, under a weak assumption on the distribution of the *real* eigenvalues, an explicit similarity can be formulated which transforms the companion matrix C_0 to the diagonal matrix, J, of its eigenvalues. First define a $2n \times 2n$ block matrix in terms of the blocks of J:

$$Z = \begin{bmatrix} \bar{\Lambda} & 0 & 0 & -I_{n-r} \\ 0 & -U_3 & I_r & 0 \\ 0 & -U_2 & -I_r & 0 \\ \Lambda & 0 & 0 & -I_{n-r} \end{bmatrix}. \tag{5}$$

Lemma 1 *Let real eigenvalues be prescribed in such a way that*

$$\det(U_2 - U_3) \neq 0. \tag{6}$$

Then Z (as defined above) is nonsingular and, with the diagonal matrix J of (1),

$$Z C_0 Z^{-1} = J. \tag{7}$$

Proof Elementary block operations can be applied to Z to reduce it to the block triangular form

$$\begin{bmatrix} -2iW & 0 & I_{n-r} & 0 \\ 0 & U_2 - U_3 & 0 & I_r \\ 0 & 0 & -I_{n-r} & 0 \\ 0 & 0 & 0 & I_r \end{bmatrix}.$$

Since $W > 0$, it is apparent that Z is nonsingular if and only if condition (6) is satisfied.

Then write C_0 in partitioned form consistent with that of Z:

$$C_0 = \begin{bmatrix} 0 & 0 & I_{n-r} & 0 \\ 0 & 0 & 0 & I_r \\ -\Omega_1^2 & 0 & 2U_1 & 0 \\ 0 & -U_2U_3 & 0 & U_2 + U_3 \end{bmatrix}.$$

Now a simple calculation with block matrices shows that $ZC_0 = JZ$. Thus, when (6) holds, Z is nonsingular and $ZC_0Z^{-1} = J$. □

Notice that the condition (6) together with our standing hypotheses ensures that the systems investigated here are *regular* in the sense of Definition 8 of [15]. These conditions also appear in Theorem 7 of [14].

3 Structure preserving similarities

The following two-part definition reflects a definition introduced in the paper [10]. The underlying idea concerns similarity transformations of C_0 which preserve the companion matrix structure (and, necessarily, the spectrum, σ). For brevity, the term "SPS" (for structure preserving similarity) is introduced.

Definition 2. A matrix $V \in \mathbb{R}^{2n \times 2n}$ is said to define an SPS of C_0 if the matrix

$$C := VC_0V^{-1} = V \begin{bmatrix} 0 & I_n \\ -\Omega^2 & 2U \end{bmatrix} V^{-1} \tag{8}$$

is a block companion matrix, i.e. C can be partitioned into $n \times n$ blocks:

$$C = \begin{bmatrix} 0 & I_n \\ C_{21} & C_{22} \end{bmatrix}.$$

□

It is clear that all matrices C of equation (8) determined by an SPS are isospectral with spectrum σ. Furthermore, the corresponding vibrating systems are isospectral *and* have real coefficients.

A simple lemma from [10] will be useful:

Lemma 2 *A nonsingular $V \in \mathbb{R}^{2n \times 2n}$ (with $n \times n$ partitions V_{ij}) defines an SPS if and only if*

$$V_{21} = -V_{12}\Omega^2, \quad \textit{and} \quad V_{22} = V_{11} + 2V_{12}U. \tag{9}$$

Proof With V nonsingular equation (8) is equivalent to $CV = V \begin{bmatrix} 0 & I_n \\ -\Omega^2 & 2U \end{bmatrix}$. Comparing blocks it is found that $C_{11} = 0$ and $C_{12} = I_n$ if and only if (9) holds. □

EXAMPLE 1. A simple class of SPS is defined by matrices

$$V = \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix},$$

where A is nonsingular. These transformations generate a narrow class of systems which are similar to the canonical system and for which the coefficients M, D, K commute. □

EXAMPLE 2. Another class of SPS is generated by nonsingular matrices V which commute with C_0. They could be described as "automorphisms" because they satisfy $VC_0V^{-1} = C_0$; they transform C_0 into itself. Our interest is in transformations for which the greatest possible freedom in the coefficients is achieved (consistent with preservation of the spectrum). □

4 Jordan pairs and SPS

A right eigenvector (say $x_j \neq 0$) can be associated with each diagonal entry of J (each eigenvalue), and these form the columns of an associated $n \times 2n$ matrix of eigenvectors, say X. More generally, if $X \in \mathbb{C}^{n\times 2n}$, the pair (X, J) (with J as in (1)) form a *Jordan pair* if $\begin{bmatrix} X \\ XJ \end{bmatrix}$ is nonsingular[1]. It is well-known (see [4] or [9], for example) that a Jordan pair, together with a mass matrix M, define a system completely.

Here, with our hypotheses on the spectrum, we may define an $n\times 2n$ matrix of eigenvectors of $L(\lambda)$ in the form

$$X = \begin{bmatrix} X_c & X_{R1} & X_{R2} & \overline{X_c} \end{bmatrix}, \tag{10}$$

where X_c is an $n \times (n-r)$ matrix of (generally) non-real eigenvectors corresponding to the eigenvalues in Λ_1, matrices X_{R1} and X_{R2} are $n\times r$ real matrices of eigenvectors corresponding to the real eigenvalues in U_2 and U_3, respectively. Note that the structure of X is consistent with that of J in (1).

The following theorem establishes a one-to-one connection between Jordan pairs constructed in this way and matrices V which define SPS transformations of C_0 as defined in Definition 2.

Theorem 3 *Let J be a diagonal Jordan form as in* (1) *for which* $\det(U_2 - U_3) \neq 0$.
(a) *If X is any matrix of the form* (10) *for which (X, J) form a Jordan pair then, with Z defined by* (5),

$$V = \begin{bmatrix} X \\ XJ \end{bmatrix} Z \tag{11}$$

defines an SPS of C_0.
(b) *Conversely, if V defines an SPS of C_0 and Z is defined by* (5), *then there is an X of the form* (10) *such that* (11) *holds (and, hence, (X, J) is a Jordan pair).*

[1]This guarantees, in particular, that every column of X (every eigenvector) is nonzero.

Proof. By definition of a Jordan pair and using Lemma 1, it is found that V of (11) is nonsingular. Then compute with block matrices to find

$$V_{21} = \left[\begin{array}{cc} (X_c + \overline{X_c})\Omega_1^2 & -(X_{R1} + X_{R2})U_2U_3 \end{array} \right],$$

$$V_{12} = \left[\begin{array}{cc} -(X_c + \overline{X_c}) & X_{R1} + X_{R2} \end{array} \right],$$

and it can be checked that $V_{21} = -V_{12}\Omega^2$. Similarly, it is found that

$$V_{11} = \left[\begin{array}{cc} X_c\overline{\Lambda} + \overline{X_c}\Lambda & -X_{R1}U_3 - X_{R2}U_2 \end{array} \right]$$

$$V_{22} = \left[\begin{array}{cc} -(X_c\Lambda + \overline{X_c\Lambda}) & X_{R1}U_2 + X_{R2}U_3 \end{array} \right],$$

and, finally, that $V_{22} = V_{11} + 2V_{12}U$. Now part (a) follows from Lemma 2 provided that V is a real matrix.

However, using (5), it follows that,

$$\begin{aligned} V &= \left[\begin{array}{cccc} X_c & X_{R1} & X_{R2} & \overline{X_c} \\ X_c\Lambda_1 & X_{R1}U_2 & X_{R2}U_3 & \overline{X_c\Lambda} \end{array} \right] Z \\ &= \left[\begin{array}{cccc} X_c\overline{\Lambda} + \overline{X_c}\Lambda & -X_{R1}U_3 - X_{R2}U_2 & X_{R1} + X_{R2} & -(X_c + \overline{X_c}) \\ (X_c + \overline{X_c})\Omega_1^2 & -(X_{R1} + X_{R2})U_2U_3 & X_{R1}U_2 + X_{R2}U_3 & -(X_c\Lambda_1 + \overline{X_c\Lambda_1}) \end{array} \right] \end{aligned} \quad (12)$$

and is clearly a real matrix (as Definition 2 requires).

