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**A note on Halanay-type stability results
for the θ -Maruyama method for stochastic delay
differential equations.**

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A note on Halanay-type stability results for the θ -Maruyama method for stochastic delay differential equations*.

Christopher T. H. Baker[†] and Evelyn Buckwar[‡]

Abstract

Using an approach that has its origins in work of Halanay, we consider stability in mean square of numerical solutions obtained from the θ -Maruyama discretization of a test stochastic delay differential equation $dX(t) = \{f(t) - \alpha X(t) + \beta X(t - \tau)\}dt + \{g(t) + \eta X(t) + \mu X(t - \tau)\} dW(t)$ interpreted in the Itô sense, where $W(t)$ denotes a Wiener process (white noise).

Our objective in this report is limited to demonstrating that we may use techniques advanced in a recent report by Baker and Buckwar to obtain criteria for asymptotic and exponential stability, in mean square, for the solutions of a recurrence

$$\begin{aligned} \tilde{X}_{n+1} - \tilde{X}_n &= \theta \{f_{n+1} - \alpha h \tilde{X}_{n+1} + \beta h \tilde{X}_{n+1-N}\} + (1-\theta) \{f_n - \alpha h \tilde{X}_n + \beta h \tilde{X}_{n-N}\} \\ &\quad + (g_n + \eta \sqrt{h} \tilde{X}_n + \mu \sqrt{h} \tilde{X}_{n-N}) \xi_n \end{aligned}$$

where $\xi_n \in N(0, 1)$ (ξ_n is normally distributed with zero mean and variance unity). This is the θ -Maruyama formula; recognizable cases arise from taking $\theta = 0$, $\theta = \frac{1}{2}$ or $\theta = 1$; for convenience we assume $\theta \in [0, 1]$.

KEYWORDS: θ -Maruyama scheme (asymptotic, exponential) stability stochastic delay differential equations stochastic difference equations multiplicative noise Halanay-type inequalities .

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*Dedicated to Donald Kershaw to commemorate his 75th birthday.

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1 Introduction

This work is a natural extension of the work of Baker & Buckwar [2, 3]. A limited familiarity with the related literature is assumed in order to avoid unnecessary repetition of basic work. We shall indicate how results for stability of solutions obtained from a θ -Mauruyama method applied to a linear stochastic delay differential equation (SDDE), that serves as a test equation, can be derived. The use of such a test equation is commonplace in numerical analysis; for deterministic delay differential equations (DDDEs) see [6] and the extensive references there; for stochastic ordinary differential equations (SODEs) see, e.g., [9, 13]; for SDDEs see [2, 7]. Here, we are prepared to accept without discussion that such simple equations generate a *test-bed* for obtaining insight.

We employ a strategy presented for stability analysis in [3], where we illustrated the investigation of numerical stability by examining the *Euler-Maruyama method*. As we remarked in [3], the technique for analyzing stability that we illustrated by reference to the Euler-Maruyama method can also be applied to some other methods, including some that are semi-implicit (*i.e.* drift-implicit). This remark is liable to be overlooked by readers and it is the authors' hope that, by providing further details here, this oversight will be less likely.

Irrespective of that, the results provided here are new, and have intrinsic interest even though the objectives we adopt are limited: We shall develop the investigation further elsewhere, and analyze stability in certain contexts where alternative approaches apparently cannot be applied.

1.1 The test equation

With $t_0, \alpha, \beta, \eta, \mu \in \mathbb{R}$ and $\tau \geq 0$, the test problem considered here reads

$$\begin{aligned} dX(t) = & \{f(t) - \alpha X(t) + \beta X(t - \tau)\} dt + \\ & + \{g(t) + \eta X(t) + \mu X(t - \tau)\} dW(t) \quad (t \geq t_0), \end{aligned} \quad (1.1a)$$

$$X(t) = \Phi(t), \quad t \in J, \quad J := [t_0 - \tau, t_0]. \quad (1.1b)$$

Notation 1.1 *We assume the standard mathematical infrastructure and notation [2, 3], notably: (i) $W(t)$ denotes a standard Wiener process; (ii) The expectation of a stochastic variable U is written $\mathbf{E}(U)$. (iii) $\Phi : J \times \Omega \rightarrow \mathbb{R}$ satisfies $\mathbf{E}(\sup_{t \in J} |\Phi(t)|^2) < \infty$; almost all sample paths are continuous and*

