

Isospectral Vibrating Systems. Part 1: The Spectral Method.

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January 23, 2005

Dedicated to Pauline van den Driessche on the occasion of her 65th birthday.

Abstract

A study is made of inverse problems for $n \times n$ systems of the form $L(\lambda) = M\lambda^2 + D\lambda + K$. This paper concerns the determination of systems in an equivalence class defined by a fixed $2n \times 2n$ admissible Jordan matrix, i.e. a class of isospectral systems. Constructive methods are obtained for complex or real systems with no symmetry constraints. It is also shown how isospectral families of complex hermitian matrices can be formed. The case of real symmetric matrices is more difficult. Some partial solutions are obtained but, in this case, the theory remains incomplete. Examples are given.

1 Introduction

Inverse eigenvalue problems are addressed in this paper in the context of vibrating systems which, for our purposes are defined as follows:

Definition 1 A (*vibrating*) *system* is a triple of $n \times n$ complex matrices $\{M, D, K\}$ for which M is nonsingular.

A system has an associated time-invariant differential operator:

$$M \frac{d^2}{dt^2} + D \frac{d}{dt} + K.$$

It is well-known that the solutions of associated differential equations can be described in terms of the *algebraic* eigenvalue/eigenvector problem $L(\lambda)x = 0$, for the quadratic matrix polynomial

$$L(\lambda) = M\lambda^2 + D\lambda + K. \tag{1}$$

This investigation will admit general coefficient matrices M , D , K of the following four types: general complex matrices, general real matrices, hermitian matrices, and real symmetric matrices.

In view of the practical importance of second order systems, and in the interests of clarity, higher order problems are not considered, although it is clear that methods developed here can be extended to this more general context. An important feature of the methods used is that “linearized” first order systems of larger size are not used; the methods are direct in this sense. A similar analysis using linearizations is the topic of another paper: Part 2 of this work ([10]).

Problems of interest include (in general terms):

- Given complete spectral data for a system (i.e. complete information on eigenvalues and eigenvectors), define a corresponding system.
- Given complete eigenvalue information only (but including all multiplicity structures) describe sets of consistent systems, i.e. show how to generate isospectral families.
- Given a system (as in Definition 1) show how to generate a family of isospectral systems.

Note carefully that the term “isospectral” is used in the strong sense that the eigenvalues *and all their partial multiplicities* are common to isospectral systems. The last of these problems is the subject of Part 2 of this work [10].

It will be shown here that, when there are no symmetry requirements on the coefficient matrices, there are essentially complete solutions for the first two problems. The analysis of this paper depends heavily on notions introduced by Gohberg et al. (see [2] and [3]) which neatly summarise all of the spectral data for a matrix polynomial. The point of view of those two works is that of the *forward* problem: Given a system (as defined above) examine the properties of eigenvalues and right and left eigenvectors of $L(\lambda)$, i.e. numbers λ_j and vectors x_j , y_j , $j = 1, 2, \dots, 2n$ such that

$$L(\lambda_j)x_j = 0, \quad \text{and} \quad L(\lambda_j)^T y_j = 0. \quad (2)$$

(The superscript “T” denotes matrix transposition.) The summarising notation referred to above is as follows: Consider three matrices $X \in \mathbb{C}^{n \times 2n}$, $J \in \mathbb{C}^{2n \times 2n}$, $Y \in \mathbb{C}^{2n \times n}$ with the following properties: X and Y both have full rank (i.e. each has rank n) and J is a matrix in Jordan canonical form. If also

$$MXJ^2 + DXJ + KX = 0 \quad \text{and} \quad J^2YM + JYD + YK = 0, \quad (3)$$

then the nonzero columns of X (rows of Y) are (respectively) right and left eigenvectors (or generalized eigenvectors¹) of $L(\lambda)$, and the eigenvalues of J are the eigenvalues of $L(\lambda)$. This formulation admits the presence of multiple eigenvalues with arbitrary Jordan structures. However, many practical situations are covered by the *semisimple* case in which J is simply a diagonal matrix of eigenvalues.

¹For simplicity, a formal definition of generalized eigenvectors is not provided, although this is necessary for a complete understanding of the structure of $L(\lambda)$ at multiple eigenvalues. See Section 1.4 of [2], or Section 14.3 of [8], for example. In the sequel, the term “eigenvectors” may refer to either eigenvectors or generalized eigenvectors

In this paper, *inverse* problems are considered (as in [7]). Thus, the topic of Sections 2 and 3 is: Given a pair of matrices (X, J) with sizes as above and J in Jordan form, what further properties ensure that they are associated with a *system* - as defined above? When this question is answered satisfactorily, the further questions are studied:

When is the system $\{M, D, K\}$ made up of real matrices (Section 5)?

When is the system made up of hermitian matrices (Sections 7-9)?

When is the system made up of real symmetric matrices (Section 10)?

A topic which is not seriously addressed here concerns the conditions under which a system has *positive definite* coefficients. This is one of the main topics of [7], but more remains to be done to generalize results obtained there to the present more general context.

Another perspective on results of this kind is to say that, if an isospectral family is known then, for one fixed member of the family, feedback structures (of displacement, velocity and acceleration) are determined which leave the underlying Jordan matrix J invariant. This brings us close to the theory developed in [6].