For the converse, observe first that, under condition (6), $C_0Z^{-1} = Z^{-1}J$, and it follows from this equation that the columns of Z^{-1} are right eigenvectors of C_0. If V defines an SPS of C_0 then, using the defining equation (8),

$$C = VZ^{-1}J(VZ^{-1})^{-1}. \quad (13)$$

Thus, the columns of VZ^{-1} are eigenvectors of C. Since C has the same spectrum as C_0 (and J), this matrix of eigenvectors can be written in the form

$$VZ^{-1} = \left[\begin{array}{c} X \\ XJ \end{array} \right] = \left[\begin{array}{cccc} X_c & X_{R1} & X_{R2} & \overline{X_c} \\ X_c\Lambda & X_{R1}U_2 & X_{R2}U_3 & \overline{X_c\Lambda} \end{array} \right]. \quad (14)$$

Thus, X has the required form and, since VZ^{-1} is nonsingular, (X, J) form a Jordan pair. □

5 Generating real isospectral systems

Computational procedures for generating isospectral families of real systems can be formulated from the preceding analysis. This is done first in the language of SPS, and then in terms of Jordan pairs.

1. Fix the diagonal matrix of eigenvalues, J, with the form (1). Form matrices Z of (5) and C_0 of (4).

2. Assign the $n \times 2n$ matrix of eigenvectors, X with the form (10) in such a way that (X, J) form a Jordan pair. (Clearly, this can be done in many ways.)

3. Compute $V = \begin{bmatrix} X \\ XJ \end{bmatrix} Z$.

4. Compute $C = VC_0V^{-1}$ and read off the sub-matrices $M^{-1}K = -C_{21}$ and $M^{-1}D = -C_{22}$.

5. Assign a nonsingular real mass matrix M and compute $K = -MC_{21}$, $D = -MC_{22}$.

The alternative procedure is based on the notion of a Jordan triple. Thus, given the Jordan pair of item 2 above, *assign* a real nonsingular mass matrix M and determine a $2n \times n$ matrix Y satisfying

$$\begin{bmatrix} X \\ XJ \end{bmatrix} Y = \begin{bmatrix} 0 \\ M^{-1} \end{bmatrix}, \tag{15}$$

and (X, J, Y) is known as a *Jordan triple.*

When Y has been determined the *moments*

$$\Gamma_j = XJ^jY, \qquad j = 0, 1, 2, 3, \tag{16}$$

can be formed, and the system coefficients are uniquely defined in terms of the moments (see Theorem 2 of [9], for example):

$$M = \Gamma_1^{-1}, \qquad D = -M\Gamma_2M, \qquad K = -M\Gamma_3M + D\Gamma_1D. \tag{17}$$

The alternative procedure for generating an isospectral family of real systems is now as follows:

1. Fix the diagonal matrix of eigenvalues, J, with the form (1).

2. Assign the $n \times 2n$ matrix of eigenvectors, X with the form (10) in such a way that (X, J) form a Jordan pair. (Clearly, this can be done in many ways.)

3. Assign a nonsingular real mass matrix M and solve equation (15) for Y.

4. Compute the moments (16) and hence the coefficients D and K from (17).

EXAMPLE 3. We will construct a 4×4 real system with 4 real eigenvalues and 4 non-real eigenvalues. Take a Jordan matrix of the form (1) with blocks

$$\Lambda = \text{diag}[-1+i,\ -4+i], \quad U2 = \text{diag}[-0.5,\ -1], \quad U3 = \text{diag}[-3, -4].$$

Then take a matrix X of the form (10) with blocks

$$X_c = \begin{bmatrix} 0.0625(1-i) & (0.6)(1-0.1i) \\ 0.2500(1-i) & (0.6)(1-0.1i) \\ 0.5625(1-i) & 0 \\ 1.0000(1-i) & (-1)(1-0.1i) \end{bmatrix}, \quad X_{R1} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad X_{R2} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ -1 & 1 \\ -1 & -1 \end{bmatrix}.$$

It is found that this data generates the real monic system with

$$D = \begin{bmatrix} -6.0008 & 6.5981 & 5.8527 & -4.4416 \\ -8.4540 & 8.7557 & 6.1483 & -4.6190 \\ 7.2668 & -8.8863 & 6.8695 & -0.9717 \\ 22.9877 & -21.5283 & -5.2094 & 8.8756 \end{bmatrix}, \; K = \begin{bmatrix} -20.7791 & 15.4356 & 19.5039 & -13.4062 \\ -22.8678 & 16.8592 & 20.4793 & -13.8052 \\ 1.2225 & -7.0538 & 11.2192 & -3.4988 \\ 35.3128 & -32.9667 & -18.4409 & 18.4076 \end{bmatrix}.$$

These calculations can then be checked by showing that the eigenvalues of this monic system are, indeed, those specified in J. □

6 Symmetric systems, part 1.

The next objective is, of course, to determine the matrices V defining an SPS of C_0, and which also generate *symmetric* systems. The question of when these coefficients satisfy positivity conditions will be considered later.

At this point it is necessary to introduce the rather subtle notion of the *sign characteristic*[2] associated with the real eigenvalues. (The reader is referred to the references of the footnote for formal definitions, but for the uninitiated, Appendix B gives an intuitive introduction to this important notion.) For systems with symmetries it is not enough to allocate arbitrary real eigenvalues; the invariants of the sign characteristic must also be specified. With our hypotheses on the spectrum, this can be accomplished by introducing the matrix

$$P = \begin{bmatrix} 0 & 0 & 0 & I_{n-r} \\ 0 & I_r & 0 & 0 \\ 0 & 0 & -I_r & 0 \\ I_{n-r} & 0 & 0 & 0 \end{bmatrix}. \tag{18}$$

Notice that, with J of the form (1), $(PJ)^* = PJ$. Thus, although J is generally not hermitian, PJ is so.

Symmetry of the coefficients of the system follows if a symmetric M is chosen and, also, $Y = PX^*$ is the only solution of (15) . For then the Γ's are hermitian and, using (17), so are D and K (and when this is the case, $(X,\ J,\ PX^*)$ is said to be a *selfadjoint Jordan triple*). If, in addition, X has the block structure of (10), then the moments and the system coefficients will be real and symmetric.

Thus, if a selfadjoint triple is to be constructed, then (see equation (15)) $XY = XPX^* = 0$. Thus, once admissible matrices J and P have been assigned, the crux of the problem is to find an X such that $XPX^* = 0$ and $X(PJ)X^*$ is nonsingular. In [9] a geometric approach is taken to the determination of such matrices X. Here, attention is focussed on *real* systems, so that the structure of (10) is also to be imposed on X. In this case $XPX^* = 0$ can be written in the form

$$X_cX_c^T + \overline{X_cX_c^T} = -X_{R1}X_{R1}^T + X_{R2}X_{R2}^T. \tag{19}$$

Now this equation simply says that the real part of the matrix $X_cX_c^T$ takes the value $\frac{1}{2}(-X_{R1}X_{R1}^T + X_{R2}X_{R2}^T)$, and does not constrain the imaginary part.

[2]See [3], [4], [9], and the expository Appendix B to this paper

Consequently, it follows from equation (19) that

$$X_c X_c^T = R_1 - iR, \tag{20}$$

where

$$R_1 := \frac{1}{2}(-X_{R1}X_{R1}^T + X_{R2}X_{R2}^T) = \begin{bmatrix} X_{R1} & X_{R2} \end{bmatrix} \begin{bmatrix} -I_r & 0 \\ 0 & I_r \end{bmatrix} \begin{bmatrix} X_{R1}^T \\ X_{R2}^T \end{bmatrix} \tag{21}$$

and R is a real symmetric matrix.

Notice also that, if the right-hand-side is designed to have rank $n-r$, then $\operatorname{rank}(X_c) \geq n-r$. (To see that equality need not be the case, consider the product $A_0 A_0^T$ where $A_0 = \begin{bmatrix} 1 & i \\ i & -1 \end{bmatrix}$.)