$\Phi(t)$ is independent of the σ -algebra generated by $W(t)$; (iv) for $t \in [t_0, \infty)$, $X(t) \equiv X(t, \Phi)$ denotes a stochastic process that “satisfies the stochastic delay differential equation (1.1a)” in the sense that

$$X(s) \Big|_{t_0}^t = \int_{t_0}^t \{f(s) - \alpha X(s) + \beta X(s-\tau)\} ds + \int_{t_0}^t \{g(s) + \eta X(s) + \mu X(s-\tau)\} dW(s), \quad (1.2)$$

where the last integral is an Itô integral. (v) The conditions assumed are such that Φ generates a unique strong solution $X(t) \equiv X(\Phi; t)$ with $\mathbf{E}(\{X(\Phi; t)\}^2) < \infty$ for bounded t .

The general discussion in [3] was presented in terms of an equation

$$X(t) = X(t_0) + \int_{t_0}^t F(s, X(s), X(s-\tau)) ds + \int_{t_0}^t G(s, X(s), X(s-\tau)) dW(s). \quad (1.3)$$

Eqn. (1.1a) is a linear inhomogeneous SDDE on $t_0 \leq t < \infty$, and we use this equation as a *test equation* for the discussion of stability and (on applying a numerical method) numerical stability. The functions $f(t)$ and $g(t)$ that give the inhomogeneous terms in (1.1a) satisfy conditions consistent with those normally assumed for F and G ; as a detail, their presence implies that the null function ($X(t) \equiv 0$ for $t \geq -\tau$) may not be a solution.

1.2 The θ -Maruyama equations

Suppose that $Nh = \tau$ where $N \in \mathbb{N}$. The Maruyama-type θ -method for (1.1), using a step $h = \tau/N$, generates the recurrence

$$\begin{aligned} \tilde{X}_{n+1} - \tilde{X}_n &= hf_n^\theta + \theta\{-\alpha h\tilde{X}_{n+1} + \beta h\tilde{X}_{n+1-N}\} + (1-\theta)\{-\alpha h\tilde{X}_n + \beta h\tilde{X}_{n-N}\} \\ &\quad + (g_n + \eta\sqrt{h}\tilde{X}_n + \mu\sqrt{h}\tilde{X}_{n-N})\xi_n, \end{aligned} \quad (1.4a)$$

where $\xi_n \in N(0, 1)$ and we use the shorthand notation

$$f_n^\theta := \theta f_{n+1} + (1-\theta)f_n. \quad (1.4b)$$

Eqn (1.4a) is a “drift-implicit” formula that (if h is not an “exceptional value” – *i.e.*, provided $1 + \theta\alpha h \neq 0$) generates the sequence $\{\tilde{X}_n\}_{n \geq 1}$, when given

$$\tilde{X}_{-\ell} = \Phi(t_0 - \ell h) \text{ for } \ell \in \mathcal{J} \text{ where } \mathcal{J} := \{0, 1, \dots, N\}. \quad (1.4c)$$

Notation 1.2 *To indicate the dependence on Φ we write $\tilde{X}_n \equiv \tilde{X}_n(\Phi)$ and our definition of stability relates to perturbations*

$$\delta\tilde{X}_n \equiv \tilde{\delta\tilde{X}}_n(\Phi) := \tilde{\delta\tilde{X}}_n(\Phi + \delta\Phi) - \tilde{\delta\tilde{X}}_n(\Phi), \quad (1.5)$$

that arise from perturbations $\delta\Phi(t_0 - \ell h)$ (for $\ell \in \mathcal{J}$) in the initial data. (A different class of definitions correspond to persistent perturbations – perturbations in the inhomogeneous terms (e.g., in $\{f_n\}$)).

