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Variation of Parameters Formulae for Volterra Integral Equations

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E.O. Agyingi*, C.T.H. Baker†

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Abstract

In this report, the authors obtain variation of parameters formulae for nonlinear Volterra integral equations and integro-differential equations. These are related to a variation of parameters formula for ordinary differential equations that is due to V.M. Alekseev. The authors' results are obtained through the use of an embedding technique.

Some doubts have been cast on previous results in the literature. Amongst other things, the authors show that a result on which much of the numerical analysis for Volterra equations has been founded in the past can be obtained by a new approach.

Keywords: Volterra integral & integro-differential equations, variation of parameters, embedding.

1 Introduction

We shall be considering systems of *Volterra integral equations* of the type:

$$\mathbf{x}(t) = \mathbf{g}(t) + \int_{t_0}^{t} \mathbf{k}(t,s,\mathbf{x}(s))ds \quad (t \geq t_0) \tag{1.1}$$

and *Volterra integro-differential equations* of the type:

$$\mathbf{x}'(t) = \mathbf{f}\left(t, \mathbf{x}(t), \int_{t_0}^{t} \mathbf{k}(t,s,\mathbf{x}(s))ds\right) \quad (t \geq t_0), \tag{1.2}$$

where $\mathbf{x}(t_0) = \mathbf{x}_0$ is prescribed. In each case the function $\mathbf{x} : [t_0, \infty) \to \mathbb{R}^m$ is the unknown function. Assumptions on $\mathbf{f}$ and $\mathbf{k}$ will be required, and these are supplied in §1.1. The case (1.2) can be represented as a special case of (1.1) on integrating the derivative.

The task that we address in the case of (1.1) is that of relating its solution $\mathbf{x}(t)$ to the solution $\mathbf{y}(t)$ of an equation of the form

$$\mathbf{y}(t) = \mathbf{g}(t) + \int_{t_0}^{t} \mathbf{k}(t,s,\mathbf{y}(s))ds + \boldsymbol{\delta}(t, \mathbf{V}\mathbf{y}(t)) \quad (t \geq t_0). \tag{1.3}$$

Here the notation $\mathbf{V}\mathbf{y}(t)$ denotes the value at t of a function $\mathbf{w} = \mathbf{V}\mathbf{y}$ where $\mathbf{V}$ is a Volterra (causal or non-anticipative) operator, acting on vector-valued functions. Examples of Volterra operators $\mathbf{V}$ are provided by the following:

$$(i)\ (\mathbf{V}\mathbf{z})(t) = \mathbf{z}(t); \quad (ii)\ (\mathbf{V}\mathbf{z})(t) = \mathbf{z}(t-1); \quad (iii)\ (\mathbf{V}\mathbf{z})(t) = \int_{t-1}^{t} \mathbf{v}(t,\sigma,\mathbf{z}(\sigma))d\sigma \tag{1.4}$$

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and in particular where

$$(\mathbf{V}\mathbf{z})(t) = \int_{t_0}^{t} \mathbf{v}(t,\sigma,\mathbf{z}(\sigma))d\sigma. \tag{1.5}$$

We shall extend the above notation in an obvious manner: in particular, if $\mathsf{z}(t,s) \equiv \mathsf{z}(t,s,t_0,\gamma)$ is given as a function of the variables s,t, parametrized by t_0, γ and with $t_0 \le t \le s$, we have (analogous to the above)

$$(i')\ (\mathbf{V}\mathsf{z})(t,t) = \mathsf{z}(t,t); \quad (ii')\ (\mathbf{V}\mathsf{z})(t,t) = \mathsf{z}(t-1,t-1); \tag{1.6}$$
$$(iii')\ (\mathbf{V}\mathsf{z})(t,t) = \textstyle\int_{t-1}^{t} \mathbf{v}(t,\sigma,\mathsf{z}(\sigma,\sigma))d\sigma,$$

respectively, and, especially (corresponding to (1.5)),

$$(\mathbf{V}\mathsf{z})(t,t) = \int_{t_0}^{t} \mathbf{v}(t,\sigma,\mathsf{z}(\sigma,\sigma))d\sigma. \tag{1.7}$$

