Isospectral Vibrating Systems,
Part 2: Structure Preserving Transformations

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January 24, 2005

Abstract

The study of inverse problems for $n \times n$ systems of the form $L(\lambda) := M\lambda^2 + D\lambda + K$ is continued. In this paper it is assumed that one vibrating system is specified and the objective is to generate isospectral families of systems, i.e. systems which reproduce precisely the eigenvalues of the given system together with their multiplicities. Two central ideas are developed and used, namely, standard triples of matrices, and structure preserving transformations.

1 Introduction

A broad spectrum of physical problems concerning vibrations is covered by classical models of second order constant-coefficient differential equations. In many cases it is natural to refer to the three coefficient matrices as the mass (M), damping (D), and stiffness matrices (K) and we will use this terminology. However, by admitting the possibility of complex coefficients (not just real matrices), the number of physical problems covered is enlarged (admitting gyroscopic effects, for example) and, not only this, it turns out that a broader mathematical analysis is obtained which casts some light on those problems in which M, D, K are real. This leads to our first definition:

Definition 1. A *(vibrating) system* is a triple of $n \times n$ complex matrices $\{M, D, K\}$ for which M is nonsingular.

When convenient, the quadratic matrix function $L(\lambda) := \lambda^2 M + \lambda D + K$ may also be described as a vibrating system.

It is well-known that the solutions of the corresponding differential equations can be described entirely in terms of the solutions of the *algebraic* eigenvalue problem: find those $\lambda \in \mathbb{C}$ and $x \in \mathbb{C}^n$ for which

$$L(\lambda)x = (\lambda^2 M + \lambda D + K)x = 0.$$

In Part 1 of this work (reference [5]) it was shown how, in some cases, the three coefficient matrices can be recovered if (well-defined) complete information is given on the eigenvalue and eigenvector structures (known as the spectral data). Here, the objective is to start with one system, say $(M_0,\ D_0,\ K_0)$, and, implicitly, a complete set of corresponding spectral data, and to show how to generate *isospectral* vibrating systems, i.e. systems $(M,\ D,\ K)$ which share the same set of eigenvalues (including their multiplicity structures). And this is to be done without explicit reference to eigenvalues and eigenvectors.

Our objective will be achieved using two parallel (but closely linked) notions. The first is the tool of "standard triples" of matrices for $L(\lambda)$ introduced and developed in great detail in [3] and earlier works cited there (see Section 3). The second is the idea of "structure preserving transformations" developed in this context in [2] (and introduced here in Section 2). Computational procedures for solution of the problem for general systems (with no symmetries) are contained in Section 5. Sections 6,7, and 8 are concerned with the important (and more complex) problems in which M, D, K are required to be hermitian (or real symmetric). It is of interest to ask when there is an isospectral system of simplest possible form - in which all three coefficients are diagonal. This is the topic of Section 9. We conclude this introduction with a discussion of the basic idea of linearisation.

It is very well-known that linear second order systems can be transformed to first order in an elementary way; generally known as "linearisation". Those linearisations associated with vibrating systems are our present concern. In the first two sections their properties under transformations of three types are reviewed, namely, "equivalence", "strict equivalence", and "similarity" in decreasing order of generality. There have been several recent publications concerning equivalence and similarity transformations of vibrating systems. The authors' formulation of these ideas together with new results appear in Sections 2 and 4 (especially Theorem 7 and its Corollary).

Definition 2. Let $L(\lambda)$ be a vibrating system. A $2n \times 2n$ matrix pencil $\lambda X - Y$ is a *linearisation* of $L(\lambda)$ if

$$\begin{bmatrix} L(\lambda) & 0 \\ 0 & I_n \end{bmatrix} = E(\lambda)(\lambda X - Y)F(\lambda) \tag{1}$$

for some matrix polynomials $E(\lambda)$, $F(\lambda)$ with constant nonzero determinants.

Notice that it follows from this definition that the matrix X of a linearisation is necessarily nonsingular. (This follows from the assumption that M is nonsingular.) The transformation of $\lambda X - Y$ on the right-hand-side of (1) is known as an *equivalence transformation*. The "equivalence class" of all pencils equivalent to $\lambda X - Y$ in this sense determines the set of all linearisations of $L(\lambda)$. A vital property of equivalence transformations is that they preserve the set of eigenvalues and their multiplicity structures. Thus a linearisation of a vibrating system shares its "spectrum" (its eigenvalues - with their multiplicities); it is *isospectral.*

Define the right and left companion matrices, C_R and C_L, of a vibrating system by:

$$C_R = \begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}D \end{bmatrix}, \qquad C_L = \begin{bmatrix} 0 & -KM^{-1} \\ I & -DM^{-1} \end{bmatrix}. \tag{2}$$

The first lemma is very well-known:

Lemma 1 *The following pencils are linearisations of the vibrating system* $L(\lambda)$*:*

1. $\lambda A - B$ *with*

$$A := \begin{bmatrix} D & M \\ M & 0 \end{bmatrix}, \quad B := \begin{bmatrix} -K & 0 \\ 0 & M \end{bmatrix}, \tag{3}$$

2.

$$\lambda I_{2n} - A^{-1}B = \lambda I_{2n} - C_R,$$

3.

$$\lambda I_{2n} - BA^{-1} = \lambda I_{2n} - C_L.$$

2 Structure preserving transformations

If E and F are nonsingular $2n$-by-$2n$ matrices in $\mathbb{C}^{2n\times 2n}$, then the pencils $\lambda A - B$ and $E(\lambda A - B)F$ are said to *strictly equivalent.* Clearly, a strict equivalence is also an equivalence in the sense used in Definition 2 *et seq.*, and so strict equivalence also preserves the spectrum. The set of all pencils strictly equivalent to $\lambda A - B$ in this sense form an equivalence class. (The relation of "strict equivalence" is reflexive, symmetric and transitive.)

Now consider the linearisation $\lambda A - B$ of (3) and let $\lambda A' - B'$ be a strictly equivalent pencil, i.e. there exist E and F such that $A' = EAF$, $B' = EBF$. The question is, when does $\lambda A' - B'$ correspond to a vibrating system? The answer is, when A', B' have the defining properties of A and B, namely, when there exist M', D', K' with M' nonsingular such that

$$A' = \begin{bmatrix} D' & M' \\ M' & 0 \end{bmatrix}, \qquad B' = \begin{bmatrix} -K' & 0 \\ 0 & M' \end{bmatrix}. \tag{4}$$

Definition 3. (cf. Garvey et al. [2].) Let $\lambda A - B$ be defined from a vibrating system as in (3). Let $\lambda A' - B'$ be obtained from $\lambda A - B$ by a strict equivalence. Then this strict equivalence is said to be *structure preserving* if and only if A', B' have the form (4) and M' is nonsingular. □

For brevity, a structure preserving strict equivalence of this kind is called an SPE. When the linearisations of two vibrating systems are connected in this way they will be said to be *related by an SPE.* Observe that strict equivalence transformations of $L(\lambda)$ itself, say $L(\lambda) \mapsto SL(\lambda)T$ where S and T are nonsingular, are included in this definition. Such a transformation corresponds to an SPE with

$$E = \begin{bmatrix} S & 0 \\ 0 & S \end{bmatrix}, \qquad F = \begin{bmatrix} T & 0 \\ 0 & T \end{bmatrix}.$$

Definition 4. (Lancaster and Prells [7].) Let C_R be the right companion matrix (2) of $L(\lambda)$. The nonsingular $2n$-by-$2n$ matrix S_R is called a *(right) structure preserving similarity* (SPS) if $S_R^{-1}C_RS_R = C_R'$ is a (right) companion matrix. □