The broad strategy suggested here is to assign all the real eigenvectors, and hence the matrix R_1. Generically, it can be expected that the eigenvectors associated with the eigenvalues in the upper half-plane will be linearly independent. Thus, the matrix $X_c \in \mathbb{C}^{n\times(n-r)}$ of (10) will have full rank, $n-r$. Now there is a standard method for finding a symmetric factorization of a complex symmetrix matrix (as required in (20)), in which the rank of the factors is equal to that of the given right-hand side. So the problem reduces to: Given R_1, with rank determined by the choice of real eigenvectors, find an R such that $R_1 - iR$ has rank $n-r$.

The "standard method" referred to above is due to Takagi in the 1920's. A quick introduction, based on the exposition and algorithm of [1], is given in Appendix A of this paper.

It is instructive to consider a simple example at this stage.

EXAMPLE 4. We construct a 2×2 system with two real eigenvalues and one complex pair. The spectral data is

$$J = \begin{bmatrix} -2+i & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2-i \end{bmatrix}, \quad P = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

(a) We first prescribe the *real* eigenvectors:

$$X_{R1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad X_{R2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Then calculate to find $R_1 = \frac{1}{2}(-X_{R1}X_{R1}^T + X_{R2}X_{R2}^T) = \begin{bmatrix} -1/2 & 0 \\ 0 & 1/2 \end{bmatrix}$.

Choosing $R = \begin{bmatrix} 0 & 1/2 \\ 1/2 & 0 \end{bmatrix}$ yields the rank one matrix

$$R_1 - iR = \frac{1}{2}\begin{bmatrix} -1 & -i \\ -i & 1 \end{bmatrix},$$

and this has the factorization (cf. equation (20)) $R_1 - iR = X_c X_c^T$ where $X_c = \frac{1}{\sqrt{2}}\begin{bmatrix} -i \\ 1 \end{bmatrix}$.

Now compute with equations (16) and (17) to find the real symmetric system

$$M = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 3 \\ 3 & -5 \end{bmatrix}, \quad K = \begin{bmatrix} 1 & 2 \\ 2 & -6 \end{bmatrix},$$

and it can be checked that the spectrum is, indeed, that prescribed.

(b) In contrast to (a), if $X_{R1} = X_{R2} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, then $R_1 = 0$ and R is chosen so that $R_1 - iR$ has rank one; say $R = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}$ so that $R_1 - iR = \begin{bmatrix} 0 & 0 \\ 0 & i \end{bmatrix}$ in which case we take

$$X_c = \begin{bmatrix} 0 \\ e^{i\pi/4} \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}}(1+i) \end{bmatrix},$$

and

$$X = \begin{bmatrix} 0 & 1 & 1 & 0 \\ \frac{1}{\sqrt{2}}(1+i) & 0 & 0 & \frac{1}{\sqrt{2}}(1-i) \end{bmatrix}.$$

Following the steps above it is found that this determines the *diagonal* system,

$$M = \begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{2} \end{bmatrix}, \quad D = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}, \quad K = \begin{bmatrix} 2 & 0 \\ 0 & \frac{-5}{2} \end{bmatrix}.$$

□

An interesting feature of this example is the fact that R is first chosen to *reduce* the rank of $R_1 - iR$ relative to that of R_1. Clearly, this was necessary, as X_c must have rank one. In the second case, R is chosen to augment the rank of R_1. Features of this kind are a major difficulty in the design of a general strategy for finding solutions to equation (19). Notice also that the necessary condition that X has full rank was guaranteed by our choice of the real eigenvectors the first time around, but not the second.

Another approach to the determination of solutions of (19) begins with assigning a *complex* matrix of eigenvectors X_c, and then solving (19) for real matrices X_{R1} and X_{R2}. This line of attack is postponed to a future investigation.

7 Symmetric systems, part 2.

The problem of finding matrices X of the form (10) which also satisfy $XPX^* = 0$ has been reformulated in the form of equation (20) where R_1 is given by (21). This matrix is to be assigned, and then a real symmetric matrix R is to be chosen in such a way that $R_1 - iR$ has rank $n - r$. To ensure that both real and non-real eigenvalues appear, it is assumed that $1 \le r \le n - 1$.

It appears that the rank of R_1 can take any value between zero (when $X_{R2} = X_{R1}$) and n (when the real eigenvectors span the whole space). A complete understanding of our problem seems to require knowledge of the connections between

$$\text{rank}(R_1), \qquad \text{rank}(R), \qquad \text{and} \qquad \text{rank}(R_1 - iR).$$

As this seems not to be well-known the details are provided in Appendix C. In particular, Theorem 9 shows that, to achieve

$$n - r = \mathrm{rank}(R_1 - iR) < \mathrm{rank}(R_1),$$

which can certainly be physically reasonable, then R must be chosen so that $\pm i$ become eigenvalues of the real symmetric pencil $R_1 - \lambda R$. Now this phenomenon arose in Example 4, apparently fortuitously! But, in fact, Theorem 9 shows that this choice of R was essentially unique.

More generally, notice that although the factor $\begin{bmatrix} -I_r & 0 \\ 0 & I_r \end{bmatrix}$ in (21) has rank $2r$, the "modified" matrix

$$\begin{bmatrix} -I_r & iI_r \\ iI_r & I_r \end{bmatrix}$$

has rank r with eigenvalues $\pm i$ repeated r times. Thus, by choosing

$$R = \begin{bmatrix} X_{R1} & X_{R2} \end{bmatrix} \begin{bmatrix} 0 & I_r \\ I_r & 0 \end{bmatrix} \begin{bmatrix} X_{R1}^T \\ X_{R2}^T \end{bmatrix}, \quad R_1 - iR = \begin{bmatrix} X_{R1} & X_{R2} \end{bmatrix} \begin{bmatrix} -I_r & iI_r \\ iI_r & I_r \end{bmatrix} \begin{bmatrix} X_{R1}^T \\ X_{R2}^T \end{bmatrix}, \tag{22}$$

we introduce eigenvalues $\pm i$ of multiplicity r into the pencil $R_1 - iR$. Modifications of this definition for R are easily devised to generate real symmetric matrices $R_1 - iR$ with rank ρ where $r \le \rho \le 2r$.

If R_1 has rank $n - r$ then, by choosing an R with the same range as R_1, an $R_1 + iR$ can be constructed with the same rank, $n - r$. However, with this construction, the necessary condition that the range of X have dimension n cannot be satisfied. Indeed, there seems to be a difficult problem here when the rank of R_1 is low. There may be a deep property that is not fully understood to the effect that, although linear dependences among the real eigenvectors are known to be possible, the dimension of the span of the real eigenvectors cannot be "too low". The parameters used in Theorem 9 of Appendix C will probably play a role in any resolution of this problem. Indeed, the sets of admissible parameters can be analysed using the canonical forms to be found in Theorem 9.2 of [11] - and described in Appendix C.

These techniques are not investigated more deeply here, and we conclude the present discussion with a numerical illustration.

EXAMPLE 5. The program here is to take the data from Example 3, which was used in the design of a real (non-symmetric) system, augment it with a sign characteristic, and design a real *symmetric* system. Thus, as in Example 3, we take $n = 4$, $r = 2$, and

$$\Lambda = \mathrm{diag}[-1 + i,\ -4 + i], \quad U_2 = \mathrm{diag}[-0.5,\ -1], \quad U_3 = \mathrm{diag}[-3, -4].$$

Now consider equation (20). Since there are two pairs of non-real eigenvalues, X_c is to be constructed with rank 2. Following the strategy leading to (22) results in

$$A = R_1 + iR = \begin{bmatrix} 2i & 1+i & -1+i & -1-i \\ & 0 & 0 & -2 \\ & & 0 & -2i \\ & & & 0 \end{bmatrix}.$$

Notice that R_1 is defined by the data of Example 3 and R is chosen as in (22). Using the Takagi algorithm we obtain

$$X_c = \begin{bmatrix} -1.0082(1+i) & 0.1281(-1+i) \\ -0.8801 & (-0.8801)i \\ (-0.8801)i & 0.8801 \\ 1.1362 & (-1.1362)i \end{bmatrix},$$

and construction of the 4×8 matrix X is complete.