For (1.2), the task is to relate the solution $\mathbf{x}(t)$ of (1.2) to the solution $\mathbf{y}(t)$ of an equation of the form

$$\mathbf{y}'(t) = \mathbf{f}\left(t,\mathbf{y}(t),\int_{t_0}^{t} \mathbf{k}(t,s,\mathbf{y}(s))ds\right) + \widehat{\boldsymbol{\delta}}(t,\mathbf{y}(t),\mathbf{V}\mathbf{y}(t)) \quad (t \ge t_0). \tag{1.8}$$

1.1 Basic assumptions and notation

Observe that the notation that we adopt is to denote vectors by bold font lower case and matrices by bold font upper case. Throughout this work, we make the following assumptions:

Assumptions 1.1 *The functions* $\mathbf{f}(t,\mathbf{u},\mathbf{w})$, $\mathbf{g}(t)$, $\mathbf{k}(t,s,\mathbf{u})$, $\mathbf{v}(t,s,\mathbf{u})$, $\mathbf{K}(t,s)$, $\mathbf{w}(t,s,\mathbf{z})$, *and* $\boldsymbol{\delta}(t,\mathbf{u})$, $\mathfrak{d}(t,\mathbf{u})$, $\widehat{\boldsymbol{\delta}}(t,\mathbf{u},\mathbf{w})$ *(encountered below) are continuously differentiable in their arguments for* $t_0 \le s \le t$, $\mathbf{u}$, $\mathbf{w},\mathbf{z}, \in \mathbb{R}^m$, $t \in [t_0,T]$.

In fact, the reader will recognize that a number of our results require the weaker conditions stated in the following.

Assumptions 1.2 *The function* $\mathbf{K}(t,s)$, *and the functions* $\mathbf{f}(t,\mathbf{u},\mathbf{v})$, $\mathbf{g}(t)$, $\mathbf{k}(t,s,\mathbf{u})$ $\mathbf{v}(t,s,\mathbf{u})$ *and* $\boldsymbol{\delta}(t,\mathbf{u})$, $\mathfrak{d}(t,\mathbf{u})$, $\widehat{\boldsymbol{\delta}}(t,\mathbf{u},\mathbf{w})$ *are continuous in their arguments for* $t_0 \le s \le t$, $\mathbf{u}$, $\mathbf{w} \in \mathbb{R}^m$, $t \in [t_0,T]$, *and* $\mathbf{f}$, $\mathbf{k}$ *and* $\boldsymbol{\delta}$, $\mathfrak{d}$, $\widehat{\boldsymbol{\delta}}$ *are uniformly Lipschitz-continuous in* $\mathbf{u}$ *and* $\mathbf{w}$.

The partial derivatives that we employ can lead to confusion in the detailed manipulation without the use of a systematic notation, and we shall rely frequently upon the following conventions. Suppose that, for each $t \ge t_0$, $\mathbf{a}(t,t_0,\boldsymbol{\alpha}) \in \mathbb{R}^n$ is a column vector dependent upon the scalars $t,t_0 \in \mathbb{R}$ and the vector $\boldsymbol{\alpha} \in \mathbb{R}^n$ and $\mathbf{b}(t,t_0,\boldsymbol{\alpha},\boldsymbol{\beta}) \in \mathbb{R}^n$ is also a column vector dependent upon the scalar vectors $\boldsymbol{\alpha},\boldsymbol{\beta} \in \mathbb{R}^n$. We define the matrix

$$\frac{\partial}{\partial \boldsymbol{\alpha}}\mathbf{a}(t,t_0,\boldsymbol{\alpha}) := \left[\frac{\partial}{\partial \alpha_1}\mathbf{a}(t,t_0,\boldsymbol{\alpha}), \frac{\partial}{\partial \alpha_2}\mathbf{a}(t,t_0,\boldsymbol{\alpha}), \ldots, \frac{\partial}{\partial \alpha_n}\mathbf{a}(t,t_0,\boldsymbol{\alpha})\right] \in \mathbb{R}^{n\times n}. \tag{1.9}$$