(Clearly, a similar definition can be made in terms of the left companion matrix.) When the companion matrices of two vibrating systems are connected in this way they will be said to be *related by an SPS*. The close relationship between SPE and SPS transformations is the subject of the easily proved result:

Theorem 2 *Two vibrating systems are related by an SPE if and only if they are related by an SPS.*

Proof. Consider a vibrating system $L_0(\lambda) = \lambda^2 M_0 + \lambda D_0 + K_0$. Let

$$A_0 := \begin{bmatrix} D_0 & M_0 \\ M_0 & 0 \end{bmatrix}, \quad B_0 := \begin{bmatrix} -K_0 & 0 \\ 0 & M_0 \end{bmatrix}, \tag{5}$$

and let $C_R^{(0)}$ be the corresponding right companion matrix. Consider the SPE $\lambda A - B = E(\lambda A_0 - B_0)F$. Since the structure is preserved A and B have the form of equations (3).This implies that A is nonsingular and

$$A^{-1}B = (EA_0F)^{-1}(EB_0F) = F^{-1}A_0^{-1}E^{-1}EB_0F = F^{-1}(A_0^{-1}B_0)F,$$

i.e. $C_R = F^{-1}C_R^{(0)}F$, so that the companion matrices C_R and $C_R^{(0)}$ are similar. Thus, the systems are related by an SPS.

Conversely, let $C_R = S^{-1}C_R^{(0)}S$ be an SPS (with the definitions above for C_R, $C_R^{(0)}$, A, B, A_0, B_0). Then

$$\lambda A - B = A(\lambda I - C_R) = A(\lambda I - S^{-1}C_R^{(0)}S) = (AS^{-1})(\lambda I - C_R^{(0)})S = (AS^{-1}A_0^{-1})(\lambda A_0 - B_0)S,$$

and this is an SPE. □

Corollary 3 *Let A, B be defined by a vibrating system as in* (3) *and let $E(\lambda A - B)F = \lambda A_0 - B_0$ be an SPE, then E defines an SPS for C_L, ($C_L = EC_L^{(0)}E^{-1}$) and F defines an SPS for C_R, ($C_R = F^{-1}C_R^{(0)}F$).*

Proof It has been shown in the above argument that, given the SPE defined by matrices E and F, the matrix F defines an SPS for C_R. In a similar way it follows that E defines an SPS for C_L. □

These results suggest that one might equally well work with SPS transformations as with SPE's (which require more parameters). Indeed, if one is interested only in *monic* systems (with $M = I$), SPS transformations tell the whole story. Otherwise, SPS transformations tell us about the products $M^{-1}D$ and $M^{-1}K$ and it remains to specify M itself, and hence D and K. In contrast, SPE's are defined in terms of the explicit coefficients M, D, and K. This point becomes more significant when working with systems for which the three coefficients are hermitian (or real symmetric) in which case the corresponding symmetry is reflected explicitly in A and B of (3).

3 Standard pairs and triples

The following notions of "standard pairs and triples" play an important unifying role in the discussion of linearisations (see [3] and earlier references given there). Our definitions differ in some respects from earlier sources. They are consistent with [5], and are formulated so that they are better suited to inverse problems - they do not refer explicitly to the matrix coefficients of the vibrating system.

Definition 5. A pair of matrices $U \in \mathbb{C}^{n\times 2n}$ and $T \in \mathbb{C}^{2n\times 2n}$ form a *standard pair* for a vibrating system if

(a) the dimension of each eigenspace of T does not exceed n, and

(b) the $2n \times 2n$ matrix $\begin{bmatrix} U \\ UT \end{bmatrix}$ is nonsingular. □

Note that the technical condition (a) is vital if T is to relate to a vibrating system (see Theorem 1.7 of [3]).

Definition 6. Three matrices $U \in \mathbb{C}^{n\times 2n}$ and $T \in \mathbb{C}^{2n\times 2n}$ and $V \in \mathbb{C}^{2n\times n}$ form a *standard triple* if $(U,\, T)$ is a standard pair, $UV = 0$ and the $n \times n$ matrix UTV is nonsingular. □

Notice that, if $(U,\, T,\, V)$ is a standard triple, then

$$\begin{bmatrix} U \\ UT \end{bmatrix} V = \begin{bmatrix} 0 \\ UTV \end{bmatrix}.$$

Since UTV is nonsingular (by definition) we may define $UTV = M^{-1}$ so that the equation

$$\begin{bmatrix} U \\ UT \end{bmatrix} V = \begin{bmatrix} 0 \\ M^{-1} \end{bmatrix} \tag{6}$$

is satisfied. In other words, given only a standard pair, the third member of a triple can be obtained by first assigning a nonsingular M and then solving equation (6) for V.

Important examples of standard triples associated with vibrating systems involve the companion matrices of (2) and are:

$$U_1 = \begin{bmatrix} I_n & 0 \end{bmatrix}, \quad T_1 = C_R = \begin{bmatrix} 0 & I_n \\ -M^{-1}K & -M^{-1}D \end{bmatrix}, \quad V_1 = \begin{bmatrix} 0 \\ M^{-1} \end{bmatrix}, \tag{7}$$

and

$$U_2 = \begin{bmatrix} 0 & M^{-1} \end{bmatrix}, \quad T_2 = C_L = \begin{bmatrix} 0 & -KM^{-1} \\ I_n & -DM^{-1} \end{bmatrix}, \quad V_2 = \begin{bmatrix} I_n \\ 0 \end{bmatrix}, \tag{8}$$

(see [3] for details). It is easy to see that, if $(U,\, T,\, V)$ is a standard triple then the three matrices obtained by a (generalized) similarity,

$$U_1 = UP^{-1}, \quad T_1 = PTP^{-1}, \quad V_1 = PV, \tag{9}$$

also form a standard triple. For example, the similarity from (U_1, T_1, V_1) to (U_2, T_2, V_2) in the examples of (7) and (8) is determined by the matrix A of equation (3), i.e. in this case $P = A = \begin{bmatrix} D & M \\ M & 0 \end{bmatrix}$.

Another important case is obtained by choosing P in (9) to be a matrix for which PTP^{-1} is in Jordan canonical form. The resulting (very special) standard triple is labeled a *Jordan triple* (see [3] and [5]).

Now let (U, T, V) be a standard triple for a vibrating system and consider the $n \times n$ matrices Γ_0, Γ_1, $\Gamma_2, \ldots$ defined by

$$\Gamma_j = UT^jV, \qquad j = 0, 1, 2, \ldots . \tag{10}$$

They are known as the *moments* of the triple. Observe that, by Definition 6, $\Gamma_0 = 0$ and Γ_1 is nonsingular. (Also, if it is known that T is nonsingular then moments can be defined for negative integers j.)