2 Exponential stability of solutions by a Halanay-type technique

There is a variety of approaches to the investigation of stability; each has its merits or demerits. Halanay [8] presented a technique for examining the stability of solutions of DDEs which was adapted to a discussion of difference equations by Tang, in his thesis [14] and in a number of related publications (e.g., [4, 5]). Baker and Buckwar [3] progressed the Halanay-type theory by applying it to establish conditions for p -th moment exponential stability of solutions of SDDEs and certain discretized versions.

2.1 Stability definitions

Our definitions of stability, asymptotic stability, and exponential stability of solutions in mean-square of solutions of (1.1) are consistent with the usual definitions to be found in the literature; *cf.* [3], or [11]; they are the analogues of the definitions of (asymptotic, exponential) stability of solutions of stochastic *recurrence relations* or *difference equations*. However, the general stability definitions associated with (1.3) and its discretization can be simplified when considering (1.1), (1.4), and we take some advantage of this simplification below.

Definition 2.1 *(Mean-square stability and related concepts)* *A solution of (1.4) is said to be*

- (a) *stable in mean-square if, for each $\varepsilon > 0$, there exists a value $\delta^+ > 0$ such that $\mathbf{E}(|\delta\tilde{X}_n|^2) < \varepsilon$ for $n \in \mathbb{N}$, whenever $\mathbf{E}(\sup_{n \in \mathcal{J}} |\delta\Phi(t_n)|^2) < \delta^+$;*
- (b) *asymptotically stable in mean-square if it is stable in mean-square and $\mathbf{E}(|\delta\tilde{X}_n|^2) \rightarrow 0$ as $n \rightarrow \infty$, whenever $\mathbf{E}(\sup_{n \in \mathcal{J}} |\delta\Phi(t_n)|^2)$ is bounded;*
- (c) *exponentially mean-square stable (with exponent ν_n^+) if it is stable in the*

mean-square and if, given $\delta^+ > 0$, there exist a finite $C > 0$, and a value $\nu_h^+ > 0$ such that, whenever $\mathbf{E}(\sup_{n \in \mathcal{J}} |\delta\Phi(t_n)|^2) < \delta^+$,

$$\mathbf{E}(|\delta\tilde{X}_n|^2) \leq C \exp\{-\nu_h^+(t_n - t_0)\} \text{ for all } n \text{ sufficiently large.}$$

The definitions of stability for the *analytical solution* $X(\Phi; t)$ of (1.1) are natural analogues of these. Thus, exponential stability is defined, where $\delta X(t) := X(\Phi + \delta\Phi; t) - X(\Phi; t)$, as follows:

Definition 2.2 *The solution $X(\Phi; t)$ of the problem (1.1) is exponentially mean-square stable (with exponent ν^+) if it is stable in the mean-square and if, given $\delta^+ > 0$, there exist a finite $C > 0$ and a value $\nu_h^+ > 0$ such that, whenever $\mathbf{E}(\sup_{t \in \mathcal{J}} |\delta\Phi(t)|^2) < \delta^+$,*

$$\mathbf{E}(|\delta X(t)|^2) \leq C \exp\{-\nu^+(t - t_0)\} \text{ for all } t \text{ sufficiently large.}$$

The restriction of the definitions to deterministic problems will be clear to the reader. For additional comments, see Appendix A.

2.2 A discrete inequality of Halanay type

We shall appeal to some results employed in [3], to which we refer for proofs.

Lemma 2.1 *Denote by $\mathcal{R}_N(\zeta; a, b)$ the polynomial in ζ :*

$$\mathcal{R}_N(\zeta; a, b) := \zeta^{N+1} - (1 - ah)\zeta^N - bh \quad (a, b \in \mathbb{R}; N \in \mathbb{N}), \quad (2.1a)$$

where $h = \tau/N > 0$. If $0 \leq \beta_h < \alpha_h$ and $0 < \alpha_h h < 1$, the polynomial $\mathcal{R}_N(\zeta; \alpha_h, \beta_h)$ has a single positive zero ζ_h^+ where

$$\zeta_h^+ \in (1 - (\alpha_h - \beta_h)h, 1), \quad \text{if } \beta_h > 0, \quad \text{and} \quad \zeta_h^+ = 1 - \alpha_h h, \quad \text{if } \beta_h = 0; \quad (2.1b)$$

further, $\zeta_h^+ = \exp(-\nu_h^+ h)$ where $0 < \nu_h^+ \leq \alpha_h$.