2 Spectral data

Observe first that the Jordan canonical form for a vibrating system cannot have arbitrary Jordan structure. For example, because $L(\lambda)$ acts on an n -dimensional space, no eigenspace can have dimension larger than n , even though J is $2n \times 2n$. In fact, as proved in Theorem 1.7 of [2], this condition on the eigenspaces is both necessary and sufficient for J to correspond to a system $L(\lambda)$ of the form (1). So let us define:

Definition 2 A $2n \times 2n$ Jordan canonical form is *admissible* (for an $n \times n$ vibrating system) if the dimension of every eigenspace of J does not exceed n .

If $X \in \mathbb{C}^{n \times 2n}$ is a candidate for a matrix of right eigenvectors, an important role will be played by the $2n \times 2n$ block matrix

$$Q := \begin{bmatrix} X \\ XJ \end{bmatrix}. \quad (4)$$

Definition 3 A *Jordan pair* is a pair of matrices (X, J) with $X \in \mathbb{C}^{n \times 2n}$, $J \in \mathbb{C}^{2n \times 2n}$ for which J is an admissible Jordan matrix and Q is nonsingular.

Example 1 The Jordan matrix $J = \text{diag}[0 \ 0 \ \lambda_1 \ \lambda_2]$ with $\lambda_1 \lambda_2 \neq 0$ is admissible and, because the matrix Q generated by

$$X = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

is nonsingular, (X, J) is a Jordan pair. □

The constructions are taken a step further to include (potentially) information about left eigenvectors.

Definition 4 A *Jordan triple* is a set of matrices (X, J, Y) for which (X, J) is a Jordan pair, $Y \in \mathbb{C}^{n \times 2n}$, and

$$\begin{bmatrix} X \\ XJ \end{bmatrix} Y = \begin{bmatrix} 0 \\ M^{-1} \end{bmatrix} \quad (5)$$

for some nonsingular $M \in \mathbb{C}^{n \times n}$.

To describe this situation geometrically, let $\text{Ker}A$ denote the “nullspace” or “kernel” of matrix A and let $\text{Im}A$ denote the “range” or “image” of A . The first row of equation (5) simply says that the columns of Y lie in $\text{Ker}X$ and, since Y also has rank n , the columns of Y form a basis for $\text{Ker}X$. Furthermore, once a matrix Y has been chosen, all other such matrices have the form YB for some nonsingular $B \in \mathbb{C}^{n \times n}$. Now the condition that XJY has rank n means that $(\text{Ker}XJ) \cap \text{Im}Y = \{0\}$. In other words, $\text{Ker}XJ$ and $\text{Im}Y$ are complementary n -dimensional subspaces of \mathbb{C}^{2n} .

The emphasis of Definition 4 can be changed by noting:

Proposition 1 *Let (X, J) be a Jordan pair, $M \in \mathbb{C}^{n \times n}$ be nonsingular, and Y be the solution of equation (5). Then (X, J, Y) is a Jordan triple.*

Let (X, J, Y) be a Jordan triple. An important role is played by the *moments* of the triple; i.e. the $n \times n$ matrices $\Gamma_0, \Gamma_1, \Gamma_2, \dots$ defined by

$$\Gamma_j = XJ^jY, \quad j = 0, 1, 2, \dots \quad (6)$$

Observe that, from Definition 4, $\Gamma_0 = 0$ and $\Gamma_1 = M^{-1}$.

3 Existence and uniqueness

Now the basic result of this paper can be proved showing how a unique vibrating system is determined by a Jordan triple.

Definition 6 A system $\{M, D, K\}$ is said to be *generated by* a Jordan triple (X, J, Y) if $M = (XJY)^{-1}$ and the equation

$$MXJ^2 + DXJ + KX = 0 \quad (7)$$

holds.

Theorem 2 *A Jordan triple (X, J, Y) generates a uniquely defined system $\{M, D, K\}$.*

Proof Use the Jordan triple to define the moments Γ_j as in equation (6). It will be verified that the system defined recursively by

$$M = \Gamma_1^{-1}, \quad D = -M\Gamma_2M, \quad K = -M\Gamma_3M + D\Gamma_1D \quad (8)$$

is generated by the Jordan triple (X, J, Y) .

First of all, using the definition of a Jordan triple together with (8) it can be verified that

$$\begin{bmatrix} X \\ XJ \end{bmatrix} \begin{bmatrix} JYM + YD & YM \end{bmatrix} = I_{2n}, \quad (9)$$

(cf. Lemma 1 of ([9])). Now observe

$$\begin{aligned} \begin{bmatrix} K & D \end{bmatrix} &= -M \begin{bmatrix} \Gamma_3 - \Gamma_2M\Gamma_2 & \Gamma_2 \end{bmatrix} M \\ &= -MXJ^2 \begin{bmatrix} JYM - YM\Gamma_2M & YM \end{bmatrix}, \\ &= -MXJ^2 \begin{bmatrix} JYM + YD & YM \end{bmatrix}, \\ &= -MXJ^2 \begin{bmatrix} X \\ XJ \end{bmatrix}^{-1}, \end{aligned} \quad (10)$$

using equation (9) at the last step. Multiply on the right by $\begin{bmatrix} X \\ XJ \end{bmatrix}$ and it follows that (7) holds. Thus, the system $\{M, D, K\}$ of equations (8) is, indeed, generated by (X, J, Y) .