The Jordan matrix is now diag$[-1+i,\ -4+i,\ -0.5,\ -1,\ -3,\ -4,\ -1-i,\ -4-i]$, and (see (18)),

$$P = \begin{bmatrix} 0 & 0 & 0 & I_2 \\ 0 & I_2 & 0 & 0 \\ 0 & 0 & -I_2 & 0 \\ I_2 & 0 & 0 & 0 \end{bmatrix}.$$

It can be verified that $XPX^* = 0$, and the formulae of (16) and (17) are applied to produce the real symmetric system:

$$M = \begin{bmatrix} 0.4496 & -0.3267 & 1.6481 & -0.2840 \\ & 0.2748 & -0.6290 & 0.0519 \\ & & 0.8991 & 0.2533 \\ & & & -0.0696 \end{bmatrix},\ D = \begin{bmatrix} -8.4914 & 4.2104 & 4.9463 & -4.5620 \\ & -1.6488 & -0.6350 & 1.7465 \\ & & 6.5591 & -1.7314 \\ & & & -0.3876 \end{bmatrix},$$

$$K = \begin{bmatrix} 0.4612 & -3.0335 & -4.5721 & 4.8244 \\ & 3.8425 & 6.4105 & -5.2257 \\ & & 12.0600 & -9.8717 \\ & & & 7.8178 \end{bmatrix}.$$

As usual, it can be verified that this sytem has the spectrum determined by J. It is interesting that, in spite of the location of all the eigenvalues in the left half-plane, *all three* coefficients are indefinite. □

Of course, the analysis simplifies if there are to be *no* real eigenvalues (as in [10]), for then X_c is nonsingular and it is only necessary to assign a nonsingular real symmetric R. On the other hand, if X_c is a solution of (19), so is $X_c\Theta$ for any real orthogonal matrix Θ. Thus, for a fixed right-hand-side of (19), a family of solutions X is obtained depending on $\frac{1}{2}(n-r)(n-r-1)$ real parameters. This is consistent with results obtained in [10] for the case $r = 0$.

The situation in which there are no *non-real* eigenvalues is also of great interest and includes the so-called over-damped, hyperbolic, and quasi-hyperbolic systems. They are the topic of Section 10 below.

8 Positivity of M, D, and K

In this section it is assumed that the spectral data is consistent with real and symmetric systems, and we examine the further conditions required to ensure positive definite (or possibly semi-definite) coefficients M, D, K. It will be convenient to make a further simplifying hypothesis, namely, that systems are to be designed which are *nonsingular*. This is equivalent to the hypothesis that K is nonsingular. This can be justified here on the grounds that, in this section, our major interest is in *stable* systems, i.e. those with all eigenvalues in the open left half of the complex plane. In other words, J of (1) is to be a stable matrix.

In this case there is a nice alternative to the formula

$$K = -M\Gamma_3 M + D\Gamma_1 D$$

of (17) (see Theorem 2 of [10], for example). Thus, given a sefadjoint triple $(X,\ J,\ PX^*)$, we have

$$\Gamma_{-1} := X(J^{-1}P)X^* = -K^{-1}. \tag{23}$$

This follows immediately from the *resolvent form* for $L(\lambda)$ expressed here in terms of any Jordan triple:

$$L(\lambda)^{-1} = X(\lambda I_{2n} - J)^{-1}Y.$$

Lemma 4 *If zero is not an eigenvalue of J and Γ_1 is nonsingular, then the inertias of $M,\ D,\ K$ are equal to those of $\Gamma_1,\ -\Gamma_2,$ and $-\Gamma_{-1}$, respectively.*

Proof. Observe that, if 0 is not an eigenvalue of J, then Γ_{-1} is well-defined, and the lemma follows from the first two relations of (17) together with (23). □

Since the moments are readily computed from a Jordan triple, this immediately suggests that the positivity of $M,\ D,\ K$ could be checked by trial and error. However, a more precise result can be proved, which generalizes Theorem 9 of [10]. Notice the important role played by positivity of the *second* moment in this result.

Theorem 5 *If J is stable (has all eigenvalues in the open left half-plane), $\Gamma_2 \le 0$, and Γ_1, Γ_{-1} are nonsingular, then $M > 0$, $D \ge 0$ and $K > 0$.*

Proof. Since Γ_1 and Γ_{-1} are nonsingular, M and K are well-defined by (17) and (23). Then the stability of J, together with Theorem 7 of [13], imply that $M > 0$ and $K > 0$. Then $D \ge 0$ follows from $\Gamma_2 \le 0$ and (17). □

Note that there is, of course, a classical converse statement for Theorem 5: If $M > 0$, $D \ge 0$ and $K > 0$ then all eigenvalues are in the (possibly closed) left half-plane.

In general, it will be difficult to apply the last result numerically *ab initio*. A major open question is:

Problem 1. Given that the spectrum is stable, what further conditions on X (the matrix of *eigenvectors*) will ensure that the coefficients of the system are positive definite?

A closely connected question seems to be:

Problem 2. What are the constraints linking the dimensions of the ranges of the matrices $\begin{bmatrix} X_{R1} & X_{R2} \end{bmatrix}$ and $\begin{bmatrix} X_c & \overline{X_c} \end{bmatrix}$?

EXAMPLE 6. Reconsider Example 4. Modify the data and take $X_{R1} = X_{R2} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, and retain the same matrices J and P. Now $R_1 = 0$ and we choose $R = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. It is found that a diagonal system with positive definite coefficients is generated:

$$M = \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix}, \quad D = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}, \quad K = \begin{bmatrix} 2 & 0 \\ 0 & 2.5 \end{bmatrix}.$$

□

9 Hyperbolic and over-damped systems

For the purpose of this discussion, consider systems with a positive definite leading coefficient, M. It is well-known that there are symmetric vibrating systems of practical interest for which all eigenvalues are real. An early and influential paper on this subject was that of Duffin [2] concerning over-damped systems. Subsequently, it has been realised that systems with real spectrum arise more generally. Thus, systems with all $2n$ eigenvalues real with n of positive type (with a $+1$ in the sign characteristic) and n of negative type (with a -1 in the sign characteristic) are said to be *quasi-hyperbolic*. If, furthermore, the sets of eigenvalues of the two types are separated (all positive type eigenvalues greater than all negative type eigenvalues) then the system is *hyperbolic*. Finally, a system which is hyperbolic and $D > 0$, $K \geq 0$ is said to be *over-damped*. Given these inequalities it is equivalent to say that a hyperbolic system is over-damped if all eigenvalues are negative. (Numerical methods for such problems are the topic of [7] and [6], for example.)

The distinction between these classes of systems is quite clear in our context of inverse problems. Let us begin with hyperbolic systems. Thus, all eigenvalues are real and the data for the inverse problem consists of diagonal matrices

$$J = \begin{bmatrix} U_2 & 0 \\ 0 & U_3 \end{bmatrix}, \quad P = \begin{bmatrix} I_n & 0 \\ 0 & -I_n \end{bmatrix}, \tag{24}$$

(cf. equations (1) and (18)). Furthermore, the smallest eigenvalue of U_2 (having positive type) exceeds the largest eigenvalue of U_3 (having negative type). Thus,

$$\max_{u_j \in U_3}(u_j) < \min_{u_k \in U_2}(u_k). \tag{25}$$

Then the necessary condition (6) of Lemma 1 is satisfied.

For the more general quasi-hyperbolic systems, U_2 and U_3 are simply specified in such a way that (6) holds but (25) does not necessarily hold.

Now an isospectral family of quasi-hyperbolic systems with spectrum defined by U_2 and U_3 is determined by full rank eigenvector matrices

$$X = \begin{bmatrix} X_{R1} & X_{R2} \end{bmatrix}$$

where X_{R1}, $X_{R2} \in \mathbb{R}^{n \times n}$ and

$$XPX^* = X_{R1}X_{R1}^T - X_{R2}X_{R2}^T = 0,$$

(cf. equations (10), (19)). This condition is easily satisfied: Since X has full rank, both X_{R1} and X_{R2} must be nonsingular, so we may take an arbitrary $A > 0$ in $\mathbb{R}^{n \times n}$ and then choose X_{R1} and X_{R2} so that

$$X_{R1}X_{R1}^T = X_{R2}X_{R2}^T = A. \tag{26}$$

Natural choices for X_{R1} and X_{R2} are then $A^{1/2}$, or a lower triangular matrix generated by a Cholesky factorization of A (see [5], for example). Having made a first choice of X_{R1} and

X_{R2}, infinitely many more candidates are generated by multiplying on the right with a real orthogonal matrix. In particular, once a nonsingular X_{R1} is chosen, one may take

$$X_{R2} = X_{R1}\Theta,$$

where Θ is real orthogonal. We adopt this strategy.