It is an important fact that *any transformation to a similar triple does not affect the moments*:

$$\Gamma_j = UT^jV = (UP^{-1})(PT^jP^{-1})(PV) = (UP^{-1})(PTP^{-1})^j(PV).$$

Thus, for a vibrating system the moments are the same whether they are found from the triples of (7), (8), or from a Jordan triple. This invariance property suggests that the moments reflect intrinsic properties of the vibrating system. Indeed, it has been shown in Theorem 2 of [5] that the moments generate a unique corresponding vibrating system through the recursive equations

$$M = \Gamma_1^{-1}, \quad D = -M\Gamma_2M, \quad K = -M(\Gamma_3)M + D\Gamma_1D. \tag{11}$$

Furthermore, for any standard pair, (U, T),

$$MUT^2 + DUT + KU = 0. \tag{12}$$

Theorem 4 *Let (U, T, V) be any standard triple and define moments by* (10)*. Then equations* (11) *generate a unique corresponding vibrating system.*

Furthermore, any two standard triples for the same system are similar (in the sense of equations (9)*).*

Proof The first statement is obtained by adapting the proof of Theorem 2 of [5] to standard triples, rather than Jordan triples. The second follows from Theorem 1.25 of [3], for example. □

Naturally, the moments generated by a standard triple may now be described without ambiguity as *the moments of a vibrating system.*

Corollary 5 *If $(U,\ T,\ V)$ is a standard triple for a vibrating system $L(\lambda)$, then $\lambda I - T$ is a linearisation for $L(\lambda)$.*

Proof Note that the standard triple (7) has C_R as the main matrix and (Lemma 1) $\lambda I - C_R$ is a linearisation. Also, if $(U,\ T,\ V)$ is any standard triple, then the second statement of the theorem implies that C_R and T are similar. Hence, $\lambda I - C_R = P(\lambda I - T)P^{-1}$ and it follows from Definition 2 that $\lambda I - T$ is a linearisation. □

It is useful to note that the transforming matrix P used in the last step of this proof can be written explicitly in the form $P = \begin{bmatrix} U \\ UT \end{bmatrix}$. To see this, first use (12) to verify that $C_R P = PT$, and hence $C_R = PTP^{-1}$.

The following technical lemma concerning the moments will assist in the proof of a theorem showing how SPE (and SPS) transformations can be obtained from standard triples.

Lemma 6 (a)
$$\begin{bmatrix} \Gamma_0 & \Gamma_1 \\ \Gamma_1 & \Gamma_2 \end{bmatrix} = A^{-1}.$$

(b)
$$\begin{bmatrix} \Gamma_1 & \Gamma_2 \\ \Gamma_2 & \Gamma_3 \end{bmatrix} = A^{-1} \begin{bmatrix} -K & 0 \\ 0 & M \end{bmatrix} A^{-1}.$$

Proof. (a) Using equations (11) we have

$$A^{-1} = \begin{bmatrix} D & M \\ M & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & M^{-1} \\ M^{-1} & -M^{-1}DM^{-1} \end{bmatrix} = \begin{bmatrix} \Gamma_0 & \Gamma_1 \\ \Gamma_1 & \Gamma_2 \end{bmatrix}.$$

(b) Using equations (11) again,

$$\begin{bmatrix} \Gamma_1 & \Gamma_2 \\ \Gamma_2 & \Gamma_3 \end{bmatrix} = \begin{bmatrix} M^{-1} & -M^{-1}DM^{-1} \\ -M^{-1}DM^{-1} & M^{-1}(DM^{-1}D - K)M^{-1} \end{bmatrix}.$$

Multiplying on left and right with A gives

$$A \begin{bmatrix} \Gamma_1 & \Gamma_2 \\ \Gamma_2 & \Gamma_3 \end{bmatrix} A = \begin{bmatrix} 0 & -KM^{-1} \\ I & -DM^{-1} \end{bmatrix} \begin{bmatrix} D & M \\ M & 0 \end{bmatrix} = \begin{bmatrix} -K & 0 \\ 0 & M \end{bmatrix},$$

from which (b) follows. □

4 Constructing SPE and SPS transformations

The next theorem is the main result of this paper. First recall the standard triple (7) for a vibrating system. To study isospectral systems it is convenient to write associated standard triples in the form (X, C_R, Y) where X and Y are still to be determined. The "main" matrix C_R is then common to two systems, so they must be isospectral. Furthermore, C_R automatically satisfies the condition (a) of Definition 5. Since triples for one system are all similar to one another there is no loss of generality in this assumption.

Now it will also be shown how SPE and SPS transformations can be constructed from standard triples. In the process, a constructive method for finding isospectral families of systems will appear.

Theorem 7 *Let a vibrating system $L_0(\lambda)$ be given and matrices A_0 and B_0 be formed as in* (5)*, and write $C_0 = A_0^{-1}B_0$. Consider any standard triple of the form (X, C_0, Y).*

Then the $2n \times 2n$ matrices $\begin{bmatrix} A_0Y & B_0Y \end{bmatrix}$ and $\begin{bmatrix} X \\ XC_0 \end{bmatrix}$ are nonsingular and

$$E := \begin{bmatrix} Y & C_0Y \end{bmatrix}^{-1} A_0^{-1}, \quad F := \begin{bmatrix} X \\ XC_0 \end{bmatrix}^{-1} \tag{13}$$

determine an SPE of the given system.

Furthermore, if the moments of the transformed system $L(\lambda)$ *are written* $\Gamma_j = XC_0^jY$, *then the transformed coefficients are*

$$M = (\Gamma_1)^{-1}, \quad D = -M\Gamma_2M, \quad K = -M\Gamma_3M + D\Gamma_1D. \tag{14}$$

Before going into the proof of this theorem, the importance of two properties of the standard triple $(X,\, C_0,\, Y)$ are emphasised. Namely, that

$$XY = 0, \quad \text{and} \quad \det(XC_0Y) \neq 0. \tag{15}$$

Keep in mind that A_0 and B_0 (and hence C_0) are prescribed by the coefficients M_0, D_0, K_0 of a given vibrating system. The design of isospectral systems depends only on these two conditions and, generally, they can be satisfied in infinitely many ways. The first can be interpreted as a biorthogonality condition imposed on the rows of X and columns of Y, and the second as a non-degeneracy condition which, generically, will be satisfied once the first condition is resolved. This interpretation is particularly useful in the discussion of systems with hermitian (or real symmetric) coefficients. Conditions (15) are also natural generalisations of corresponding conditions used in Part I of this work, [5]. There, the analysis is confined to the case of *Jordan triples*, a special case of the present formulation. In Part I the data consists of suitable sets of eigenvalue and eigenvector data. Here, the data consists of the coefficients M_0, D_0, K_0 of a given system and the spectral information remains implicit.

Proof of Theorem 7. Consider the following product and use (15) to obtain

$$\begin{bmatrix} X \\ XC_0 \end{bmatrix} \begin{bmatrix} Y & C_0Y \end{bmatrix} = \begin{bmatrix} 0 & XC_0Y \\ XC_0Y & XC_0^2Y \end{bmatrix}$$

But we also have $\det(XC_0Y) \neq 0$ and it follows that the product is nonsingular. So each factor on the left is nonsingular and E, F can be defined as in (13). So we may define the strict equivalence

$$\lambda A - B = E(\lambda A_0 - B_0)F. \tag{16}$$

It is to be shown that this is an SPE.

Since $A = EA_0F$ it follows that A is nonsingular and

$$\begin{aligned} A &= EA_0F = \begin{bmatrix} Y & C_0Y \end{bmatrix}^{-1} \begin{bmatrix} X \\ XC_0 \end{bmatrix}^{-1}, \\ &= \begin{bmatrix} XY & XC_0Y \\ XC_0Y & XC_0^2Y \end{bmatrix}^{-1} = \begin{bmatrix} \Gamma_0 & \Gamma_1 \\ \Gamma_1 & \Gamma_2 \end{bmatrix}^{-1}. \end{aligned}$$

Now use Part (a) of Lemma 6 to obtain

$$A = \begin{bmatrix} D & M \\ M & 0 \end{bmatrix},$$

and the structure of A_0 is preserved.