Theorem 2.1 *Suppose, for some fixed integer $N \geq 0$, that $t_n = t_0 + nh$ for some $h > 0$ and $\{v_n\}_{n=0}^\infty$ is a sequence of positive numbers that satisfies, where*

$$0 \leq \beta_h < \alpha_h \text{ and } 0 < \alpha_h h < 1, \quad (2.2a)$$

the relation

$$\frac{v_{n+1} - v_n}{h} \leq -\alpha_h v_n + \beta_h \max_{\ell \in \mathcal{J}} v_{n+\ell} \text{ for } n \in \mathbb{N} \quad (2.2b)$$

with $N = 0$ if $\beta_h = 0$. Then $v_n \leq \{\max_{\ell \in \mathcal{J}} v_\ell\} \exp\{-\nu_h^+(t_n - t_0)\}$ where $\nu_h^+ > 0$ is the value occurring in Lemma 2.1.

Theorem 2.1 (see [3]) is similar in spirit to a result obtained by Halanay [8] in the context of DDDEs. The form of the result $0 < \nu_h^+ \leq \alpha_h$ explains the presence of the scaling factor $1/h$ in (2.2b) – so that $\frac{v_{n+1} - v_n}{h}$ is then an approximation to a derivative – although α_h may depend upon h .

3 Deterministic insight

Results for deterministic problems yield insight. Consider the DDDE

$$x'(t) = f(t) - \alpha x(t) + \beta x(t - \tau) \quad (\alpha, \beta \in \mathbb{R}). \quad (3.1)$$

A necessary and sufficient condition for exponential stability of solutions of (3.1) is that the zeros of the function $\mathcal{Q}(\zeta; \alpha, \beta, \tau) := \zeta - \alpha - \beta \exp(-\zeta\tau)$ lie in the left half-plane (i.e., $\Re(\zeta) \leq -\nu^+$ for some $\nu^+ > 0$); a sufficient condition is

$$|\beta| < -\alpha.$$

With $Nh = \tau$ where $N \in \mathbb{N}$, the θ -method for the DDDE (3.1) reads

$$x_{n+1} - x_n = hf_n^\theta - \alpha h\{\theta x_{n+1} + (1-\theta)x_n\} + \beta h\{\theta x_{n+1-N} + (1-\theta)x_{n-N}\}. \quad (3.2)$$

Perturbing $\{x_\ell\}_{\ell \in \mathcal{J}}$, we find the consequent perturbations $\{\delta x_\ell\}_{\ell \geq 1}$ satisfy

$$\delta x_{n+1} = \frac{1 + (1-\theta)\alpha h}{1 + \theta\alpha h} x_n + \frac{\beta h \theta}{1 + \alpha h \theta} \delta x_{n+1-N} + \frac{(1-\theta)\beta h}{1 + \alpha h \theta} \delta x_{n-N}, \quad (3.3)$$

for $n \geq 0$. For $h > 0$ we define

$$\varrho_0 := \frac{1 - (1-\theta)\alpha h}{1 + \theta\alpha h} \quad \text{and} \quad \varrho_1 := \frac{\beta h \theta}{1 + \alpha h \theta}, \quad \varrho_2 := \frac{(1-\theta)\beta h}{1 + \alpha h \theta}. \quad (3.4)$$

Now, $\delta x_{n+1}^2 - \delta x_n^2$ can be expressed $\left(\{\varrho_0 + 1\}\delta x_n + \varrho_1 \delta x_{n+1-N} + \varrho_2 \delta x_{n-N}\right) \times \left(\{\varrho_0 - 1\}\delta x_n + \varrho_1 \delta x_{n+1-N} + \varrho_2 \delta x_{n-N}\right)$; we deduce that

$$\begin{aligned} \delta x_{n+1}^2 - \delta x_n^2 &= \{\varrho_0^2 - 1\} \delta x_n^2 + 2\varrho_0\varrho_1 \delta x_n \delta x_{n+1-N} + 2\varrho_0\varrho_2 \delta x_n \delta x_{n-N} \\ &\quad + 2\varrho_1\varrho_2 \delta x_{n+1-N} \delta x_{n-N} + \varrho_1^2 \delta x_{n+1-N}^2 + \varrho_2^2 \delta x_{n-N}^2. \end{aligned} \quad (3.5)$$