Then it is easily verified that the following formulae hold: For the moments,

$$\begin{aligned}\Gamma_1 &= X_{R1}(U_2 - \Theta U_3 \Theta^T)X_{R1}^T,\\ \Gamma_2 &= X_{R1}(U_2^2 - \Theta U_3^2 \Theta^T)X_{R1}^T,\\ \Gamma_{-1} &= X_{R1}(U_2^{-1} - \Theta U_3^{-1} \Theta^T)X_{R1}^T.\end{aligned} \tag{27}$$

And, for the coefficients:

$$\begin{aligned}M &= X_{R1}^{-T}(U_2 - \Theta U_3\Theta^T)^{-1}X_{R1}^{-1}\\ D &= -X_{R1}^{-T}(U_2 - \Theta U_3\Theta^T)^{-1}(U_2^2 - \Theta U_3^2\Theta^T)(U_2 - \Theta U_3\Theta^T)^{-1}X_{R1}^{-1}\\ K &= -X_{R1}^{-T}(U_2^{-1} - \Theta U_3^{-1}\Theta^T)^{-1}X_{R1}^{-1}.\end{aligned} \tag{28}$$

It is immediately apparent that X_{R1} merely determines a simultaneous congruence applied to the three system coefficients. Once the spectrum is specified in the form of U_2 and U_3, the coefficients are determined (to within this simultaneous congruence) by the choice of Θ. Thus, the *inertias* of $M,\ D,\ K$ do not depend on X_{R1}. (A similar phenomenon arises in the case when all eigenvalues are *non-real*, see equations (34)-(36) and Theorem 13 of [10]).

Theorem 5 now provides criteria for generating families of real hyperbolic systems:

Corollary 6 *Assume that, Λ and W do not appear in equation* (1) *(i.e. all eigenvalues are real) and $U_2 < 0,\ U_3 < 0$ are chosen so that $det(U_2 - U_3) \neq 0$. Let*
(a) $X_{R1} \in R^{n\times n}$ *be nonsingular,*
(b) Θ *be a real orthogonal matrix for which*

$$U_2^2 \leq \Theta U_3^2 \Theta^T \tag{29}$$

and $U_2 - \Theta U_3\Theta^T$, $U_2^{-1} - \Theta U_3^{-1}\Theta^T$ are nonsingular.
Then (in equations (28)*), $M > 0,\ D \geq 0,\ K > 0$, and the system $M\lambda^2 + D\lambda + K$ is quasi-hyperbolic.*

If, in addition, (25) *holds, then the system is over-damped.*

Proof. The condition (29) and the equation for Γ_2 in (27) ensure that $\Gamma_2 \geq 0$. Also, from (27), Γ_1 and Γ_{-1} are nonsingular. So the result follows from Theorem 5. □

EXAMPLE 7. Clearly, given the hypotheses of the corollary on U_2 and U_3, $X_{R1} = I_n$ and $\Theta = I_n$ are admissible choices. If we write

$$U_2 = \text{diag}[\mu_1^{(2)},\ \mu_2^{(2)}, \cdots, \mu_n^{(2)}], \quad U_3 = \text{diag}[\mu_1^{(3)},\ \mu_2^{(3)}, \cdots, \mu_n^{(3)}],$$

equations (28) determine the diagonal system with diagonal entries

$$\frac{\lambda^2 - (\mu_j^{(2)} + \mu_j^{(3)})\lambda + \mu_j^{(2)}\mu_j^{(3)}}{\mu_j^{(2)} - \mu_j^{(3)}} = \frac{(\lambda - \mu_j^{(2)})(\lambda - \mu_j^{(3)})}{\mu_j^{(2)} - \mu_j^{(3)}}.$$

EXAMPLE 8. Take $X_{R1} = I_4$ and

$$U_2 = \text{diag}[-1,\ -2,\ -3,\ -4], \qquad U_3 = \text{diag}[-5,\ -6,\ -7,\ -8].$$

Consider the orthogonal matrix

$$\Theta = \frac{1}{10}\begin{bmatrix} 2 & -8 & 4 & -4 \\ 8 & -2 & -4 & 4 \\ 4 & 4 & -3 & -8 \\ 4 & 4 & 8 & 2 \end{bmatrix}$$

and verify that the hypotheses of the corollary are satisfied. Apply the formulae of (28) to generate the overdamped system (with truncated decimal form)

$$M = \begin{bmatrix} 0.1886 & 0.0269 & -0.0168 & -0.0051 \\ & 0.2896 & 0.0690 & 0.0707 \\ & & 0.2694 & 0.0808 \\ & & & 0.42342 \end{bmatrix}, \ D = \begin{bmatrix} 1.3771 & 0.0808 & -0.0673 & -0.0253 \\ & 2.1582 & 0.3451 & 0.4242 \\ & & 2.6162 & 0.5657 \\ & & & 4.3939 \end{bmatrix},$$

$$K = \begin{bmatrix} 1.1886 & 0.0539 & -0.0505 & -0.0202 \\ & 3.1582 & 0.4141 & 0.5657 \\ & & 5.4242 & 0.9697 \\ & & & 10.7879 \end{bmatrix},$$

with eigenvalues -1, -2, -3, -4 of positive type, and -5, -6, -7, -8 of negative type. An associated matrix of eigenvectors has the form

$$X = \begin{bmatrix} I & \Theta \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0.2 & -0.8 & 0.4 & -0.4 \\ 0 & 1 & 0 & 0 & 0.8 & -0.2 & -0.4 & 0.4 \\ 0 & 0 & 1 & 0 & 0.4 & 0.4 & -0.2 & -0.8 \\ 0 & 0 & 0 & 1 & 0.4 & 0.4 & 0.8 & 0.2 \end{bmatrix}.$$

□

10 Conclusions

Real damped vibrating systems defined by $n \times n$ coefficient matrices M, D, and K have been studied, with the simplifying hypothesis of semi-simple Jordan structure, i.e. associated $2n \times 2n$ Jordan canonical forms, J, are diagonal. A corresponding primitive companion matrix, C_0, is formulated (equation (4)) and plays a significant role. The equivalence between "structure preserving similarities" of C_0 and Jordan pairs for the system has been established in Theorem 3.

These constructions have been used to find solutions to the inverse problem: Given J, find consistent real viobrating systems (see Section 5). The corresponding problem for consistent *symmetric* real systems (with mixed real and non-real spectrum) is more complicated. A partial solution for this problem (taking advantage of Takagi's factorization of symmetric complex matrices) is the subjext of Sections 6 and 7. To ensure that coefficient matrices M, D, K have definiteness properties is more difficult again, but significant insight is provided in Section 8 in terms of "moments" of the system.

Similar methods applied to systems having *only* real eigenvalues are more tractable (and include systems of hyperbolic and over-damped types). They are developed in Section 9, where parametrizations of isospectral systems by real orthogonal matrices arise naturally. This compares nicely with similar results of [10] for systems at the other (elliptic) extreme with *no* real eigenvalues.

The analysis developed here requires knowledge of the relationship between the ranks of a complex symmetric matrix and its real and imaginary parts. This is clarified in Theorem 8 of Appendix C. The result seems to be new and may be of more general interest.

11 APPENDIX A: Takagi's factorization

A method for making the factorization needed in Sections 6 and 7 is attributed to Takagi in the 1920's. For a given complex symmetric matrix $A \in \mathbb{C}^{n\times n}$ of rank $n-r$, a factorization $A = X_c X_c^T$ is produced in which X_c also has rank $n-r$. Computer programs are now available for this task (see Bunse-Gerstner and Gragg [1]). There is also a careful discussion in Section 4.4.4 of Horn and Johnson [8]. Here, an introduction is made for the relatively simple case in which nonzero singular values of the right-hand-side of (20) (i.e. when $A = R_1 - iR$) are distinct. It is based on the presentation of [1].