For the transformation of B_0, observe that

$$B_0 = E^{-1}BF^{-1} = \begin{bmatrix} A_0Y & B_0Y \end{bmatrix} B \begin{bmatrix} X \\ XC_0 \end{bmatrix}.$$

Multiply on the left and on the right by

$$\begin{bmatrix} X \\ XC_0 \end{bmatrix} A_0^{-1} \quad \text{and} \quad \begin{bmatrix} Y & C_0Y \end{bmatrix},$$

respectively, to obtain

$$\begin{bmatrix} \Gamma_1 & \Gamma_2 \\ \Gamma_2 & \Gamma_3 \end{bmatrix} = \begin{bmatrix} \Gamma_0 & \Gamma_1 \\ \Gamma_1 & \Gamma_2 \end{bmatrix} B \begin{bmatrix} \Gamma_0 & \Gamma_1 \\ \Gamma_1 & \Gamma_2 \end{bmatrix}.$$

Using Part (a) of Lemma 6 it follows that

$$B = A \begin{bmatrix} \Gamma_1 & \Gamma_2 \\ \Gamma_2 & \Gamma_3 \end{bmatrix} A.$$

Finally, Part (b) of Lemma 6 yields

$$B = \begin{bmatrix} -K & 0 \\ 0 & M \end{bmatrix}.$$

Thus, the structure of B_0 is also preserved and we have an SPE. □

Notice also that an application of Corollary 3 gives:

Corollary 8 *With E and F defined as in Theorem 7, E defines an SPS for C_L, ($C_L = EC_L^{(0)}E^{-1}$) and F defines an SPS for C_R, ($C_R = F^{-1}C_R^{(0)}F$).*

Example 1. Consider the vibrating system defined by

$$M_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad D_0 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad K_0 = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix},$$

with (truncated) eigenvalues

$$-0.9567 \pm 0.6412i, \quad \text{and} \quad -0.0433 \pm 1.2272i.$$

(Note that, although D is singular, the damping is pervasive.) Making the (almost arbitrary) choice

$$X = \begin{bmatrix} 1 & 2 & 0 & -3 \\ 0 & 1 & 1 & 2 \end{bmatrix},$$

it is found that $\begin{bmatrix} X \\ XC_0 \end{bmatrix}$ is nonsingular. Since C_0 is a companion matrix, condition (a) of Definition 5 is satisfied and it follows that (X, C_0) form a standard pair.

Assign the mass matrix $M = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, and compute the third member of the standard triple (cf. equation (6)):

$$Y = \frac{1}{5}\begin{bmatrix} -5 & 0 \\ 1 & 3 \\ 1 & -7 \\ -1 & 2 \end{bmatrix}.$$

Now moments Γ_1, Γ_2 and Γ_3 can be computed (as in equation (10)), and the new (exact) coefficients come from (14):

$$M = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} -0.8 & -5.4 \\ 1.6 & 2.8 \end{bmatrix}, \quad K = \begin{bmatrix} -1.4 & -4.0 \\ -0.2 & -2 \end{bmatrix}.$$

Calculations confirm that the spectrum is unchanged.

Alternatively, after finding Y as above, E and F could be computed from equations (13) and then the coefficients of the system are obtained from $A = EA_0F$, $B = EB_0F$. $\square$

5 Algorithms

The computational procedures for finding isospectral systems resulting from Theorem 7 are summarised in this section. Some simplifications arise if one is interested only in monic systems. The reader will easily make these adjustments when required.

ALGORITHM 1. **The SPE method.**

DATA: Coefficients M_0, D_0, K_0 of a vibrating system and corresponding matrices

$$A_0 := \begin{bmatrix} D_0 & M_0 \\ M_0 & 0 \end{bmatrix}, \quad B_0 := \begin{bmatrix} -K_0 & 0 \\ 0 & M_0 \end{bmatrix}.$$

STEP 1. Compute the matrix C_R of equation (2).

STEP 2. Choose an $n \times 2n$ matrix X for which $\begin{bmatrix} X \\ XC_R \end{bmatrix}$ is nonsingular.

STEP 3. Choose a new mass matrix M and solve for Y:

$$\begin{bmatrix} X \\ XC_R \end{bmatrix} Y = \begin{bmatrix} 0 \\ M^{-1} \end{bmatrix}.$$

STEP 4. Compute

$$E := \begin{bmatrix} Y & C_RY \end{bmatrix}^{-1} A_0^{-1}, \quad F := \begin{bmatrix} X \\ XC_R \end{bmatrix}^{-1}$$

STEP 5. Compute $A = EA_0F$, and EB_0F and read off the new coefficients D and K.
END

ALGORITHM 2. **The moment method.**

DATA: Coefficients M_0, D_0, K_0 of a vibrating system.

STEPS 1, 2, and 3: As steps 1, 2, and 3 of Algorithm 1.

STEP 4. Compute the moments $\Gamma_2 = XC_R^2Y$, $\Gamma_3 = XC_R^3Y$.

STEP 5. Compute the new coefficients

$$D = -M\Gamma_2 M, \qquad K = -M(\Gamma_3 - \Gamma_2 M\Gamma_2)M.$$

END

Of course, these summaries cannot be described as efficient general purpose algorithms. At a superficial level, Algorithm 1 may seem more direct. On the other hand, Algorithm 2 may gain in efficiency because it is formulated in terms of matrices of size n rather than $2n$, as in the first algorithm.

6 Systems with symmetries

In this section the study of systems isospectral with a given system (M_0, D_0, K_0) is continued, but now all of these matrices are hermitian (in particular, they may be real and symmetric). Isospectral systems are to be generated which preserve the symmetry of (M_0, D_0, K_0).

Here, an important role is played by the nonsingular hermitian matrix

$$A_0 = \begin{bmatrix} D_0 & M_0 \\ M_0 & 0 \end{bmatrix}. \tag{17}$$

Continuing the practice of Section 4, C_0 will denote the right companion matrix associated with the given system.

Matrix A_0 is always indefinite with inertia $\{n, n, 0\}$, and it has the important property that it symmetrises C_0 in the sense that $A_0C_0 = C_0^*A_0 = (A_0C_0)^*$. (The * denotes the complex-conjugate transposed matrix.) The matrix C_0 is said to be *self-adjoint in the indefinite inner product defined by* A_0, see [4]. Indeed, it is easy to prove the more general statement:t

Lemma 9 *The matrix* A_0 *of* (17) *has the property that, for* $j = 0, 1, 2, \ldots$,

$$A_0C_0^j = (A_0C_0^j)^* = (C_0^j)^*A_0.$$

Let (X, C_0, Y) be a standard triple generated as in the algorithms of Section 5, for example. It will now be shown that if X and Y satisfy a geometric constraint, then the system generated will be self-adjoint (in the sense that the coefficients are hermitian). But first note the following lemma:

Lemma 10 *Let* (X, T, Y) *be a standard triple for a vibrating system* (M, D, K). *Then* M, D *and* K *are hermitian if and only if the moments* Γ_1, Γ_2, *and* Γ_3 *are hermitian.*

Proof. The proof is an easy verification using the formulae of (11) (or see [5]). □

Proposition 11 *Let (X, C_0, Y) be a standard triple. Then the corresponding vibrating system has hermitian coefficients if $Y = A_0^{-1}X^*$.*

Proof If $Y = A_0^{-1}X^*$ then $X = Y^*A_0$ and

$$\Gamma_j = XC_0^jY = Y^*(A_0C_0^j)Y.$$

But, by Lemma 9, $A_0C_0^j = (A_0C_0^j)^*$ and it follows that Γ_j is hermitian for each j. Then it follows from Lemma 10 that the corresponding system has hermitian coefficients. □

Now the condition $X = Y^*A_0$ is reformulated in terms of X alone, and the geometric interpretation will be clarified. Since $X^* = A_0Y$, the definition of Y in terms of X and C_0 (from Section 3) gives

$$X^* = A_0 \begin{bmatrix} X \\ XC_0 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ M^{-1} \end{bmatrix},$$

whence

$$\begin{bmatrix} X \\ XC_0 \end{bmatrix} A_0^{-1}X^* = \begin{bmatrix} 0 \\ M^{-1} \end{bmatrix}.$$

Thus, hermitian coefficients are generated provided the two following conditions are satisfied:

$$\begin{aligned} XA_0^{-1}X^* &= 0, & (18) \\ X(C_0A_0^{-1})X^* &= M^{-1}. & (19) \end{aligned}$$

These equations say that, *first, the subspace ImX^* must be isotropic with respect to the known indefinite matrix A_0^{-1} and, at the same time, (if M is to be positive definite) this subspace must be positive with respect to the hermitian matrix $C_0A_0^{-1}$.*

(Here, it is simply convenient to suppose that the ubiquitous condition $M > 0$ applies.) Naturally, equations (18) and (19) are the symmetrised form of the fundamental equations (15). They are also the analogues of very similar conditions formulated in [5] in terms of canonical forms for A_0 and C_0.