If $uv \neq 0$, $|suv| \leq \frac{1}{2}\{v^2 + s^2u^2\}$, with equality if, and only if, $s = v/u$. Thus $|uv| = \inf_{s \in (0, \infty)} \frac{1}{2s}\{v^2 + s^2u^2\} \leq \frac{1}{2}\left\{\frac{v^2}{r} + ru^2\right\}$ for all $r \in (0, \infty)$. Then,

$$|\varrho_r \varrho_s \delta x_j \delta x_k| \leq \frac{|\varrho_r \varrho_s|}{2} \left\{r_{jk} \delta x_j^2 + \frac{1}{r_{jk}} \delta x_k^2\right\} \text{ for arbitrary } r_{jk} \in (0, \infty), \quad (3.6)$$

with equality for some r_{jk} . From (3.5) and (3.6) we obtain the inequality

$$\begin{aligned} \delta x_{n+1}^2 - \delta x_n^2 \leq & \quad (3.7) \\ \{\varrho_0^2 - 1\} \delta x_n^2 & + |\varrho_0 \varrho_1| \left\{ \frac{1}{r} \delta x_n^2 + r \delta x_{n-N+1}^2 \right\} + |\varrho_0 \varrho_2| \left\{ \frac{1}{r'} \delta x_n^2 + r' \delta x_{n-N}^2 \right\} \\ & + |\varrho_1 \varrho_2| \left\{ \frac{1}{r''} \delta x_{n+1-N}^2 + r'' \delta x_{n-N}^2 \right\} + \varrho_1^2 \delta x_{n+1-N}^2 + \varrho_2^2 \delta x_{n-N}^2, \end{aligned}$$

for arbitrary $r, r', r'' \in (0, \infty)$. Hence,

$$\frac{\delta x_{n+1}^2 - \delta x_n^2}{h} \leq -A^{\natural} \delta x_n^2 + B^{\natural} \max_{\ell \in \mathcal{J}} \delta x_{n-\ell}^2 \quad (\mathcal{J} = \{0, 1, \dots, N\}), \quad (3.8)$$

where, for arbitrary positive numbers $\{r, r'\}$ we may set

$$A_h^{\natural} \equiv A_h^{\natural}(r, r') = -\frac{1}{h} \left\{ \varrho_0^2 - 1 + \frac{|\varrho_0 \varrho_1|}{r} + \frac{|\varrho_0 \varrho_2|}{r'} \right\}, \quad (3.9a)$$

and (setting $r'' = 1$)

$$B_h^{\natural} \equiv B_h^{\natural}(r, r') = \frac{1}{h} \{ |\varrho_0 \varrho_1| r + |\varrho_0 \varrho_2| r' + (|\varrho_1| + |\varrho_2|)^2 \}. \quad (3.9b)$$

(Note that A_h^{\natural} and B_h^{\natural} are functions of h and are $\mathcal{O}(1)$ as $h \rightarrow 0$.) We deduce:

Theorem 3.1 *For the deterministic case, the recurrence (3.2) is exponentially stable if for any choice of positive r, r' , the values in (3.9) satisfy the conditions $hA^{\natural} \in (0, 1)$ and $0 \leq B^{\natural} < A^{\natural}$.*

For further comment see Appendix B.

4 Simulation of stability of $X(\Phi; t)$ via that of $X_n(\Phi)$.

It is natural to ask to what extent the stability of $\{X_n(\Phi)\}$ corresponds to the stability of the true solution $X(\Phi; t)$ that it is assumed to approximate. For (1.1) we have the following result (see, e.g., [3]).