To establish uniqueness observe first that, certainly, M is uniquely defined by the Jordan triple (X, J, Y) . So suppose that both

$$MXJ^2 + D_1XJ + K_1X = 0 \quad \text{and} \quad MXJ^2 + D_2XJ + K_2X = 0.$$

It follows that

$$\begin{bmatrix} K_1 & D_1 \\ K_2 & D_2 \end{bmatrix} \begin{bmatrix} X \\ XJ \end{bmatrix} = - \begin{bmatrix} M \\ M \end{bmatrix} XJ^2.$$

But it has been seen above that $\begin{bmatrix} X \\ XJ \end{bmatrix}$ is invertible. Hence

$$\begin{bmatrix} K_1 & D_1 \\ K_2 & D_2 \end{bmatrix} = -MXJ^2 \begin{bmatrix} X \\ XJ \end{bmatrix}^{-1},$$

and uniqueness follows. □

Theorem 2 is closely linked with Theorem 2.4 of [2]. In that work, one begins with the system coefficients, then standard (or Jordan) triples are generated, and it is shown in Theorem 2.4 how the coefficients can be recovered from a standard triple. Here, we begin with three matrices forming a Jordan triple according to Definition 4 and show that such a triple generates a unique system. Note that the formula (10) defines one associated pair K, D , but there may be other pairs not covered by this formula. The uniqueness argument shows that this cannot happen. Note also that a given system $\{M, D, K\}$ certainly has associated Jordan triples, but there is no uniqueness in this direction (due to some remaining flexibility in the normalisation of eigenvectors).

4 Left eigenvectors

It will be clarified in this section how the role of left eigenvectors fits into the terminology of Jordan triples. It will be useful to introduce a square matrix of zeros and ones of the form:

$$P_0 = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & & & 1 & 0 \\ \vdots & & & & \vdots \\ 0 & 1 & & & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix},$$

and observe that, if J_0 is a typical Jordan block with the size of P_0 , then J_0P_0 is symmetric and $P_0J_0P_0 = J_0^T$.

Now if J is a general (multi-block) Jordan matrix, form a block diagonal matrix P with diagonal blocks P_0 as above matching the sizes of the diagonal blocks of J . Then again it is found that JP is symmetric. Thus,

$$JP = PJ^T \quad \text{and} \quad PJP = J^T. \tag{11}$$

Let us call such a matrix *the sip matrix associated with J* (sip abbreviates “standard involutory permutation”). (Notice that, $P^2 = I$ and, if J is diagonal, then $P = I$.)

Using the definition of a Jordan triple it is easily verified that:

Lemma 3 *If (X, J, Y) is a Jordan triple generating the system $\{M, D, K\}$ and P is the sip matrix associated with J , then $(Y^T P, J, P X^T)$ is also a Jordan triple and generates $\{M^T, D^T, K^T\}$.*

Thus, using (7) the equation

$$M^T Y^T P J^2 + D^T Y^T P J + K^T Y^T P = 0$$

holds. Multiply on the right with P and take the transpose to obtain:

Corollary 4 *If (X, J, Y) is a Jordan triple generating the system $\{M, D, K\}$, then both of the equations (3) hold.*

This shows that the formal definition of Y in (5) does, indeed, determine a matrix whose rows are left eigenvectors of $L(\lambda)$.

5 Real systems

It is easily seen that, for systems with real coefficients, the eigenvalues are either real numbers, or they appear in complex conjugate pairs. Consequently the number of real eigenvalues (counted with algebraic multiplicities) is even - say $2r$, where $0 \leq r \leq n$. Then there are $n - r$ conjugate pairs of non-real eigenvalues. Let J_1 be the Jordan matrix associated with the non-real eigenvalues with positive imaginary part, and let J_2 be the real $2r \times 2r$ Jordan matrix associated with the real eigenvalues. Clearly, for the real eigenvalues, there will be corresponding real eigenvectors, and these determine the columns of an $n \times 2r$ matrix X_2 .

Thus, let X_1 be a complex matrix of size $n \times (n - r)$ whose columns are to determine eigenvectors associated with J_1 . Then the conjugate eigenvalues will have a corresponding matrix of eigenvectors, $\overline{X_1}$. Let X_2 be a real matrix of size $n \times 2r$ whose columns determine the eigenvectors associated with the real eigenvalues, and form

$$X = [X_1 \quad X_2 \quad \overline{X_1}], \quad J = \begin{bmatrix} J_1 & 0 & 0 \\ 0 & J_2 & 0 \\ 0 & 0 & \overline{J_1} \end{bmatrix}. \quad (12)$$

Thus, $X \in \mathbb{C}^{n \times 2n}$ and $J \in \mathbb{C}^{2n \times 2n}$. Now it will be shown that, for real systems, when a Jordan pair X and J have these forms, Y necessarily has an analogous structure.

Proposition 5 *If (X, J) is a Jordan pair of the form (12), $M^{-1} \in \mathbb{R}^{n \times n}$, and Y is determined by (5), then*

$$Y = \begin{bmatrix} Y_1 \\ Y_2 \\ \overline{Y_1} \end{bmatrix}, \quad (13)$$

where $Y_1 \in \mathbb{C}^{(n-r) \times n}$, and $Y_2 \in \mathbb{R}^{2r \times n}$.