A procedure for generating hermitian (or real and symmetric) isospectral systems can now be formulated as follows:

ALGORITHM 3. **Hermitian, or real symmetric systems.**

STEP 1. From the given hermitian (or real symmetric) system $\{M_0, D_0, K_0\}$ with $M_0 > 0$ formulate

$$C_0 = \begin{bmatrix} 0 & I \\ -M_0^{-1}K_0 & -M_0^{-1}D_0 \end{bmatrix}$$

and

$$A_0^{-1} = \begin{bmatrix} 0 & M_0^{-1} \\ M_0^{-1} & -M_0^{-1}D_0M_0^{-1} \end{bmatrix}.$$

STEP 2. Compute an n-dimensional subspace $\mathbb{S}$ which is A_0^{-1}-isotropic and is also $C_0A_0^{-1}$-positive.

STEP 3. Form the columns of matrix X^* from basis vectors for $\mathbb{S}$ and compute $Y = A_0^{-1}X^*$.

STEP 4. Compute the moments

$$\Gamma_1 = XC_0Y, \qquad \Gamma_2 = XC_0^2Y, \qquad \Gamma_3 = XC_0^3Y.$$

STEP 5. Compute the new coefficients

$$M = \Gamma_1^{-1}, \qquad D = -M\Gamma_2M, \qquad K = -M\Gamma_3M + D\Gamma_1D.$$

END

From the computational point of view, the most serious challenge here is to complete Step 2. To the authors' knowledge algorithms for this step are not generally available. The first essential is a general-purpose algorithm for finding a family of n-dimensional subspaces which are isotropic with respect to a given $2n \times 2n$ matrix with inertia $\{n, n, 0\}$. Then it is necessary to scan these subspaces to find one (or more) satisfying the positivity condition of Step 2. If the context is that of real and symmetric systems then C_0 and A_0 are real, of course, and the algorithms may be in real arithmetic.

Analysis of the relevant algorithms may be assisted by using the spectral decompositions:

$$C_0 = \begin{bmatrix} X \\ XJ \end{bmatrix} J \begin{bmatrix} X \\ XJ \end{bmatrix}^{-1}, \qquad A_0^{-1} = \begin{bmatrix} X \\ XJ \end{bmatrix} P \begin{bmatrix} X \\ XJ \end{bmatrix}^{*}.$$

Definitions of J and P can be found in [5] and the complete theory in [4], for example.

Example 2. It is shown here that, by defining $\mathbb{S} = \text{span}\{e_1, e_2, \ldots, e_n\}$ at Step 2 of Algorithm 3, the original problem can be reproduced. First, it is easily seen that this subspace is A_0^{-1}-isotropic and $C_0A_0^{-1}$-positive. Indeed, it is found that $X = \begin{bmatrix} I & 0 \end{bmatrix}$ and $X(C_0A_0^{-1})X^* = M_0^{-1}$ so that (from (19)), $M = M_0$.

Step 3 gives $Y = \begin{bmatrix} 0 \\ M_0^{-1} \end{bmatrix}$.

At Step 4 it is found that

$$\Gamma_1^{(1)} = M_0^{-1}, \quad \Gamma_2 = -M_0^{-1}D_0M_0^{-1}, \quad \Gamma_3 = \left(-M_0^{-1}K_0 + (M_0^{-1}D_0)^2\right) M_0^{-1}$$

and then that $M = M_0$, $D = D_0$, $K = K_0$. □

Example 3. Data:

$$M_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad D_0 = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \quad K_0 = \begin{bmatrix} 4 & 3 \\ 3 & 6 \end{bmatrix}.$$

The truncated eigenvalues are found to be (in truncated form), $-0.6959 \pm 1.1943i$, and $-0.8041 \pm 2.6841i$. Step 1 yields

$$C_0 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -4 & -3 & -1 & 0 \\ -3 & -6 & 0 & -2 \end{bmatrix}, \quad A_0^{-1} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -2 \end{bmatrix}.$$

For Step 2 it is found that, for example, the image (or range) of the (truncated) matrix

$$X^* = \begin{bmatrix} 0.2923 & -1.3743 \\ -1.6097 & -0.1742 \\ -0.4729 & -0.1959 \\ -0.1742 & 0.4204 \end{bmatrix}$$

is A_0^{-1}-isotropic and $C_0A_0^{-1}$-positive. Then step 3 yields

$$Y = \begin{bmatrix} -0.4729 & -0.1959 \\ -0.1742 & 0.4204 \\ 0.7651 & -1.1784 \\ -1.2613 & -1.0151 \end{bmatrix}.$$

The calculations of Steps 3 and 4 produce

$$M = \begin{bmatrix} 11.9099 & -6.6010 \\ -6.6010 & 4.2471 \end{bmatrix}, \quad D = \begin{bmatrix} 46.9830 & 14.7497 \\ 14.7497 & -31.3384 \end{bmatrix}, \quad K = \begin{bmatrix} 47.5991 & 56.5223 \\ 56.5223 & 69.3272 \end{bmatrix}.$$

Then it can be verified that, indeed, this system and the system M_0, D_0, K_0 are isospectral.
□

7 Monic hermitian systems with $D \geq 0$

Hermitian systems arise in many practical problems and, frequently, the conditions $M > 0$ and $D \geq 0$ hold. In this case, the system is easily reduced to *monic* form, i.e. with $M = I$. In this section it is shown that, under these conditions, a direct attack can be made on the problem (of generating isospectral systems) using the methods of the preceding section. Note first of all that, in the monic case,

$$A_0^{-1} = \begin{bmatrix} 0 & I \\ I & -D \end{bmatrix}, \quad C_0 = \begin{bmatrix} 0 & I \\ -K & -D \end{bmatrix}. \tag{20}$$

First, let us examine the possibility of generating an A_0^{-1}-isotropic subspace of the form $\mathrm{Im}X^*$ with $X = \begin{bmatrix} I & Z \end{bmatrix}$ for some $n \times n$ matrix Z. It is easily seen that such a subspace is A_0^{-1}-isotropic if and only if

$$0 = Z + Z^* - ZDZ^*. \tag{21}$$

Suppose that D has rank h and consider a factorization $D = WW^*$ where W is an $n \times h$ matrix of full rank.

Solution sets for (21) are easily formulated in the two extreme cases $h = 0$ and $h = n$:

- When $h = 0$ then $D = 0$ and every skew-symmetric matrix Z is a solution of (21).
- When $h = n$ it is easily verified that every matrix of the form

  $$Z = (W^{-1})^*(I - U)W^{-1}, \tag{22}$$

  where U is a unitary matrix, is a solution of equation (21).

The next proposition gives a generalization of these families of solutions to the general case $0 \leq h \leq n$.