Theorem 4.1 *(i) Every solution of (1.1) is exponentially mean-square stable when $|\beta| < \alpha - \{|\eta|^2 + |\mu|^2\}$. (ii) If $\mu = 0$ then every solution of (1.1) is exponentially mean-square stable when $|\beta| < \alpha - \frac{1}{2}|\eta|^2$.*

For additional comment see Appendix C. We seek an analogue of Theorem 4.1 for stability of the numerical solutions. To analyze mean-square stability we

first derive a relationship between the expectations $\{\mathbf{E}(\delta\tilde{X}_n^2)\}$. The objective is to derive a suitable relationship

$$\mathbf{E}(\delta\tilde{X}_{n+1}^2) - \mathbf{E}(\delta\tilde{X}_n^2) \leq -\alpha_h h \mathbf{E}(\delta\tilde{X}_n^2) + \beta_h h \max_{\ell \in \mathcal{J}} \mathbf{E}(\delta\tilde{X}_{n-\ell}^2). \quad (4.1)$$

There are various ways of obtaining such a relationship. To commence, we establish the following lemma.

Lemma 4.1 *Provided that $1 + \theta\alpha h \neq 0$,*

$$\delta\tilde{X}_{n+1} = (\varrho_0 + \varpi_0 \xi_n) \delta\tilde{X}_n + \varrho_1 \delta\tilde{X}_{n+1-N} + (\varrho_2 + \varpi_2 \xi_n) \delta\tilde{X}_{n-N}, \quad (4.2a)$$

where

$$\varrho_0 = \frac{1 - (1 - \theta)\alpha h}{1 + \theta\alpha h}, \quad \varrho_1 = \frac{\theta\beta h}{1 + \theta\alpha h}, \quad \varrho_2 = \frac{(1 - \theta)\beta h}{1 + \theta\alpha h}, \quad (4.3a)$$

$$\varpi_0 = \frac{\eta\sqrt{h}}{1 + \theta\alpha h}, \quad \varpi_2 = \frac{\mu\sqrt{h}}{1 + \theta\alpha h}. \quad (4.3b)$$

Proof: From (1.4a), we derive the relation $\delta\tilde{X}_{n+1} - \delta\tilde{X}_n = \frac{-\alpha h}{1 + \theta\alpha h} \delta\tilde{X}_n + \frac{\beta h}{1 + \theta\alpha h} \{\theta \delta\tilde{X}_{n+1-N} + (1 - \theta) \delta\tilde{X}_{n-N}\} + \{\frac{\mu\sqrt{h}}{1 + \alpha\theta h} \delta\tilde{X}_n + \frac{\mu\sqrt{h}}{1 + \theta\alpha h} \delta\tilde{X}_{n-N}\} \xi_n$.

Lemma 4.2

$$\begin{aligned} \mathbf{E}(\delta\tilde{X}_{n+1}^2) - \mathbf{E}(\delta\tilde{X}_n^2) &= \{\varrho_0^2 - 1 + \varpi_0^2\} \mathbf{E}(\delta\tilde{X}_n^2) + \\ &+ 2\varrho_0 \varrho_2 \mathbf{E}(\delta\tilde{X}_n \delta\tilde{X}_{n-N}) + 2\varrho_0 \varrho_1 \mathbf{E}(\delta\tilde{X}_n \delta\tilde{X}_{n+1-N}) + \\ &+ \{\varrho_2^2 + \varpi_2^2\} \mathbf{E}(\delta\tilde{X}_{n-N}^2) + \varrho_1^2 \mathbf{E}(\delta\tilde{X}_{n+1-N}^2). \end{aligned} \quad (4.4)$$

Proof: The expression $\{\delta\tilde{X}_{n+1} - \delta\tilde{X}_n\} \{\delta\tilde{X}_{n+1} + \delta\tilde{X}_n\}$ can be written

$$\begin{aligned} &\{(\varrho_0 + 1 + \varpi_0 \xi_n) \delta\tilde{X}_n + \varrho_1 \delta\tilde{X}_{n+1-N} + (\varrho_2 + \varpi_2 \xi_n) \delta\tilde{X}_{n-N}\} \times \\ &\{(\varrho_0 - 1 + \varpi_0 \xi_n) \delta\tilde{X}_n + \varrho_1 \delta\tilde{X}_{n+1-N} + (\varrho_2 + \varpi_2 \xi_n) \delta\tilde{X}_{n-N}\}. \end{aligned}$$