In the proofs of our results[1], we employ suffices to denote partial derivatives; typically: $\mathbf{a}_1(t,t_0,\boldsymbol{\alpha}) := \frac{\partial}{\partial t}\mathbf{a}(t,t_0,\boldsymbol{\alpha})$; $\mathbf{a}_2(t,t_0,\boldsymbol{\alpha}) := \frac{\partial}{\partial t_0}\mathbf{a}(t,t_0,\boldsymbol{\alpha})$; $\mathsf{A}_3(t,t_0,\boldsymbol{\alpha}) := \frac{\partial}{\partial \boldsymbol{\alpha}}\mathbf{a}(t,t_0,\boldsymbol{\alpha})$ (a matrix of partial derivatives derived from a vector will be designated in bold capital sanserif font); and $\mathsf{B}_4(t,t_0,\boldsymbol{\alpha},\boldsymbol{\beta}) := \frac{\partial}{\partial \boldsymbol{\beta}}\mathbf{b}(t,t_0,\boldsymbol{\alpha},\boldsymbol{\beta})$. The meaning of, for example,

$$\mathbf{a}_1(t_0,t_0,\boldsymbol{\alpha}) := \frac{\partial}{\partial t}\mathbf{a}(t,t_0,\boldsymbol{\alpha})\Big|_{t=t_0},$$

[1] In formal statements of results we shall employ the fuller notation, as typified by that in (1.9).

$$\mathbf{a}_1(t,t,\boldsymbol{\alpha}) := \frac{\partial}{\partial s}\mathbf{a}(s,t_0,\boldsymbol{\alpha})\Big|_{s=t,t_0=t},$$

and

$$\mathbf{B}_4(t,t_0,\boldsymbol{\alpha},\boldsymbol{\alpha}) := \frac{\partial}{\partial \boldsymbol{\beta}}\mathbf{b}(t,t_0,\boldsymbol{\alpha},\boldsymbol{\beta})\Big|_{\boldsymbol{\beta}=\boldsymbol{\alpha}},$$

etc. is clear in this suffix notation.

2 Previous results

Results for ODEs and for linear Volterra integral or integro-differential equations can be found in the literature and we consider some of these results here. We then review attempts at extending the theory to nonlinear Volterra integral or integro-differential equations.

2.1 VPFs for ODEs

As a trivial example of (1.1) we have

$$\mathbf{x}(t) = \mathbf{x}_0 + \int_{t_0}^{t} \mathfrak{f}(s,\mathbf{x}(s))ds \tag{2.1}$$

from which we obtain the differential equation

$$\mathbf{x}'(t) = \mathfrak{f}(t,\mathbf{x}(t)) \quad (t \geq t_0) \tag{2.2a}$$

with the initial condition

$$\mathbf{x}(t_0) = \mathbf{x}_0. \tag{2.2b}$$

The solution $\mathbf{x}(t)$ of (2.2) may be written $\mathbf{x}(t,t_0,\mathbf{x}_0)$. Results relating the solution of (2.2) to the solution of the problem

$$\mathbf{y}'(t) = \mathfrak{f}(t,\mathbf{y}(t)) + \mathfrak{d}(t,\mathbf{y}(t)), \quad (t \geq t_0), \tag{2.3a}$$

with the initial condition

$$\mathbf{y}(t_0) = \mathbf{x}_0, \tag{2.3b}$$

are much more familiar in the literature than the problems that we address. We observe that (2.3) can be re-expressed as

$$\mathbf{y}(t) = \mathbf{x}_0 + \int_{t_0}^{t} \mathfrak{f}(s,\mathbf{y}(s))ds + \boldsymbol{\delta}(t,\mathbf{V}\mathbf{y}(t)), \tag{2.4a}$$

$$\boldsymbol{\delta}(t,\mathbf{V}\mathbf{y}(t)) = \int_{t_0}^{t} \mathfrak{d}(s,\mathbf{y}(s))ds \tag{2.4b}$$

$(t \geq t_0)$. Amongst the variation of parameters formulae for (2.2) is that of Alekseev [2], which we recall to orientate the reader.