Proof Define the $2n \times 2n$ permutation matrix

$$N = \begin{bmatrix} 0 & 0 & I_{n-r} \\ 0 & I_{2r} & 0 \\ I_{n-r} & 0 & 0 \end{bmatrix},$$

and observe that $N^{-1} = N$, $XN = \bar{X}$, $JN = N\bar{J}$. Write equation (5) in the form

$$\begin{bmatrix} X \\ XJ \end{bmatrix} N(N^{-1}Y) = \begin{bmatrix} 0 \\ M^{-1} \end{bmatrix}$$

to obtain

$$\begin{bmatrix} \bar{X} \\ \bar{X}J \end{bmatrix} (N^{-1}Y) = \begin{bmatrix} 0 \\ M^{-1} \end{bmatrix}.$$

Now take complex conjugates recalling that M is real to obtain

$$\begin{bmatrix} X \\ XJ \end{bmatrix} (N^{-1}\bar{Y}) = \begin{bmatrix} 0 \\ M^{-1} \end{bmatrix}.$$

However, the solution of (5) is unique, so $Y = N^{-1}\bar{Y}$, or $NY = \bar{Y}$, which implies (13). \square

6 Isospectral systems defined by J

When the size of a system, n , is fixed then, by definition, all isospectral systems have the same associated $2n \times 2n$ Jordan matrix, J . Observe that, in general, M , D , and K are determined by $3n^2$ complex parameters. Also, when J is fixed there are $4n^2$ parameters in X and Y , but the condition $XY = 0$ effectively reduces this number to $3n^2$, matching the number of parameters to be determined.

So, an admissible Jordan matrix J is given and the objective is to determine the class \mathbb{S} of all systems (M, D, K) having J as a Jordan canonical form. We consider first the case of a general complex Jordan form and systems with complex coefficient matrices. Members of this class can be generated in three steps:

- Step 1C: Determine an $X \in \mathbb{C}^{n \times 2n}$ such that (X, J) form a Jordan pair. (Generically, this step involves choosing an arbitrary matrix X of the right size. This is because a random choice of X will have rank n and, with an admissible J , the matrix Q of (4) will be nonsingular. In practice, physical intuition, or experimental data, can generally play a role here in matching “mode shapes” to columns of X .)
- Step 2C: For an X generated by Step 1C find a Y such that $XY = 0$ and XJY is nonsingular. (As remarked above, this has the geometric interpretation: find a Y (of the right size) such that $\text{Ker}(XJ) \cap \text{Im}(Y) = \{0\}$. Algorithmically, a basis for $\text{Ker}(XJ)$ can be computed, and then columns of Y are formed by basis vectors for a subspace which is complementary to $\text{Ker}(XJ)$. Alternatively, when the Jordan pair (X, J) has been determined, assign a suitable leading coefficient M and solve equation (5) for Y .)

- Step 3C: Apply formulae (8) to find M , D , and K .

Observe that, when an admissible J is specified, there is an open set of candidates for the matrix X of a Jordan pair. Then for each such X there is a family of matrices Y completing a Jordan triple (and hence an isospectral system) determined by equation (5).

To generate *real* systems observe first that an admissible J must have the form given in (12). Then,

- Step 1R: Determine an X with the structure prescribed in (12) and such that (X, J) form a Jordan pair. As above, random choices of X (except for the structure imposed by (12)) will generally suffice for this step.
- Step 2R: For an X generated by Step 1R, assign a real nonsingular matrix $M \in \mathbb{R}^{n \times n}$ and solve equation (5) for Y (which, by Proposition 5, will automatically have the form (13)).
- Step 3R: Apply the formulae of (8) for D and K .

Example 2 Some simple, but degenerate, real systems are constructed in this example. They are degenerate in the sense that the three coefficient matrices obtained can be simultaneously diagonalized.

Let $J_1 = U + iW$ be an $n \times n$ diagonal matrix with U and W real and $W > 0$. Then define the $2n \times 2n$ matrix J as in (12). In this way, an entirely non-real spectrum is specified and the matrix J_2 does not appear in (12). Now make the primitive eigenvector assignment implicit in $X = \begin{bmatrix} I & I \end{bmatrix}$. It is easily verified that $\begin{bmatrix} X \\ XJ \end{bmatrix}$ is nonsingular, so that (X, J) form a Jordan pair and Step 1R above is complete.

In Step 2R allow M to be any real nonsingular matrix and solve equation (5) for Y to obtain

$$Y = \frac{i}{2} \begin{bmatrix} -I \\ I \end{bmatrix} W^{-1} M^{-1},$$

and observe that this is consistent with Proposition 5.

Now compute to find $\Gamma_1 = M^{-1}$, $\Gamma_2 = 2UM^{-1}$, and $\Gamma_3 = (3U^2 - W^2)M^{-1}$. Then Step 3R yields $D = -MU$ and $K = M(U^2 + W^2)$. Thus the system generated is

$$M (\lambda^2 I - 2\lambda U + (U^2 + W^2)).$$

□

7 Hermitian isospectral systems

The next theorem allows us to investigate symmetry properties in terms of the moments, as an alternative to working directly with the coefficients of the system.

Theorem 6 *Let the system $\{M, D, K\}$ be generated by a Jordan triple (X, J, Y) . Then the moments $\Gamma_1, \Gamma_2, \Gamma_3$ of equation (6) are all real, hermitian, or real symmetric according as all of M, D, K are also real, hermitian, or real symmetric, respectively.*

Proof If the moments have one of the three properties under discussion, it is apparent from equations (8) that the same is true of M , D and K .