First, for any $n \times h$ matrix W of rank h define a *unitary completion* of W to be an $n \times (n-h)$ matrix $\widehat{W}$ for which $W^*\widehat{W} = 0$ and $\widehat{W}^*\widehat{W} = I_{n-h}$. Then it can be shown that the matrix

$$E := \left[\begin{array}{cc} W & \widehat{W} \end{array} \right]$$

is nonsingular.

Proposition 12 *Given the full-rank factorization $D = WW^*$, let $\widehat{W}$ be a unitary completion of W, and construct the nonsingular matrix E as above. Then for any unitary matrix U of size h, any skew-hermitian matrix S of size $n-h$, and an arbitrary matrix N of size $(n-h) \times h$, the matrix*

$$Z = (E^{-1})^* \left[\begin{array}{cc} I_h - U & UN^* \\ -N & \frac{1}{2}N^*N - S \end{array} \right] E^{-1} \tag{23}$$

is a solution of (21).

Proof. The proof is by verification. Substitute for Z from (23) in (21) and simplify. $\square$

It can be shown that the number of free (complex) parameters in the representation (23) is $\frac{1}{2}n(n-1)$, and that this is *independent of h.* The parameters will be further constrained on applying the second condition, namely, that the subspace $\mathbb{S}$ =Im(X^*) is $C_0A_0^{-1}$ positive. Note that, in the monic case,

$$C_0A_0^{-1} = \left[\begin{array}{cc} I & -D \\ -D & D^2 - K \end{array} \right],$$

so, with $X = \left[\begin{array}{cc} I & Z \end{array} \right]$, the positivity condition becomes

$$I - XD - DX^* + X(D^2 - K)X^* > 0.$$

Example 4. Consider the monic hermitian system given by $D_0 = \left[\begin{array}{cc} 2 & i \\ -i & 2 \end{array} \right]$ and $K_0 =$ diag$(1, -1)$. The eigenvalues (rounded) are $\lambda_1 = -3.0523, \lambda_2 = 0.4476$ and $\lambda_3 = \overline{\lambda_4} = -0.6977 + 0.4952i$.

Since D_0 is positive definite there is a solution of (21) of the form $Z = (W^{-1})^*(I_2 - U)W^{-1}$ where U is unitary. From the singular value decomposition, $D = W_0\Sigma^2W_0^*$, it is found that $D = WW^*$ with

$$W = W_0\Sigma = \left[\begin{array}{cc} i/\sqrt(2) & i\sqrt{3/2} \\ -1/\sqrt{2} & \sqrt{3/2} \end{array} \right]$$

and the choice

$$U = \left[\begin{array}{cc} \cos(2\pi/3) & -\sin(2\pi/3) \\ \sin(2\pi/3) & \cos(2\pi/3) \end{array} \right]$$

yields

$$Z = (W^{-1})^*(I_2 - U)W^{-1} = \begin{bmatrix} 1 & 0 \\ i & 1 \end{bmatrix}.$$

It is now a matter of computation to verify that, for $X = [I_2\ Z]$, we have

$$XPX^* = I_2 - ZD_0 - D_0X^* + Z(D_0^2 - K_0)Z^* = I_2$$

and hence $\mathrm{Im} X^*$ is $C_0A_0^{-1}$-positive. After first computing Γ_2 and Γ_3 in Step 4 of Algorithm 3, new monic system matrices are obtained:

$$D = \mathrm{diag}(4, 0), \quad K = \begin{bmatrix} 3 & i \\ -i & 0 \end{bmatrix}.$$

This monic system has, indeed, the same eigenvalues as the initial system. □

In this section we have focused on the case of standard triples $(X,\ C_0,\ Y)$ for which $X = [I_n\ Z]$, i.e. that the first n-by-n block of X is nonsingular. In the next section the case where the *second* n-by-n block is nonsingular will lead to another family of monic isospectral systems.

8 Isospectral families of hermitian, or real symmetric systems

The notations of Section 7 are maintained here. Once again, as the context is hermitian systems with positive definite leading coefficient, there is no significant loss in assuming that the system is already reduced to monic form.

In generating a parametrized set of isospectral systems, keep in mind that, from (18) and (19), such a system corresponds to an n dimensional subspace of $\mathbb{C}^{2n}$, namely $\mathrm{Im}\, X^*$. Decompose

$$X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}, \quad \text{where} \quad X_1,\ X_2 \in \mathbb{C}^{n \times n}.$$

Using C_0 and A_0^{-1} for the monic system, the isotropy condition (18) takes the form

$$X_1^*X_2 + X_2^*X_1 = X_2^*D_0X_2. \tag{24}$$

Now a general complex matrix C can be written in the form $C = A + iB$ where $A = \frac{1}{2}(C + C^*)$ and $B = \frac{1}{2i}(C - C^*)$ are hermitian. Using this, it follows from the last equation that

$$X_1^*X_2 = \frac{1}{2}(X_2^*D_0X_2) + S \tag{25}$$

where S is an arbitrary skew-hermitian matrix.

In applying this technique, it is important to recognize that it is the n-dimensional subspace $\mathrm{Im}(X^*)$ which entirely determines the perturbed isospectral system. Furthermore, $\mathrm{Im}\,(X^*)$ $=\mathrm{Im}\,(X^*A)$ for any nonsingular $n \times n$ matrix A. Using this idea, if we confine attention to subspaces for which X_2 is nonsingular then we can, without further loss of generality, set $X_2 = I$. However, it is convenient to introduce another real parameter $\epsilon \neq 0$ and set $X_2 = \epsilon I$. In this case, using (25) (and absorbing a real parameter into S), we may take $X_1 = \frac{1}{2}\epsilon D_0 + S$ and

$$X^* = \begin{bmatrix} \frac{1}{2}\epsilon D_0 + S \\ \epsilon I \end{bmatrix}. \tag{26}$$

Now reconsider Algorithm 3, and notice first that in Step 1 we can set $M_0 = I$.

For Step 2 a parametrized set of subspaces is to be defined with the necessary isotropic and positivity properties. Let S be an arbitrary skew-hermitian matrix ($n \geq 2$), ϵ be a real parameter, and define X by (26). Then

$$XA_0^{-1}X^* = \begin{bmatrix} \frac{1}{2}\epsilon D_0 - S & \epsilon I \end{bmatrix} \begin{bmatrix} 0 & I \\ I & -D_0 \end{bmatrix} \begin{bmatrix} \frac{1}{2}\epsilon D_0 + S \\ \epsilon I \end{bmatrix} = 0$$

and the isotropic property is verified independently of ϵ and S.

Then, with a little computation,

$$\begin{aligned} X(C_0A_0^{-1})X^* &= \begin{bmatrix} \frac{1}{2}\epsilon D_0 - S & \epsilon I \end{bmatrix} \begin{bmatrix} I & -D_0 \\ -D_0 & -K_0 + D_0^2 \end{bmatrix} \begin{bmatrix} \frac{1}{2}\epsilon D_0 + S \\ \epsilon I \end{bmatrix} \\ &= (\frac{1}{2}\epsilon D_0 + S)(\frac{1}{2}\epsilon D_0 + S)^* - \epsilon^2 K_0, \end{aligned}$$

and it is apparent from this last expression that, provided only that ϵ and S are chosen so that $\frac{1}{2}\epsilon D_0 + S$ is nonsingular, $X(C_0A_0^{-1})X^* > 0$ for all sufficiently small ϵ.