Theorem 2.1 (Alekseev) *Let $\mathfrak{f}$, $\mathfrak{d} \in C[[t_0,\infty)\times\mathbb{R}^n,\mathbb{R}^n]$, and let the first order partial derivative $\frac{\partial \mathfrak{f}}{\partial \mathbf{x}}$ exist and be continuous on $[t_0,\infty)\times\mathbb{R}^n$. There exists a unique solution $\mathbf{x}(t,t_0,\mathbf{x}_0)$ for $t \geq t_0$ of (2.2a) satisfying (2.2b). If there exists a unique solution $\mathbf{y}(t,t_0,\mathbf{x}_0)$ of (2.3a) and (2.3b), then $\mathbf{y}(t,t_0,\mathbf{x}_0)$ satisfies the integral equation*

$$\mathbf{y}(t,t_0,\mathbf{x}_0) = \mathbf{x}(t,t_0,\mathbf{x}_0) + \int_{t_0}^{t} \mathbf{U}(t,s,\mathbf{y}(s,t_0,\mathbf{x}_0))\mathfrak{d}(s,\mathbf{y}(s,t_0,\mathbf{x}_0))ds \tag{2.5}$$

for $t \geq t_0$ where $\mathbf{U}(t,t_0,\mathbf{x}_0) = \frac{\partial}{\partial \mathbf{x}_0}\mathbf{x}(t,t_0,\mathbf{x}_0)$. □

The following lemma (Lakshmikantham & Leela [10]) proves useful in establishing a proof of the above theorem.

Lemma 2.1 *Suppose that* $\mathfrak{f} \in C[[t_0, \infty) \times \mathbb{R}^n, \mathbb{R}^n]$ *possesses continuous first order partial derivatives with respect to the second variable, that is,*

$$\frac{\partial}{\partial \mathbf{x}} \mathfrak{f}(t, \mathbf{x}) \in C[[t_0, \infty) \times \mathbb{R}^n, \mathbb{R}^n].$$

The solution $\mathbf{x}(t, t_0, \mathbf{x}_0)$ *of (2.2) exist for* $t \geq t_0$*; let the matrix* $\mathbf{H}(t, t_0, \mathbf{x}_0)$ *be defined as the Jacobian*

$$\mathbf{H}(t, t_0, \mathbf{x}_0) = \frac{\partial}{\partial \mathbf{x}} \mathfrak{f}(t, \mathbf{x}(t, t_0, \mathbf{x}_0)).$$

Then (i) $\mathbf{U}(t, t_0, \mathbf{x}_0) = \dfrac{\partial}{\partial \mathbf{x}_0} \mathbf{x}(t, t_0, \mathbf{x}_0)$ *exists and is the solution of the equation*

$$\frac{\partial}{\partial t} \mathbf{U}(t, t_0, \mathbf{x}_0) = \mathbf{H}(t, t_0, \mathbf{x}_0) \mathbf{U}(t, t_0, \mathbf{x}_0) \tag{2.6a}$$

such that

$$\mathbf{U}(t_0, t_0, \mathbf{x}_0) = \mathbf{I}; \tag{2.6b}$$

(ii) $\mathbf{v}(t) \equiv \mathbf{v}(t, t_0, \mathbf{x}_0) = \dfrac{\partial}{\partial t_0} \mathbf{x}(t, t_0, \mathbf{x}_0)$ *exists and is the solution of the equation*

$$\mathbf{v}'(t) = \mathbf{H}(t, t_0, \mathbf{x}_0) \mathbf{v}(t) \tag{2.7a}$$

that satisfies $\mathbf{v}(t, t_0, \mathbf{x}_0) = -\mathbf{U}(t, t_0, \mathbf{x}_0) \mathfrak{f}(t_0, \mathbf{x}_0)$*, for* $t \geq t_0$*; thus,*

$$\mathbf{v}(t_0, t_0, \mathbf{x}_0) = -\mathfrak{f}(t_0, \mathbf{x}_0). \tag{2.7b}$$

□

Definition 2.1 *The function* $\mathbf{U}(t, t_0, \mathbf{x}_0)$ *is termed the* fundamental matrix *for (2.7a).*

We seek a similar result to Theorem 2.1, for Volterra equations. In establishing the main theorem of this paper, we shall use an approach that is found in the proof of Lemma 2.1.