A moment's calculation shows that this relation is of the form

$\delta\tilde{X}_{n+1}^2 - \delta\tilde{X}_n^2 = \{\mathbf{a}_0 + \mathbf{a}_1 \xi_n + \mathbf{a}_2 \xi_n^2\} \delta\tilde{X}_n^2 + \{\mathbf{b}'_0 + \mathbf{b}'_1 \xi_n + \mathbf{b}'_2 \xi_n^2\} \delta\tilde{X}_n \delta\tilde{X}_{n+1-N} + \{\mathbf{b}''_0 + \mathbf{b}''_1 \xi_n + \mathbf{b}''_2 \xi_n^2\} \delta\tilde{X}_n \delta\tilde{X}_{n-N} + \{\mathbf{c}'_0 + \mathbf{c}'_1 \xi_n + \mathbf{c}'_2 \xi_n^2\} \delta\tilde{X}_{n+1-N}^2 + \{\mathbf{c}''_0 + \mathbf{c}''_1 \xi_n + \mathbf{c}''_2 \xi_n^2\} \delta\tilde{X}_{n-N}^2$ with $\mathbf{b}'_2 = \mathbf{c}'_1 = \mathbf{c}'_2 = 0$. (Here, the coefficients \mathbf{a}_i etc. are functions of $\{\varrho_i\}$ and $\{\varpi_j\}$ and hence of α, β, η, μ , and h .) Since $\delta\tilde{X}_r$ and ξ_n ($\xi_n \in N(0, 1)$)

are independent for $r \leq n$, we have $\mathbf{E}(\xi_n \delta \tilde{X}_r \delta \tilde{X}_s) = 0$, and $\mathbf{E}(\xi_n^2 \delta \tilde{X}_r \delta \tilde{X}_s) = \mathbf{E}(\delta \tilde{X}_r \delta \tilde{X}_s)$, if $-N \leq r, s \leq n$ ($r, s \in \mathbb{N}$). Thus, we arrive at a relation

$$\begin{aligned} \mathbf{E}(\delta \tilde{X}_{n+1}^2) - \mathbf{E}(\delta \tilde{X}_n^2) &= \{\mathbf{a}_0 + \mathbf{a}_2\} \mathbf{E}(\delta \tilde{X}_n^2) + \\ &+ \mathbf{b}'_0 \mathbf{E}(\delta \tilde{X}_n \delta \tilde{X}_{n-N}) + \mathbf{b}''_0 \mathbf{E}(\delta \tilde{X}_n \delta \tilde{X}_{n+1-N}) + \\ &+ \{\mathbf{c}'_0 + \mathbf{c}'_2\} \mathbf{E}(\delta \tilde{X}_{n-N}^2) + \mathbf{c}''_0 \mathbf{E}(\delta \tilde{X}_{n+1-N}^2). \end{aligned} \quad (4.5)$$

Evaluating the coefficients in (4.5) we establish the lemma.

4.1 Application of the general Halanay-type theory

The following theorem provides a delay-difference inequality and, using Theorem 2.1, a consequent condition for exponential mean-square stability.

Theorem 4.2 *Given arbitrary positive numbers $\{r, r'\}$, set*

$$A_h(r, r') = A_h^\natural(r, r'), \quad B_h(r, r') = B_h^\natural(r, r') + (\varpi_0^2 + \varpi_2^2) \quad (4.6)$$

where $A_h^\natural(r, r')$ and $B_h^\natural(r, r')$ are the values (3.9) occurring in the deterministic case. Then

$$\frac{\mathbf{E}(\delta \tilde{X}_{n+1}^2) - \mathbf{E}(\delta \tilde{X}_n^2)}{h} \leq -A_h \mathbf{E}(\delta \tilde{X}_n^2) + B_h \max_{\ell \in \mathcal{J}} \mathbf{E}(\delta \tilde{X}_{n-\ell}^2), \text{ where } \mathcal{J} = \{0, 1, \dots, N\}, \quad (4.7)$$

and a solution $\{\tilde{X}_n(\Phi)\}$ is exponentially stable in mean-square if, for any $r, r' \in (0, \infty)$, $0 < hA_h(r, r') < 1$ and $0 \leq B_h(r, r') < A_h(r, r')$; the exponential exponent is then bounded by $A(r, r')$.