Then it is found in Step 3 (with $M = I$) that

$$Y = \begin{bmatrix} 0 & I \\ I & -D_0 \end{bmatrix} \begin{bmatrix} \frac{1}{2}\epsilon D_0 + S \\ \epsilon I \end{bmatrix} = \begin{bmatrix} \epsilon I \\ -\frac{1}{2}\epsilon D_0 + S \end{bmatrix}.$$

Now all the necessary information is generated to complete Steps 4 and 5, and hence a family of isospectral systems determined by $\epsilon \in \mathbb{R}$ and skew-hermitian matrices S.

It is clear that, if the given system is real and symmetric, then *real and symmetric isospectral systems are determined by choosing a real skew-symmetric matrix S.* The following examples are of this kind.

Example 5. (Data are taken from Example 7 of [7].) Let $M_0 = I_4$,

$$D_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0.5 \end{bmatrix}, \quad K_0 = \begin{bmatrix} 5 & -2 & 0 & 0 \\ -2 & 4 & -2 & 0 \\ 0 & -2 & 5 & -3 \\ 0 & 0 & -3 & 3 \end{bmatrix}.$$

The spectrum of this system is

$$-0.3951 \pm 2.8146i, \quad -0.4247 \pm 2.3936i, \quad -0.3302 \pm 1.5813i, \quad -0.3500 \pm 0.4081i.$$

Assign the arbitrary skew-symmetric matrix

$$S = \begin{bmatrix} 0 & 1 & 1 & 1 \\ -1 & 0 & 1 & 1 \\ -1 & -1 & 0 & 1 \\ -1 & -1 & -1 & 0 \end{bmatrix},$$

and it is found experimentally that the positivity condition is satisfied for $0 \le \epsilon \le \frac{1}{8}$.

For the purpose of comparison, this technique has been applied to generate isospectral systems at $\epsilon = 1/16$ and at $\epsilon = 1/8$. The results are then scaled to produce monic systems. First compare the damping matrices $D(\epsilon)$:

$$D(1/16) = \begin{bmatrix} .8210 & -.0985 & .2144 & .0484 \\ & 1.0410 & -.1285 & -.0760 \\ & & .5334 & .2769 \\ & & & .6046 \end{bmatrix},$$

$$D(1/8) = \begin{bmatrix} 1.440 & -.6155 & .8599 & .0396 \\ & 1.5039 & -.4797 & -.4254 \\ & & .3818 & 1.1442 \\ & & & -.3297 \end{bmatrix},$$

The stiffness matrices are:

$$K(1/16) = \begin{bmatrix} 3.7321 & -1.7603 & -.5248 & 3.6499 \\ & 5.0111 & -1.2465 & -.2472 \\ & & 3.3212 & -.5021 \\ & & & 5.0464 \end{bmatrix},$$

$$K(1/8) = \begin{bmatrix} 5.8149 & -2.9548 & -.8040 & 6.0288 \\ & 5.3074 & -.1764 & -2.2789 \\ & & 2.3600 & -.2299 \\ & & & 7.4087 \end{bmatrix}.$$

□

The technique developed and illustrated here can include Example 3 as a limiting case (i.e. in the limit as $\epsilon \to 0$) and, consequently, determines smooth isospectral continuations of the given system. However, some care must be taken if n is odd, for then the perturbing skew-symmetric matrix S is necessarily singular.

Now Example 1 is revisited to construct real and symmetric isospectral systems.

Example 6. As in Example 1 take

$$M_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad D_0 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad K_0 = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix},$$

with (truncated) eigenvalues

$$-0.9567 \pm 0.6412i, \quad \text{and} \quad -0.0433 \pm 1.2272i.$$

Since $n = 2$, we take $S = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and, in effect, there is only one free parameter, namely ϵ. It is easily seen that the positivity condition is satisfied if $-0.6 \le \epsilon \le 0.5$.

The damping and stiffness matrices are tabulated below for $\epsilon = -0.1$ and $\epsilon = 0.1$. Observe that the singular damping matrix D_0 is perturbed to a positive definite matrix by a negative shift of the parameter, but not by a positive shift.

$$D(-0.1) = \begin{bmatrix} .8988 & -.9235 \\ -.9235 & 1.1012 \end{bmatrix} \quad K(-0.1) = \begin{bmatrix} 1.6477 & .0504 \\ .0504 & 1.2153 \end{bmatrix}$$

$$D(0.1) = \begin{bmatrix} 1.1013 & -1.1276 \\ -1.1276 & .8987 \end{bmatrix} \quad K(0.1) = \begin{bmatrix} 2.4715 & -.0503 \\ -.0503 & .8103 \end{bmatrix}$$

□

9 Systems in reduced form

A wide class of systems of practical interest are intimately related to a diagonal system, or to a system which is "close to diagonal". They will be investigated in this section.

Observe first that, for each companion matrix appearing in a standard triple there is a class of isospectral systems. If a relation $\leftrightarrow$ is defined on $n \times n$ systems by saying $L_1(\lambda) \leftrightarrow L_2(\lambda)$ when they are isospectral, then $\leftrightarrow$ determines an *equivalence relation*, and it is natural to ask for canonical systems in each equivalence class. The fact that this is an equivalence relation is most easily verified by noting that all companion matrices associated with one of these classes have the same Jordan form (and this is the true canonical representation).

Definition 7 A system $L(\lambda)$ is said to be *reduced* or in *reduced form* if

$$L(\lambda) = (I\lambda - J_1)(I\lambda - J_2) = \lambda^2 - \lambda(J_1 + J_2) + J_1 J_2 \tag{27}$$

where J_1 and J_2 are in Jordan canonical form.

(Definition 7 includes a similar notion of "canonical systems" introduced in [7] under more restrictive hypotheses. There is a thorough treatment of the conditions under which $L(\lambda)$ in (27) is self-adjoint in [6].)

Not all equivalence classes contain such a reduced form. For example, a system with $n = 3$ and three distinct double eigenvalues each with an associated block of size two has no reduced form in its equivalence class (such an example is easily constructed). For this reason, the word "reduced" is used here, rather than "canonical". If J_1 and J_2 are consistent with a real, or an hermitian system then, of course, there is interest in isospectral families and reduced forms which preserve these properties.

In many cases, when there is a reduced form (27) the class has an associated Jordan matrix

$$J = \begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix}, \tag{28}$$

but this is not always the case. For example, when $n = 1$ the system $L(\lambda) = \lambda^2$ is already in reduced form (with $J_1 = J_2 = 0$), but

$$J = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \neq \begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix}.$$

To put this another way, it is **not** the case that a structure like (28) for J implies that there is a reduced form like (27) in the associated equivalence class. For example, if $n = 1$ and $J_1 = J_2 = \lambda_0$, then

$$(\lambda I - J_1)(\lambda I - J_2) = (\lambda - \lambda_0)^2,$$

but the Jordan form of this system is $\begin{bmatrix} \lambda_0 & 1 \\ 0 & \lambda_0 \end{bmatrix}$ and not $\begin{bmatrix} \lambda_0 & 0 \\ 0 & \lambda_0 \end{bmatrix}$.

Sufficient conditions for a Jordan form with the structure of (28) to have a reduced form in its equivalence class of systems are contained in:

Proposition 13 *An isospectral equivalence class contains a reduced system if the class has a corresponding Jordan matrix J of the form* (28) *where J_1 and J_2 are $n \times n$ Jordan forms and either* (a) *J_1 and J_2 have no common eigenvalues, or* (b) *J_1 and J_2 commute and* $\det(J_1 - J_2) \neq 0$.

Proof Case (a) follows because the first factor in (27) is nonsingular at each eigenvalue of the second, and it vice versa. This ensures that chains of right (left) eigenvectors for J_2 (resp. J_1) are inherited by $L(\lambda)$.