1. Let A denote the (given) complex symmetric right-hand-side of (20) and assume that $\operatorname{rank}(A) = n-r$. Form the singular value decomposition $A = U\Sigma V^*$, where U and V are unitary matrices and
$$\Sigma = \operatorname{diag}\left[\begin{array}{cccc} \sigma_1 & \sigma_2 & \cdots & \sigma_n \end{array}\right]$$
with $\sigma_1 > \sigma_2 > \cdots > \sigma_{n-r} > 0$ and $\sigma_{n-r+1} = \ldots = \sigma_n = 0$ (see [5] for further details).

2. Let u_j and v_j denote the columns of U and V, respectively and compute $q_j^2 := u_j^T v_j$, $j = 1, 2, \ldots, n-r$ (note that q_j^2 will generally be complex).

3. Form $\Sigma_1 := \operatorname{diag}\left[\begin{array}{cccc} \sigma_1 & \sigma_2 & \cdots & \sigma_{n-r} \end{array}\right]$ of size $(n-r)\times(n-r)$ and form the $n\times(n-r)$ matrix $U_0 = \left[\begin{array}{cccc} u_1 & u_2 & \cdots & u_{n-r} \end{array}\right]$.

4. Compute a matrix $Q = \operatorname{diag}\left[\begin{array}{cccc} q_1 & q_2 & \cdots & q_{n-r} \end{array}\right]$.

5. Compute $X_c = U_0 \bar{Q} \Sigma_1^{1/2}$ (of size $n \times (n-r)$).

Let us quickly confirm that this produces the required symmetric factorization. Since A is symmetric,
$$A = U\Sigma V^* = \overline{V}\Sigma U^T.$$
But the singular vectors for the (distinct) nonzero singular values are unique to within a scalar multiplier of modulus one. Thus, there are numbers $\omega_1, \ldots, \omega_{n-r}$ such that
$$v_j = \omega_j \overline{u_j}, \quad |\omega_j| = 1, \quad j = 1, 2, \ldots, n-r.$$

Defining U_0 as in item 3, $V_0 = \left[\begin{array}{ccc} v_1 & \cdots & v_{n-r}\end{array}\right]$ and $\Omega = \text{diag}\left[\begin{array}{cc}\omega_1 & \ldots\omega_{n-r}\end{array}\right]$, we have $V_0 = \overline{U_0}\Omega$, or

$$V_0^* = \overline{\Omega}U_0^T.$$

Furthermore, $u_j^T v_j = \omega_j(u_j^T\overline{u_j}) = \omega_j$ for $j = 1, 2, \ldots, n-r$, so that (see item 4), $\Omega = Q^2$.

Now compute

$$\begin{aligned} X_c X_c^T &= (U_0\overline{Q}\Sigma^{\frac{1}{2}})(\Sigma^{\frac{1}{2}}\overline{Q}U_0^T), \\ &= U_0\Sigma\overline{Q}^2 U_0^T = U_0\Sigma(\overline{\Omega}U_0^T), \\ &= U_0\Sigma V_0^* = U\Sigma V^* = A. \end{aligned}$$

□

The purpose of the next example is simply to illustrate this scheme. Calculations are completed in MATLAB.

EXAMPLE 9. Let $A = \begin{bmatrix} i & i \\ i & 0 \end{bmatrix}$ and note that $n = 2$, $r = 0$ (so no singular values are equal to zero). The MATLAB singular value decomposition yields (with truncated numbers)

$$U = \begin{bmatrix} -0.8507 & -0.5257 \\ -0.5257 & 0.8507 \end{bmatrix} i, \qquad V = \begin{bmatrix} -0.8507 & 0.5257 \\ -0.5257 & -0.8507 \end{bmatrix}.$$

Also, $\sigma_1^2 = (1+\sqrt{5})/2$, and $\sigma_2^2 = (1-\sqrt{5})/2$. It is found that $q_1^2 = i$ and $q_2^2 = -i$ and then $q_1 = \frac{\sqrt{2}}{2}(1+i)$, $q_2 = \frac{\sqrt{2}}{2}(1-i)$. Finally,

$$X_c = U\bar{Q}\Sigma^{1/2} = \begin{bmatrix} -0.7651(1+i) & 0.2923(1-i) \\ -0.4729(1+i) & -0.4729(1-i) \end{bmatrix}.$$

It can be verified that, indeed $X_c X_c^T = A$. □

12 APPENDIX B: An expository example

Consider the problem of constructing 2×2 symmetric systems with the simple (mixed) spectrum: 1, -1, i, -i. One such system is obvious, namely, the monic system

$$L(\lambda) = \begin{bmatrix} \lambda^2 - 1 & 0 \\ 0 & \lambda^2 + 1 \end{bmatrix}. \tag{30}$$

Let us first examine the *forward* problem for this system. The sign characteristic associated with the real eigenvalues $\{1, -1\}$ of this sytem is $\{+1, -1\}$. (The best way to see this is to use Theorem 12.5 of [4].) Clearly, this system has associated matrices

$$X = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}, \quad J = \begin{bmatrix} i & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -i \end{bmatrix} \quad P = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}. \tag{31}$$

Since $\det\begin{bmatrix} X \\ XJ \end{bmatrix} \neq 0$, $(X,\, J)$ form a Jordan pair. However,

$$Y := \begin{bmatrix} X \\ XJ \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ M \end{bmatrix} \neq PX^*,$$

so $(X,\, J)$ is not part of a *selfadjoint triple.* The eigenvectors must be re-normalised to achieve this. (Such a re-normalisation leaves the moments and the coefficients invariant.) It is found that if we set $\kappa = e^{-i\pi/4}$ and define

$$X = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 1 & 1 & 0 \\ \kappa & 0 & 0 & \overline{\kappa} \end{bmatrix},$$

then $Y = PX^*$. It is now a matter of computation to verify that the equations (16) and (17) lead back to the coefficients of the system (30).

Now consider the inverse problem in which the data consists of matrices J and P of (31). Observe first that X_{R1} and X_{R2} will be 2×1 vectors and, to find an X_c of rank one, it is convenient to assume that X_{R1} and X_{R2} are linearly dependent (see Section 6). Indeed, with $\alpha, \beta \in \mathbb{R}$ let us take $X_{R1} = X_{R2} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$. Then $R_1 = 0$ in (20) and we may choose $R = \begin{bmatrix} 0 & 0 \\ 0 & \gamma \end{bmatrix}$ where $\gamma \in \mathbb{R}$, so that the equation for X_c becomes $X_c X_c^T = \begin{bmatrix} 0 & 0 \\ 0 & -i\gamma \end{bmatrix}$.

It is easily verified that $X_c = \begin{bmatrix} 0 \\ \gamma^{\frac{1}{2}}\kappa \end{bmatrix}$ is a solution of this equation. Thus, a complete eigenvector matrix X is

$$X = \begin{bmatrix} 0 & \alpha & \alpha & 0 \\ \gamma^{\frac{1}{2}}\kappa & \beta & \beta & \overline{\gamma^{\frac{1}{2}}\kappa} \end{bmatrix}. \tag{32}$$

Now compute to find $\Gamma_0 = 0$ and

$$\Gamma_1 = 2\begin{bmatrix} \alpha^2 & \alpha\beta \\ \alpha\beta & \beta^2 + \gamma \end{bmatrix}, \quad \Gamma_2 = 0, \quad \Gamma_3 = 2\begin{bmatrix} \alpha^2 & \alpha\beta \\ \alpha\beta & \beta^2 - \gamma \end{bmatrix},$$

and then, assuming $\alpha\gamma \neq 0$,

$$M = \Gamma_1^{-1} = (2\alpha^2\gamma)^{-1} \begin{bmatrix} \beta^2 + \gamma & -\alpha\beta \\ -\alpha\beta & \alpha^2 \end{bmatrix}, \quad D = 0, \quad K = (2\alpha^2\gamma)^{-1} \begin{bmatrix} \beta^2 - \gamma & -\alpha\beta \\ -\alpha\beta & \alpha^2 \end{bmatrix}.$$

Thus, a three-parameter family of isospectral systems is obtained. The system (30), with which this discussion began, is obtained by taking $\alpha = 1/\sqrt{2},\ \beta = 0,\ \gamma = 1/2$. In contrast with matrix X of (31), (32) evaluated at these parameter values is a member of a *selfadjoint* triple.