Conversely, suppose first that M , D , K are all real. Then, using Proposition 5 there is a Jordan triple with the structure of equations (12) and (13) (and $X_1 \in \mathbb{C}^{n \times (n-r)}$, $X_2 \in \mathbb{R}^{n \times 2r}$, etc.). Then, for $j = 1, 2, 3$,

$$\begin{aligned} \Gamma_j &= XJ^jY = \begin{bmatrix} X_1 & X_2 & \overline{X_1} \end{bmatrix} \begin{bmatrix} J_1^j & 0 & 0 \\ 0 & J_2^j & 0 \\ 0 & 0 & \overline{J_1^j} \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \\ \overline{Y_1} \end{bmatrix}, \\ &= (X_1 J_1^j Y_1 + \overline{X_1 J_1^j Y_1}) + X_2 J_2^j Y_2, \end{aligned}$$

and is obviously real, as required.

If M , D , K are hermitian and there is a Jordan triple (X, J, Y) , then there is also a standard triple² (Y^*, J^*, X^*) (Corollary 1 of Chapter 14 of [8]). Since the moments are independent of the choice of standard triple,

$$\Gamma_j = XJ^jY = Y^*(J^*)^jX^* = (XJ^jY)^* = \Gamma_j^*,$$

and so the moments are hermitian.

If M , D , K are real and symmetric then, combining the above results, the moments are both real and hermitian, i.e. they are real and symmetric. \square

Systems with hermitian or, more importantly, real symmetric coefficients are of great practical importance. For such systems, (as with real systems) Jordan triples of special form can be generated. These special forms are discussed here, based on material from Chapter 10 of [2].

Observe first of all that, if a system has hermitian (generally complex) coefficients then, again, the eigenvalues must either be real or appear in complex conjugate pairs. This means that an associated Jordan canonical form can be constructed exactly as in Section 5. Thus, a Jordan matrix may be supposed to have the form given in (12).

The same is not true of the matrix of right eigenvectors. For example, the eigenvectors associated with real eigenvalues will generally be complex. At this stage, we can only say that X has the block form

$$X = \begin{bmatrix} X_1 & X_2 & X_3 \end{bmatrix}, \quad (14)$$

where X_1 and X_3 are $n \times (n-r)$, X_2 is $n \times 2r$, and all three blocks may be complex (including the block X_2 associated with the real eigenvalues).

Experience with the *forward* spectral problem tells us that, for a hermitian problem with nonsingular M , there is a Jordan triple (X, J, Y) with X and J as above and

$$Y = \begin{bmatrix} P_1 X_3^* \\ \hat{P}_2 X_2^* \\ P_1 X_1^* \end{bmatrix} = P X^*, \quad (15)$$

²A standard triple (U, T, V) is related to a Jordan triple (X, J, Y) by similarity, i.e. for some nonsingular S , $U = YS$, $T = S^{-1}JS$, $V = S^{-1}Y$.

if we define

$$P = \begin{bmatrix} 0 & 0 & P_1 \\ 0 & \hat{P}_2 & 0 \\ P_1 & 0 & 0 \end{bmatrix}. \quad (16)$$

Here, P_1 is the sip matrix associated with J_1 (as defined in Section 4, see (11)). To define the matrix \hat{P}_2 , first consider the sip matrix P_2 associated with J_2 , and suppose that there are k diagonal Jordan blocks in J_2 . Each block is to be multiplied by a number $\varepsilon_j = \pm 1$, $j = 1, 2, \dots, k$ (the associated *sign characteristic*³). The resulting matrix is \hat{P}_2 . The sign characteristic and the multiplicities of the real eigenvalues cannot be assigned arbitrarily. They must satisfy constraints specified in Section 4 of [1] (see also Proposition 10.12 of [2]). For example, if $M > 0$ and all real eigenvalues are semisimple, then $\sum_{j=1}^{2r} \varepsilon_j = 0$.

However, given these conditions, a significant unsolved inverse problem remains:

Problem For a given nonsingular hermitian M , describe those Jordan pairs (X, J) with J of the form (12) for which the solution Y of equation (5) has the form (15).

A Jordan triple with these properties is said to be *self-adjoint*. It is easily verified that the moments Γ_j , and hence the coefficients of the system, are all hermitian when the Jordan triple is selfadjoint. The good news is that this issue can be avoided, as described in the next section.

8 A geometric/computational approach

To generate an hermitian system with a given Jordan form as described in (12), first assign a corresponding matrix P as in (16). Without prescribing X of (14) numerically, suppose that Y has the form (15). It follows that

$$XY = \begin{bmatrix} X_1 & X_2 & X_3 \end{bmatrix} \begin{bmatrix} 0 & 0 & P_1 \\ 0 & \hat{P}_2 & 0 \\ P_1 & 0 & 0 \end{bmatrix} \begin{bmatrix} X_1^* \\ X_2^* \\ X_3^* \end{bmatrix} = 0, \quad (17)$$

i.e. the $n \times 2n$ complex matrix X must satisfy

$$XPX^* = 0. \quad (18)$$

In other words, $\text{Im}(X^*)$ must be an n -dimensional subspace of \mathbb{C}^{2n} which is self-orthogonal (or P -isotropic) with respect to the indefinite matrix P . Note that, in the definition of P , P_1 is determined by the structure of J_1 , but the definition of \hat{P}_2 depends on both J_2 and a suitable choice of sign-characteristic.

An X , and hence a subspace $\text{Im}(X^*)$, is to be chosen so that $\det(XJY) = \det(XJPX^*) \neq 0$. Then the moments

$$\Gamma_j = XJ^jY = XJ^jPX^*, \quad j = 1, 2, 3$$

are computed and hermitian coefficients are generated by equations (8). However, if it is required that M be positive definite, as is often the case, then the P -isotropic subspace must

³For the sake of brevity, this rather complex issue is not developed in detail.

be chosen so that, in addition, it is JP -positive, i.e.

$$M^{-1} = X(JP)X^* > 0. \quad (19)$$

From a computational point of view, recall that J is given, P is defined in terms of J and an assigned sign-characteristic, and the problem reduces to finding an algorithm for finding n -dimensional P -isotropic subspaces in \mathbb{C}^{2n} under the constraint that $\det(XJPX^*) \neq 0$ or that (19) is satisfied.