2.2 VPFs for linear VIEs, based upon resolvents

The linear version of (1.1) reads

$$\mathbf{x}(t) = \mathbf{g}(t) + \int_{t_0}^{t} \mathbf{K}(t, s) \mathbf{x}(s) ds \quad (t \geq t_0). \tag{2.8}$$

Variation of parameters formulae are easily obtained in the linear case. Corresponding to $\mathbf{K}(t, s)$ is the *resolvent kernel* $\mathbf{R}(t, s)$, which satisfies

$$\mathbf{R}(t, s) = \mathbf{K}(t, s) + \int_{s}^{t} \mathbf{K}(t, u) \mathbf{R}(u, s) du, \quad t \geq s, \tag{2.9}$$

with $\mathbf{R}(t, s) = 0$ for $s > t$. The solution of (2.8) can be expressed as

$$\mathbf{x}(t) = \mathbf{g}(t) + \int_{t_0}^{t} \mathbf{R}(t, s) \mathbf{g}(s) ds, \quad (t \geq t_0). \tag{2.10}$$

In view of (2.10), the relation between the solution $\mathbf{x}(t)$ of (2.8) and any solution $\mathbf{y}(t)$ of an equation

$$\mathbf{y}(t) = \mathbf{g}(t) + \int_{t_0}^{t} \mathbf{K}(t, s) \mathbf{y}(s) ds + \boldsymbol{\delta}(t, \mathbf{V}\mathbf{y}(t)) \quad (t \geq t_0), \tag{2.11}$$

where

$$\mathbf{V}\mathbf{y}(t) = \int_{t_0}^{t} \mathbf{v}(t, s, \mathbf{y}(s))ds, \tag{2.12}$$

is given by

$$\mathbf{y}(t) = \mathbf{x}(t) + \boldsymbol{\delta}(t, \mathbf{V}\mathbf{y}(t)) + \int_{t_0}^{t} \mathbf{R}(t, s)\boldsymbol{\delta}(s, \mathbf{V}\mathbf{y}(s))ds, \quad (t \geq t_0). \tag{2.13}$$

As a special case, if we have

$$\boldsymbol{\delta}(t, \mathbf{V}\mathbf{y}(t)) = \int_{t_0}^{t} \mathbf{K}(t, s)\mathbf{q}(s, \mathbf{y}(s))ds,$$

in (2.11), then the relation (2.13) becomes

$$\mathbf{y}(t) = \mathbf{x}(t) + \int_{t_0}^{t} \mathbf{R}(t, s)\mathbf{q}(s, \mathbf{y}(s))ds. \tag{2.14}$$

The relations (2.13) and (2.14) are examples of variation of parameters formulae (VPFs) for linear Volterra integral equations.

As an alternative to the use of $\mathbf{R}(t, s)$, let us consider the equation

$$\mathbf{U}(t, u) = \mathbf{I} + \int_{u}^{t} \mathbf{K}(t, s)\mathbf{U}(s, u)ds, \quad (t \geq u), \tag{2.15}$$

the solution $\mathbf{U}(t, u)$ of which is sometimes known as the *differential resolvent*[2] for (2.8), with $\mathbf{U}(t, u) = 0$ for $u > t$. From (2.15), we have $\mathbf{U}(t, t) = \mathbf{I}$ and we have

$$\mathbf{U}(t, u) = \mathbf{I} + \int_{u}^{t} \mathbf{R}(t, s)ds, \quad (t \geq u), \tag{2.16}$$

where $\mathbf{R}(t, s)$ is the resolvent (satisfying (2.9)). Differentiating equation (2.16), we obtain

$$\frac{\partial}{\partial u}\mathbf{U}(t, u) = -\mathbf{R}(t, u), \tag{2.17}$$

which when substituted into equation (2.10) and employing integration by parts, yields

$$\mathbf{x}(t) = \mathbf{U}(t, t_0)\mathbf{g}(t_0) + \int_{t_0}^{t} \mathbf{U}(t, s)\mathbf{g}'(s)ds, t \geq t_0, \tag{2.18}$$

provided $\mathbf{g}'(s)$ exists. Substituting equation (2.17) into equation (2.13), we obtain the following variation of parameters formula.