To illustrate the role played by the sign characteristic consider the following special cases:

- CASE 1: $\alpha = 1,\ \beta = 0,\ \gamma = 1.\quad M_1\lambda^2 + K_1 = \frac{1}{2}\begin{bmatrix} \lambda^2 - 1 & 0 \\ 0 & \lambda^2 + 1 \end{bmatrix}$,
- CASE 2: $\alpha = 1,\ \beta = -1,\ \gamma = 1.\quad M_2\lambda^2 + K_2 = \frac{1}{2}\begin{bmatrix} 2\lambda^2 & \lambda^2 + 1 \\ \lambda^2 + 1 & \lambda^2 + 1 \end{bmatrix}$,

- CASE 3: $\alpha = 1,\ \beta = 0,\ \gamma = -1.\quad M_3\lambda^2 + K_3 = \frac{1}{2}\begin{bmatrix} \lambda^2 - 1 & 0 \\ 0 & -(\lambda^2+1) \end{bmatrix}.$

Theorem 12.5 of [4] tells us that the sign characteristic determines the derivatives of the eigenvalue functions $\mu_1(\lambda)$, $\mu_2(\lambda)$ of the matrix function $L(\lambda)$ at the points where they cross the real axis. Obviously, in Case 1, the nature of these eigenvalue functions corresponds to the first sketch of Figure 1, and the eigenvalue functions are $\mu_1(\lambda) = \frac{1}{2}(\lambda^2-1),\ \mu_2(\lambda) = \frac{1}{2}(\lambda^2+1)$. The sign characteristic $\{-1,\ +1\}$ corresponds to the sign of the derivative of $\mu_1(\lambda)$ at the points $\lambda = -1$ and $\lambda = +1$, respectively.

The system of Case 2 has similar structure but now $\mu_1(\lambda) = \frac{1}{2}(\lambda^2 - 1),\ \mu_2(\lambda) = \lambda^2 + 1$.

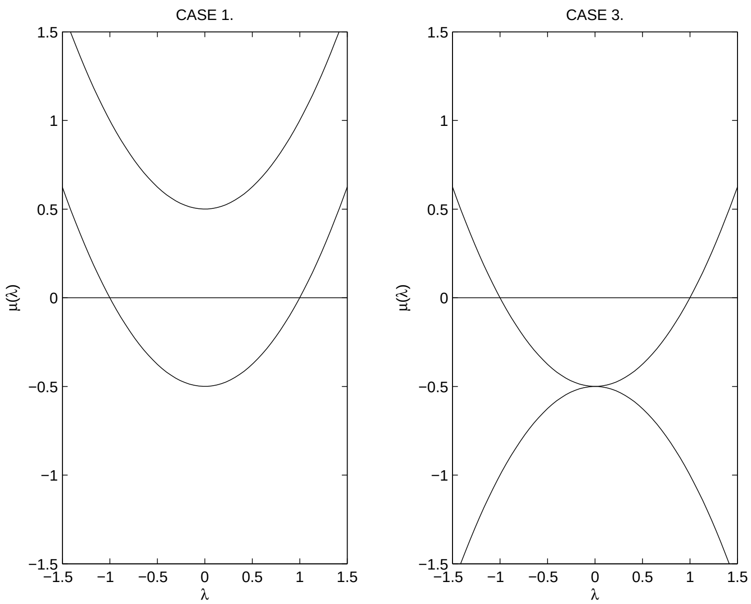

Figure 1: Eigenvalue functions $\mu_j(\lambda)$.

In contrast, Case 3 has $\mu_1(\lambda) = \frac{1}{2}(\lambda^2 - 1),\ \mu_2(\lambda) = -\frac{1}{2}(\lambda^2 + 1)$, and M is indefinite (Case 3 of Figure 1). However, it is clear from the sketches that both the real spectrum of $L(\lambda)$ and the sign characteristic are the same in every case.

13 APPENDIX C: The rank of complex symmetric matrices

Let M be a complex symmetric matrix in $\mathbb{C}^{n\times n}$. Thus, there are real symmetric matrices A and B such that $X = A + iB$. Our objective is to show how the ranks of matrices X, A and B are connected. Notice that there are no other hypotheses on A and B such as invertibility, or positivity.

The rank of a square matrix is invariant under congruence transformations so, if $S \in \mathbb{R}^{n\times n}$ is nonsingular, then rank$X$ =rank(SXS^T). Our problem will be resolved by applying a congruence with matrix S which simultaneously reduces A and B to a canonical form. The canonical forms in question can be found in the recent work [11], and are to be described here. It is convenient to use the language of spectral analysis and consider our problem in the context of the reduction of the *pencil* $A + \lambda B$ by real congruence.

The general canonical forms are quite complicated. They are block-diagonal with blocks of several different types as follows:

Square matrices F_m of size m with ones on the NE-SW diagonal and zeros elsewhere (also known as the *sip* matrices).

Matrices G_m:

$$G_m = \begin{bmatrix} 0 & \cdots & \cdots & 1 & 0 \\ \vdots & & & 0 & 0 \\ \vdots & & & & \vdots \\ 1 & 0 & & & \vdots \\ 0 & 0 & \cdots & \cdots & 0 \end{bmatrix} = \begin{bmatrix} F_{m-1} & 0 \\ 0 & 0 \end{bmatrix},$$

and matrices H_{2m}:

$$H_{2m} = \begin{bmatrix} 0 & 0 & & \cdots & & & 1 & 0 \\ 0 & & & & & & 0 & -1 \\ \vdots & & & 1 & 0 & & & \\ & & & 0 & -1 & & & \\ & & \cdot & & & & & \vdots \\ 1 & 0 & & & & & & 0 \\ 0 & -1 & & & & \cdots & 0 & 0 \end{bmatrix}.$$

Then the canonical form for A is a direct sum of blocks of (up to) five distinct types, say

$$A = A_1 \oplus A_2 \oplus A_3 \oplus A_4 \oplus A_5,$$

and similarly for B. The blocks A_r and B_r will have the same size for each r and are as follows:

1. $A_1 = B_1 = 0$ (a square zero matrix).

2. $A_2 = \sum_j \oplus G_{2\varepsilon_j+1}, \qquad B_2 = \sum_j \oplus \begin{bmatrix} 0 & 0 & F_{\varepsilon_j} \\ 0 & 0 & 0 \\ F_{\varepsilon_j} & 0 & 0 \end{bmatrix}.$

3. $A_3 = \sum_j \oplus \delta_j F_{k_j}, \qquad B_3 = \sum_j \oplus \delta_j G_{k_j},$
   where each δ_j is ± 1 and, together, they define the sign characteristic of the eigenvalue of $A + \lambda B$ at infinity (if any).

4. $A_4 = \sum_j \oplus(\eta_j \alpha_j F_{l_j} + G_{l_j}), \qquad B_4 = \sum_j \oplus \eta_j F_{l_j},$
   where each η_j is ± 1 and, together, they define the sign characteristic associated with the real eigenvalues (if any).

5. $A_5 = \sum_j \oplus(\mu_j F_{2m_j} + \nu_j H_{2m_j} + \begin{bmatrix} F_{2m_j-2} & 0 \\ 0 & 0_2 \end{bmatrix}), \qquad B_5 = \sum_j \oplus F_{2m_j},$
   where $\mu_j \pm i\nu_j$ with $\nu_j \neq 0$ are the complex eigenvalues (if any), and 0_2 is the 2×2 zero matrix.