We observe that the condition $0 < hA_h(r, r') < 1$ is the same condition

$$0 < -\{\varrho_0^2 - 1 + \frac{|\varrho_0 \varrho_1|}{r} + \frac{|\varrho_0 \varrho_2|}{r'}\} < 1$$

that was required in the deterministic case in Theorem 3.1. It is clear that to emulate the result $|\beta| < -\alpha + |\eta|^2 + |\mu|^2$, with $\alpha \in (0, \infty)$ that holds in the case of the test equation (1.1) (cf. Theorem 4.1 (i)) it is desirable that $\varrho_0 \rightarrow 0$ as $\alpha \rightarrow \infty$ (rather than $|\varrho_0| \rightarrow 1$ as $\alpha \rightarrow \infty$); i.e., an underlying L -stable deterministic θ -formula appears preferable to an A -stable one. However, an L -stable formula can be “over-stable” (“stable when the DDE is unstable”).

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Appendices

A Comments on stability

Remark A.1 Stability definitions are often (*cf.* [2]) provided in a form that relate to the stability of a *null solution*, which would here require us to consider the homogeneous case $f(t) = g(t) = 0$. (Our perturbations satisfy the *homogeneous* equation.) Note that to cast stability definitions in terms of behaviour of solutions (rather than perturbations) is acceptable when addressing the stability of null solutions but it carries a risk of misconceptions in other contexts, see Remark C.1(a).

When discussing solutions of (1.1) and the associated difference equations (1.4), exponential mean-square stability is equivalent to asymptotic mean-square stability and implies mean-square stability. Under the assumed conditions, all solutions of (1.4a) have the same stability properties. For more general nonlinear problems, different solutions can have different stability properties. \square

B Stability of a finite-term recurrence

Remark B.1 If we suppose that $\alpha \in \mathbb{R}$, $\beta = 0$ and $h > 0$ in (3.1), the θ -method for $x'(t) = f(t) - \alpha x(t)$ gives a recurrence $x_{n+1} - x_n = h\{\theta f_{n+1} + (1 - \theta)f_n\} - \alpha h\{\theta x_{n+1} + (1 - \theta)x_n\}$. The recurrence has a solution if $1 + \theta\alpha h \neq 0$, and a solution $\{x_n\}$ to the recurrence is “asymptotically and exponentially stable” if $\varrho_0 \in (-1, 1)$.

Remark B.2 We recognize

$$(1 + \theta\alpha h)\mu^{N+1} - (1 + (1 - \theta)\alpha h)\mu^N + \theta\beta h\mu + (1 - \theta)\beta h$$

as the *stability polynomial* for the recurrence (3.2). The boundary locus method permits the use of this stability polynomial to compute exact stability regions, but we have proceeded by an alternative argument to demonstrate a different technique that extends to other test equations. \square

Remark B.3 In Theorem 3.1, for arbitrary $D \in (0, A_h^{\natural})$, we can replace A_h^{\natural} and B_h^{\natural} in the condition by $A_h^{\natural} - D$ and $B_h^{\natural} + D$. This is because $\delta x_n^2 \leq \max_{\ell \in \mathcal{J}} \delta x_{n-\ell}^2$. \square

C Comments on stability results

Remark C.1 (a) A solution of the *inhomogeneous* equation (1.1a) can be exponentially unbounded in mean square and still exponentially mean square stable. (b) Theorem 4.1 gives a sufficient but not a necessary condition; a proof appears in [2]. (c) In their discussion, in the special case $\eta = 0$, of the Euler-Maruyama method for SDDEs, Cao *et al.* [7] consider (asymptotic) stability with the conditions $|\beta| < \alpha - \frac{1}{2}|\mu|^2$, $\eta = 0$. (d) The case of asymptotic stability for $\beta = \mu = 0$ (the non-delay case) is in [9]. \square printed April 30, 2004