Although the computational aspect of the problem is not pursued here, the state of the art in numerical analysis may admit the design of numerical algorithms for finding families of matrices X satisfying (18) and (19).

Example 3 We generate a system with eigenvalues $\pm i$, 2, and -1 and assign the sign-characteristic +1, -1 to the eigenvalues 2 and -1, respectively. Thus, (see (16) and (12))

$$P = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \quad J = \begin{bmatrix} i & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -i \end{bmatrix}.$$

Let α be an angle and $c = \cos \alpha$, $s = \sin \alpha$, respectively. It is easily verified that the image (or range) of the α -dependent matrix

$$X^* = \begin{bmatrix} c/\sqrt{2} & (1+s)/\sqrt{2} \\ 1 & 0 \\ -s & c \\ -c/\sqrt{2} & (1-s)/\sqrt{2} \end{bmatrix}$$

is P -isotropic for any α . Choosing $\alpha = \pi/4$ it is found that $X(PJ)X^* > 0$. Now it is only necessary to apply (6) and (8) to find that the hermitian system:

$$M = \begin{bmatrix} 1 & 1+i\sqrt{2} \\ 1-i\sqrt{2} & 5 \end{bmatrix}, \quad D = \begin{bmatrix} -3 & -3-i\sqrt{2} \\ -3+i\sqrt{2} & 3 \end{bmatrix}, \quad K = \begin{bmatrix} 2 & 2-i2\sqrt{2} \\ 2+i2\sqrt{2} & 4 \end{bmatrix}$$

has the given spectrum. Furthermore, it is clear that an α -dependent family of isospectral hermitian systems can be generated. \square

To generate a *real and symmetric* system (i.e. for which M , D , K are all real and symmetric) it is necessary to superimpose structures described above and those of real systems described in Section 5. Thus, there must be a corresponding self-adjoint Jordan triple of the form:

$$X = [X_1 \quad X_2 \quad \overline{X_1}], \quad J = \begin{bmatrix} J_1 & 0 & 0 \\ 0 & J_2 & 0 \\ 0 & 0 & \overline{J_1} \end{bmatrix}, \quad Y = \begin{bmatrix} P_1 X_1^T \\ \hat{P}_2 X_2^T \\ P_1 X_1^* \end{bmatrix}, \quad (20)$$

in which X_2 is real.

It is easily verified that the structures of (20) ensure that the moments Γ_j and hence the coefficients of the system are hermitian and real, i.e. real and symmetric. However, it is not immediately clear how these constraints can be imposed on the problem of solving (18) for X subject to (19). A different approach to this problem is taken in [7].

9 Hermitian systems: a case study

In this section we apply the general constructions discussed above to systems with relatively simple structure; namely, those with no real eigenvalues. Thus, isospectral families of hermitian systems with no real spectrum are to be devised. Such systems are still of considerable interest for applications and provide a useful class of examples for the more general theory.

In this case, the eigenvector matrix and the Jordan matrix have the forms

$$X = [X_1 \quad X_2], \quad J = \begin{bmatrix} J_1 & 0 \\ 0 & \overline{J_1} \end{bmatrix}, \quad (21)$$

where $X_1, X_2 \in \mathbb{C}^{n \times n}$ are nonsingular⁴ and we may take it that the eigenvalues of J_1 are in the (open) upper-half-plane.

The conditions (18) and (19) take the form

$$X_1 P_1 X_2^* + (X_1 P_1 X_2^*)^* = 0 \quad (22)$$

and

$$X(JP)X^* = X_1 J_1 P_1 X_2^* + X_2 \overline{J_1} P_1 X_1^* > 0. \quad (23)$$

Now the first of these relations implies that $X_1 P_1 X_2^* = iR$ for some hermitian matrix R . Our strategy is going to be: *Given X_1, J_1 (and hence P_1), parametrize an isospectral family by choice of the hermitian matrix R .* Thus, we write

$$X_2 = -iR(X_1^*)^{-1}P_1, \quad (24)$$

and, with a little calculation (using the fact that $P_1 \overline{J_1} P_1 = J_1^*$) the positivity condition (23) becomes

$$i\{(X_1 J_1 X_1^{-1})R - R(X_1 J_1 X_1^{-1})^*\} > 0,$$

or, defining $A = i(X_1 J_1 X_1^{-1})$,

$$X(JP)X^* = AR + RA^* > 0. \quad (25)$$

But this is just a Lyapunov inequality and in principle, can be solved for R by assigning an $H > 0$ and solving the *Lyapunov equation*

$$AR + RA^* = H \quad (26)$$

for R . (Notice that, by definition, A has its spectrum in the left-half-plane and, when J_1 is diagonal, there is an explicit formula for R ; see p. 100 of [5].)

Thus, for the given (non-real) spectrum of J , a procedure for generating a family of corresponding isospectral systems can be summarized as follows:

1. Assign a nonsingular $X_1 \in \mathbb{C}^{n \times n}$ and a Jordan matrix J_1 with spectrum in the open upper-half-plane.