It is clear that $\mathrm{rank}(A) = \sum_{r=1}^{5} \mathrm{rank}(A_r)$, $\mathrm{rank}(B) = \sum_{r=1}^{5} \mathrm{rank}(B_r)$, and $\mathrm{rank}(A+iB) = \sum_{r=1}^{5} \mathrm{rank}(A_r + iB_r)$. However, the ranks of these component matrices are easily obtained from those of the F's, G's and H's. Notice, in particular, that for the first four types the component diagonal blocks are triangular, and the rank can be read off by observation. For the fifth type, the structures will be clear if we just examine the case $m_j = 3$ more closely. It is easily seen that, if we define

$$\Delta_j := \begin{bmatrix} \nu_j & \mu_j \\ \mu_j & -\nu_j \end{bmatrix},$$

then, in this case,

$$A_5 = \begin{bmatrix} 0_2 & F_2 & \Delta_j + iF_2 \\ F_2 & \Delta_j + iF_2 & 0_2 \\ \Delta_j + iF_2 & 0_2 & 0_2 \end{bmatrix},$$

so that $\det(A_5) = 0$ if and only if $\det(\Delta_j + iF_2) = 0$ (and this is the case whatever the value of m_j). However,

$$\det(\Delta_j + iF_2) = -(\mu_j^2 + \nu_j^2) - 2i\mu_j + 1,$$

and this vanishes if and only if $\mu_j = 0$ and $\nu_j = \pm 1$, i.e. the corresponding complex eigenvalue pair is $\pm i$.

Notice also that, when $\mu_j + i\nu_j = i$, then

$$\Delta_j + iF_2 = \begin{bmatrix} 1 & i \\ i & -1 \end{bmatrix},$$

a matrix of rank one.

The important conclusion of this argument is:

Proposition 7

$$\mathrm{rank}(A_5) = 2\sum_j m_j, \tag{33}$$

where the sum is over all j associated with non-real eigenvalues, if and only if there are no eigenvalue pairs $\pm i$.

If there are such pairs, then the rank of A_5 is decreased from the value (33) *by the algebraic multiplicity of the eigenvalue i (or $-i$).*

It will be convenient to denote the algebraic multiplicty of the eigenvalue i by $a(i)$ so that, in general,

$$\mathrm{rank}(A_5) = 2\sum_j m_j - a(i).$$

Now it can be seen that:

1. $\mathrm{rank}(A_1) = \mathrm{rank}(B_1) = \mathrm{rank}(A_1 + iB_1) = 0$.

2. $\mathrm{rank}(A_2) = \mathrm{rank}(B_2) = \mathrm{rank}(A_2 + iB_2) = \sum_j 2\varepsilon_j$.

3. $\mathrm{rank}(A_3) = \sum_j k_j$, $\quad \mathrm{rank} B_3 = \sum_j (k_j - 1)$, $\quad \mathrm{rank}(A_3 + iB_3) = \sum_j k_j$.

4. $\mathrm{rank}(A_4) = \sum_{j:\alpha_j \neq 0} l_j + \sum_{j:\alpha_j = 0}(l_j - 1)$, $\quad \mathrm{rank}(B_4) = \sum_j l_j$, $\quad \mathrm{rank}(A_4 + iB_4) = \sum_j l_j$.

5. $\mathrm{rank}(A_5) = 2\sum_j m_j - a(i)$, $\quad \mathrm{rank}(B_5) = 2\sum_j m_j$, $\quad \mathrm{rank}(A_5 + iB_5) = 2\sum_j m_j$.

Now consider how the ranks of A and B can differ from that of $A + iB$. Items 1 and 2 produce no differences. Due to Item 3, however, the rank of B_3 (and hence B) is less than the other two ranks by one for *each canonical block* associated with the eigenvalue at infinity (if there is such an eigenvalue), i.e. the geometric multiplicity of the eigenvalue at infinity , say $g(\infty)$.

Similarly, it can be deduced from Item 4 that $\mathrm{rank}(A_4)$ is less than $\mathrm{rank}(B_4)$ and $\mathrm{rank}(A_4 + iB_4)$ by the geometric multiplicity of the zero eigenvalue, say $g(0)$. The case of blocks of the fifth type is covered by Proposition 7.

Since $\mathrm{rank}(A) = \sum_{r=1}^{5} \mathrm{rank}(A_r)$, and $\mathrm{rank}(B) = \sum_{r=1}^{5} \mathrm{rank}(B_r)$, the results can be brought together in the following form (and we keep in mind that $a(i)$, $g(o)$ and $g(\infty)$ refer to eigenvalues of the pencil $A + \lambda B$):

Theorem 8

$$\mathrm{rank}(A + iB) = \mathrm{rank}(A) - a(i) + g(0) = \mathrm{rank}(B) - a(i) + g(\infty).$$

EXAMPLES: Let us illustrate with Example 4 of the main text. The first case arising there is $A + iB = \frac{1}{2}\begin{bmatrix} -1 & -i \\ -i & 1 \end{bmatrix}$, so that

$$A + iB = \frac{1}{2}\begin{bmatrix} -1 & -\lambda \\ -\lambda & 1 \end{bmatrix}.$$

We have $a(i) = 1,\ g(0) = g(\infty) = 0$ and $\mathrm{rank}(A + iB) = 1$.

The second case is $A + iB = \begin{bmatrix} 0 & 0 \\ 0 & i \end{bmatrix}$, so that

$$A + iB = \begin{bmatrix} 0 & 0 \\ 0 & \lambda \end{bmatrix}.$$

Notice first that, in this case, $A_1 = B_1 = [0]$. Then $a(i) = 0,\ g(0) = 1,\ g(\infty) = 0$ and $\mathrm{rank}(A + iB) = 1$. □

References

[1] Bunse-Gerstner, A., and Gragg, W. B., *Singular value decompositions of complex symmetric matrices*, Journal of Computational and Applied Mathematics, **21**, 1988, 41-54.

[2] Duffin, R.J., *A minimax theory for overdamped networks*, Journal of Rational mechanics and Analysis, **4**, 1955, 221-233.

[3] Gohberg, I., Lancaster, P. and Rodman, L., *Spectral analysis of selfadjoint matrix polynomials*, Annals of Mathematics, **112**, 1980, 34-71.

[4] Gohberg, I., Lancaster, P. and Rodman, L., *Matrix Polynomials*, Academic Press, New York, 1982.

[5] Golub, G., and Van Loan, C., *Matrix Computations*, 3rd. Edition, Johns Hopkins University Press, Baltimore, 1996.

[6] Guo, Chun-Hua, and Lancaster, P., *Algorithms for hyperbolic quadratic eigenvalue problems*, Mathematics of Computation, (to appear).

[7] Higham, N.J., Tisseur, F., and Van Dooren, P. M., *Detecting a definite Hermitian pair and a hyperbolic or elliptic quadratic eigenvalue problem, and associated nearness problems*, Linear Algebra and its Applications, **351/352**, 2002, 455-474.

[8] Horn, R. A., and Johnson, C. R., *Matrix Analysis*, Cambridge University Press, Cambridge, 1985.

[9] Lancaster, P., *Isospectral vibrating systems. Part 1: The spectral method*, Linear Algebra and its Applications, (to appear).

[10] Lancaster, P., and Prells, U., *Inverse problems for vibrating systems*, Journal of Sound and Vibration, (to appear).

[11] Lancaster, P., and Rodman, L., *Canonical forms for hermitian matrix pairs under strict equivalence and congruence*, SIAM Review, (to appear).

[12] Lancaster, P., and Tismenetsky, M., *The Theory of Matrices*, Academic Press, Orlando, 1985.

[13] Lancaster, P., and Tismenetsky, M., *Inertia characteristics of self-adjoint matrix polynomials*, Linear Algebra and its Applications, **52/53**, 1983, 479-496.

[14] Lancaster, P. and Ye, Q., *Inverse spectral problems for linear and quadratic matrix pencils*, Linear Algebra and its Applications, **107**, 1988, 293-309.

[15] Prells, U., and Lancaster, P., *Isospectral vibrating systems. Part 2: Structure preserving transformations* Operator Theory: Advances and Applications, (to appear).

[16] Tisseur, F., and Meerbergen, K., *The quadratic eigenvalue problem*, SIAM Review, **43**, 2001, 235-286.