For case (b) observe that the determinantal condition implies that the matrix

$$X = \begin{bmatrix} I & I \\ J_1 & J_2 \end{bmatrix}$$

is nonsingular. Then (with L as in (27))

$$C_R X = \begin{bmatrix} J_1 & J_2 \\ J_1^2 + (J_2 J_1 - J_1 J_2) & J_2^2 \end{bmatrix}$$

and

$$XJ = \begin{bmatrix} J_1 & J_2 \\ J_1^2 & J_2^2 \end{bmatrix}.$$

Since J_1 and J_2 commute it follows that $C_R X = XJ$ (so that $C_R = XJX^{-1}$). Thus, J is characteristic of this equivalence class. $\square$

Notice that in case (b) J_1 and J_2 may have common eigenvalues.

Definition 8. A system is said to be *regular* if the following properties hold:

1. The non-real eigenvalues (if any) arise in conjugate pairs with the same multiplicities. (Denote the corresponding Jordan matrices by J_c and $\overline{J_c}$.)
2. The real eigenvalues, if any, (say $2r$ in number) can be divided into two subsets of size r with corresponding $r \times r$ Jordan matrices J_s and J_t.
3. $\det(J_s - J_t) \neq 0$.
4. J_s and J_t commute.

Property 1 is enjoyed by two important classes of systems, namely, those with real coefficient matrices and those with hermitian coefficients (and this is the main reason for introducing this definition). Of course, Properties 2, 3, and 4 are void if there are no real eigenvalues. Property 2 excludes some systems; for example, those with exactly one real eigenvalue which is not semisimple and has algebraic multiplicity two. Notice that there are *no constraints* on the multiplicities of the non-real eigenvalues (other than the fact that their algebraic multiplicity cannot exceed $n-r$).

Given Properties 1-4, it is possible to construct an associated $2n \times 2n$ Jordan matrix as in (28) with

$$J_1 = \begin{bmatrix} J_c & 0 \\ 0 & J_s \end{bmatrix}, \quad J_2 = \begin{bmatrix} \overline{J_c} & 0 \\ 0 & J_t \end{bmatrix}.$$

Now form the reduced system $L(\lambda)$ of (27) and it is easily seen that either or both of the conditions (a) and (b) of Theorem 10 hold. Then it is easily verified that:

Proposition 14 *The isospectral equivalence class of a regular system contains a real reduced system:*

$$D = -(J_1+J_2) = -\begin{bmatrix} J_c + \overline{J_c} & 0 \\ 0 & J_s + J_t \end{bmatrix}, \quad K = J_1 J_2 = \begin{bmatrix} J_c\overline{J_c} & 0 \\ 0 & J_sJ_t \end{bmatrix}.$$

If in addition, all eigenvalues are semisimple, then there is a real diagonal reduced system.

One might ask whether a reduced system will be included in the isospectral families of Sections 8 and 9. As the parametrized systems are all hermitian it is, of course, necessary that the reduced form be symmetric so, as in Proposition 12, the answer can be "yes" if all real eigenvalues are semisimple or if there are no real eigenvalues, as in:

Example 7. The systems generated in Example 6 include a reduced system - up to a congruence transformation. If we choose (truncated) $\epsilon = -0.1728$ then we find

$$X^* = \begin{bmatrix} \frac{\epsilon}{2}D_0 + S \\ \epsilon I_2 \end{bmatrix} = \begin{bmatrix} -0.0864 & 0.9136 \\ -1.0864 & -0.0864 \\ -0.1728 & 0 \\ 0 & -0.1728 \end{bmatrix}$$

and

$$V_1 = \begin{bmatrix} \epsilon I_n \\ S - \frac{\epsilon}{2}D_0 \end{bmatrix} = \begin{bmatrix} -0.1728 & 0 \\ 0 & -0.1728 \\ 0.0864 & 1.0864 \\ -0.9136 & 0.0864 \end{bmatrix}$$

from which we may compute

$$\begin{aligned} \Gamma_1 &= U_1TV_1 = \begin{bmatrix} 0.8123 & 0.0149 \\ 0.0149 & 1.1280 \end{bmatrix}, \\ \Gamma_2 &= U_1T^2V_1 = \begin{bmatrix} -0.6545 & 0.8422 \\ 0.8422 & -1.3157 \end{bmatrix}, \\ \Gamma_3 &= U_1T^3V_1 = \begin{bmatrix} 0.0075 & -1.7931 \\ -1.7931 & 0.8645 \end{bmatrix}, \end{aligned}$$

and hence we have

$$M = \begin{bmatrix} 1.2314 & -0.0163 \\ -0.0163 & 0.8867 \end{bmatrix}, \; D = \begin{bmatrix} 1.0265 & -0.9519 \\ -0.9519 & 1.0591 \end{bmatrix}, \; K = \begin{bmatrix} 1.7653 & 0.0699 \\ 0.0699 & 1.2395 \end{bmatrix}.$$

It can now be verified that $KM^{-1}D = DM^{-1}K$ and hence the three matrices M, D, K can be diagonalised by a congruence transformation [1]. Define a matrix of eigenvectors of the undamped system:

$$X_0 := \begin{bmatrix} -0.5655 & -0.7018 \\ 0.8165 & -0.6792 \end{bmatrix}.$$

Because of the commutativity assumption above, this matrix will also diagonalise D. Indeed, $X_0^T M X_0 = I_2$, $X_0^T D X_0 = \mathrm{diag}(1.9134, 0.0866) = -\Lambda - \Lambda^*$ and

$$X_0^T K X_0 = \mathrm{diag}(1.3264, 1.5079) = \Lambda\Lambda^*$$

where $\Lambda := \mathrm{diag}(-0.9567 + 0.6412i, -0.0433 + 1.2272i)$.

Instead of this indirect derivation of the reduced system it is also possible to obtain the same result by using $U_d^* := U_1^*(X_0^T)^{-1}$ which is A_0^{-1}-isotropic and leaves $M_0 = I_2$ invariant, i.e. $U_d^* C_0 A_0^{-1} U_d = I_2$. □

10 Conclusions

The spectral theory of vibrating systems has been reviewed and and re-examined from the point of view of inverse spectral problems: i.e. the construction of systems with spectral characteristics defined implicitly via a given system. In Part 1 of this work (see [7]) the spectral characteristics were prescribed explicitly in the form of complete sets of eigenvalue and eigenvector data. If no symmetry properties are required of the systems generated, the problem has a relatively easy solution summarised in the five-step procedures summarised in Section 5.

If symmetries are imposed on the coefficients of all systems considered, then the situation is more involved. However, an explicit construction is given in Section 6 for the determination of isospectral families of symmetric systems (whether complex hermitian or real symmetric). If efficient general-purpose algorithms are to be created, then an efficient procedure is required for the determination of n-dimensional subspaces of a $2n$-dimensional space which are neutral with respect to a known real symmetric indefinite matrix (see Step 2 of Algorithm 3 in Section 6). The construction of symmetric systems with positivity conditions imposed on the coefficient matrices remains an essentially open problem, although methods developed in [7] should cast some light on this problem. Special cases of problems with positivity constraints have been discussed in Sections 7 and 8.

Finally, in Section 9 it has been clarified under what circumstances an isospectral family may contain a diagonal (or "close to diagonal") system.

Acknowledgements

The first named author gratefully acknowledges Prof. Seamus D. Garvey for many stimulating discussions and the support of the EPSRC in this work through the research grant

GR/S31679/01 on "Model Compaction and Model Reduction Methods for Large-Scale Dynamic Systems in Engineering". The second named author gratefully acknowledges support from Prof. N. J. Higham of the University of Manchester and from the Natural Sciences and Engineering Research Council of Canada.

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