⁴This is a technical point and has to do with the fact that, implicitly, $L(\lambda)$ must be positive definite whenever λ is real. See [1], [2], for example.

2. Compute $A = i(X_1 J_1 X_1^{-1})$, and determine an hermitian matrix R for which (25) holds.
3. Compute X_2 from (24) and set $X = [X_1 \ X_2]$.
4. Compute $Y = PX^*$.
5. Compute moments from (6) and then (hermitian) system coefficients from (8).

Example 4 By taking $X_1 = I$ (as in Example 2) it is assumed again that there is an orthonormal system of n eigenvectors. A general Jordan structure is admitted for the eigenvalues in the upper-half-plane. This determines J_1 and P_1 and hence $A = iJ_1$ in (25), and admits the determination of a Lyapunov solution, R , for the inequality (25).

Now the choice of X_2 is constrained by (24): $X_2 = -iRP_1$, so that $X = [I \ -iRP_1]$. Then

$$Y = PX^* = \begin{bmatrix} 0 & P_1 \\ P_1 & 0 \end{bmatrix} \begin{bmatrix} I \\ iP_1 R \end{bmatrix} = \begin{bmatrix} iR \\ P_1 \end{bmatrix},$$

and it is found that, indeed, $XY = 0$.

Then it follows that

$$\Gamma_1 = AR + RA^*, \quad \Gamma_2 = -i\{(A^2 R) - (A^2 R)^*\}, \quad \Gamma_3 = -A^3 R - (A^3 R)^*.$$

It is easily seen that these three moments are all hermitian, and so the same is true of the system coefficients generated by (8). Note also that, since $M = \Gamma_1^{-1}$, the leading coefficient is necessarily positive definite.

Finally, note that if J is diagonal then R can be chosen diagonal, and the same is true of the three coefficient matrices. The assumption that $X_1 = I$ and the choice of a diagonal matrix, R , lead to systems of this special form. \square

Example 5 This is a special case of Example 4. Take

$$J_1 = \begin{bmatrix} -1 + i & 0 & 0 \\ 0 & -\frac{1}{4} + \frac{1}{2}i & 1 \\ 0 & 0 & -\frac{1}{4} + \frac{1}{2}i \end{bmatrix},$$

(and note the nonlinear elementary divisor). It is found that, with

$$R = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & -1 \end{bmatrix},$$

inequality (25) is satisfied.

After computing $\Gamma_1, \Gamma_2, \Gamma_3$ it is found that

$$M = \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 0.25 & (0.25)i \\ 0 & (-0.25)i & 1.25 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.125 & -0.25 + (0.125)i \\ 0 & -0.25 - (0.125)i & 0.625 \end{bmatrix},$$

$$K = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.0781 & -0.0625 - (0.0469)i \\ 0 & -0.0625 + (0.0469)i & 0.3906 \end{bmatrix}.$$

□

However, there is an alternative treatment of the Lyapunov equation (26). Indeed, we have $M^{-1} = X(JP)X^* = AR + RA^*$. Thus, by first assigning a hermitian positive definite leading coefficient, M , the Lyapunov equation

$$AR + RA^* = M^{-1} \quad (27)$$

is to be solved to obtain its unique solution, R_0 , say. This leads to the modified strategy:

1. Assign a nonsingular $X_1 \in \mathbb{C}^{n \times n}$ and a Jordan matrix J_1 with spectrum in the open upper-half-plane.
2. Assign a desired positive definite hermitian leading coefficient M .
3. Compute $A = i(X_1 J_1 X_1^{-1})$, and find the unique solution R_0 of (27).
4. Compute X_2 from (24) and set $X = [X_1 \ X_2]$.
5. Compute $Y = PX^*$.
6. Compute moments from (6) and then the remaining (hermitian) system coefficients D and K from (8).

Example 6 The data includes the same matrix J_1 (and hence P_1) of Example 5. But now a random choice of X_1 is taken (whose entries have real and imaginary parts randomly distributed between -1 and +1):

$$X_1 = \begin{bmatrix} 0.9003 - .1106i & -0.0280 + 0.8436i & -0.0871 - 0.1886i \\ -0.5377 + 0.2309i & 0.7826 + 0.4764i & -0.9630 + 0.8709i \\ 0.2137 + 0.5839i & 0.5242 - 0.6475i & 0.6428 + 0.8338i \end{bmatrix}.$$

We determine a *monic* system by assigning $M = I_3$. The matrix $A = i(X_1 J_1 X_1^{-1})$ is computed and then a negative definite hermitian matrix R is found by solving the Lyapunov equation (27): $AR + RA^* = I$. Thus,

$$R = \begin{bmatrix} -30.79 & -35.87 - (38.59)i & 30.04 - (33.67)i \\ & -102.57 & -9.52 - (85.93)i \\ & & -73.61 \end{bmatrix}.$$

After completing Steps 4, 5, and 6 above, a monic system is obtained with hermitian matrices:

$$D = \begin{bmatrix} 0.8418 & -0.1325 + (2.4736)i & 2.8893 - (1.7446)i \\ & 1.8486 & 8.9022 - (1.0351)i \\ & & 0.3096 \end{bmatrix},$$

$$K = \begin{bmatrix} 1.7086 & -1.3086 - (2.6204)i & -2.2202 + (0.6795)i \\ & 43.1191 & -1.8776 - (48.5119)i \\ & & 55.5069 \end{bmatrix}.$$

Notice that, even though all eigenvalues of the system are in the left half-plane, D is not positive definite. It frequently occurs that, with these inverse constructions for stable systems, D is indefinite but, nevertheless, the damping is “pervasive”. □

10 Real symmetric systems: another case study

The line of thought followed in Section 9 will bring us back to results discussed in [7]. Nevertheless, it will be useful to re-develop the ideas in this context.

As in Section 9, it is assumed that we are to build systems with no real eigenvalues; but now they are to be real and symmetric. Thus, the two fundamental conditions (18) and (19) are to be satisfied, but with the more limited spectral structures imposed by equations (20):

$$X = [X_1 \overline{X_1}], \quad J = \begin{bmatrix} J_1 & 0 \\ 0 & J_1 \end{bmatrix},$$

where $X_1 \in \mathbb{C}^{n \times n}$ is nonsingular. For our further convenience it is assumed that all eigenvalues are semisimple, so that $P_1 = I$ and $P = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$. Equation (18) now reduces to

$$X_1 X_1^T + (X_1 X_1^T)^* = 0$$

(cf. (22)). It follows that $X_1 X_1^T = iR$ where R is hermitian. But $X_1 X_1^T$ is also symmetric, so $X_1 X_1^T = iR$ for some real symmetric matrix R .

Write X_1 in real and imaginary parts: $X_1 = X_R + iX_I$ and, since the real part of $X_1 X_1^T$ is zero, it follows that

$$X_R X_R^T = X_I X_I^T. \quad (28)$$

Lemma 7 *If $X_1 = X_R + iX_I$ is nonsingular and (28) holds, then X_R and X_I are nonsingular.*

Proof Let X_1 be nonsingular and suppose that $a^T X_R = 0$ for some $a \in \mathbb{R}^n$. Then, $X_R^T a = 0$ and, from (28), $a^T X_I X_I^T = 0$ as well. Hence

$$a^* X_1 X_1^* a = a^T (X_R + iX_I)(X_R^T - iX_I^T) a = 0.$$

But X_1 nonsingular implies that $X_1 X_1^* > 0$ and so $a = 0$. Thus X_R is nonsingular and, from (28), so is X_I . \square

Now define matrix $\Theta \in \mathbb{R}^{n \times n}$ by $\Theta = -X_R^{-1} X_I$, and it is found (using (28)) that

$$\Theta \Theta^T = X_R^{-1} (X_I X_I^T) X_R^{-T} = X_R^{-1} (X_R X_R^T) X_R^{-T} = I,$$

i.e. Θ is a *real orthogonal* matrix and X_1 has the form

$$X_1 = X_R (I - i\Theta). \quad (29)$$

(If the semisimple hypothesis is not made, then $P \neq I$ and interest is focussed on *P-orthogonal matrices*, as studied in some detail by Higham [4].)

Turn now to the positivity condition (19). The argument of Section 9 utilising a Lyapunov equation no longer applies and so we outline the argument developed in [7]. It is easily seen that, if we write $J_1 = U + iW$ (with real diagonal $U < 0$ and $W > 0$), then with X_1 in the form (29) $X J P X^* > 0$ is equivalent to $U + W \Theta^T + \Theta W - \Theta U \Theta^T > 0$, or,

$$\begin{bmatrix} I & \Theta \end{bmatrix} \begin{bmatrix} U & W \\ W & -U \end{bmatrix} \begin{bmatrix} I \\ \Theta^T \end{bmatrix} > 0.$$

Thus, in geometric terms, the admissible orthogonal matrices Θ (and hence X_1) are characterized by the fact that the n -dimensional subspace $\text{Im} \begin{bmatrix} I \\ \Theta^T \end{bmatrix}$ is positive with respect to the indefinite

matrix $\begin{bmatrix} U & W \\ W & -U \end{bmatrix}$.

As in Section 8, a resolution of an inverse problem is expressed in terms of properties of subspaces with respect to an indefinite inner product. Numerical examples can be found in reference [7].

11 Conclusions

The spectral theory of vibrating systems has been reviewed and re-examined from the point of view of inverse spectral problems: i.e. the construction of systems with given spectral characteristics defined by a Jordan matrix, a matrix of eigenvectors and, for hermitian systems, a sign characteristic associated with the real eigenvalues. A fundamental theorem ensuring existence and uniqueness of systems with suitable spectral data sets has been established (Theorem 2). If no symmetry properties are required of the systems generated, the problem has a relatively easy solution summarised in the three-step procedures of Section 6 (for complex, and for real systems).

If symmetries are imposed on the coefficients of the systems generated, then the situation is more involved. One line of attack (Section 8) requires efficient procedures for the determination of n -dimensional subspaces of a $2n$ -dimensional space which are neutral with respect to one known real symmetric indefinite matrix and positive with respect to another (equations (18) and (19)). For hermitian systems with no real eigenvalues more computational procedures are described in Section 9, and illustrated with examples.

Constructions for real symmetric systems are more complex. When there is no real spectrum, more detailed properties and technique are described in Section 10 (where there is common ground with ideas first developed in [7]). The generation of systems with positivity conditions imposed on the coefficient matrices remains an essentially open problem, although methods developed in [7] are promising.

ACKNOWLEDGEMENTS The author is grateful to Uwe Prells of the University of Nottingham for helpful comments and discussion and to Professor N. J. Higham of the University of Manchester for a Fellowship during which much of this work was completed. Also, this research was supported in part by a grant from the Natural Sciences and Engineering Research Council of Canada.

A. S. Markus and an anonymous referee contributed penetrating remarks which led to significant improvements in the exposition.

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