Theorem 2.2 *The solutions of (2.8) and (2.11) are related by the equation*

$$\mathbf{y}(t) = \mathbf{x}(t) + \boldsymbol{\delta}(t, \mathbf{V}\mathbf{y}(t)) - \int_{t_0}^{t} \frac{\partial}{\partial s}\mathbf{U}(t, s)\boldsymbol{\delta}(s, \mathbf{V}\mathbf{y}(s))ds, t \geq t_0. \tag{2.19}$$

or (when $\frac{d}{ds}\boldsymbol{\delta}(s, \mathbf{V}\mathbf{y}(s))$ *exists)*

$$\mathbf{y}(t) = \mathbf{x}(t) + \mathbf{U}(t, t_0)\boldsymbol{\delta}(t_0, \mathbf{V}\mathbf{y}(t_0)) + \int_{t_0}^{t} \mathbf{U}(t, s)\frac{d}{ds}\boldsymbol{\delta}(s, \mathbf{V}\mathbf{y}(s))\, ds, \; t \geq t_0, \tag{2.20}$$

[2]The term differential resolvent appears in the literature as the fundamental solution of a Volterra integro-differential equation.

2.3 VPFs for linear VIDEs

The linear versions of (1.2) and the perturbed system (1.8) are

$$\mathbf{x}'(t) \quad = \quad \mathbf{A}(t)\mathbf{x}(t) + \int_{t_0}^{t} \mathbf{K}(t,s)\mathbf{x}(s)ds, \quad t \geq t_0, \tag{2.21}$$

with an initial condition $\mathbf{x}(t_0) = \mathbf{x}_0 \in \mathbb{R}^m$, and

$$\mathbf{y}'(t) = \mathbf{A}(t)\mathbf{y}(t) + \int_{t_0}^{t} \mathbf{K}(t,s)\mathbf{y}(s)ds + \widehat{\boldsymbol{\delta}}(t, \mathbf{y}(t), \mathbf{V}\mathbf{y}(t)),\ t \geq t_0, \tag{2.22}$$

with an initial condition $\mathbf{y}(t_0) = \mathbf{x}_0 \in \mathbb{R}^m$, respectively and where $\mathbf{V}\mathbf{y}$ is defined as in (2.12). Here, $\mathbf{A} : [t_0, \infty) \to \mathbb{R}^{m\times m}$.

The linear VIDE (2.21) is equivalent to a linear VIE of the form (2.8), that is,

$$\mathbf{x}(t) = \mathbf{x}_0 + \int_{t_0}^{t} \mathcal{K}(t,s)\mathbf{x}(s)ds,$$

with $\mathbf{g}(t) = \mathbf{x}_0$ and

$$\mathcal{K}(t,s) = \mathbf{A}(s) + \int_{s}^{t} \mathbf{K}(\sigma, s)d\sigma. \tag{2.23}$$

Associated with (2.21), we introduce the the $m \times m$ matrix-valued function $\mathcal{U}(t, t_0)$ such that

$$\mathcal{U}'(t,t_0) \quad = \quad \mathbf{A}(t)\mathcal{U}(t,t_0) + \int_{t_0}^{t} \mathbf{K}(t,s)\mathcal{U}(s,t_0)ds, \quad t \geq t_0 \tag{2.24a}$$

(where $\mathcal{U}'(t, t_0)$ denotes the derivative with respect to t, $\frac{\partial}{\partial t}\mathcal{U}(t, t_0)$) such that

$$\mathcal{U}(t_0, t_0) = \mathbf{I}. \tag{2.24b}$$

With the kernel function given by $\mathcal{K}(t, s)$, equation (2.24a) is equivalent to (2.15) and for a corresponding *resolvent* $\mathcal{R}(t, s)$ satisfying $\mathcal{R}(t, s) = \mathcal{K}(t, s) + \int_s^t \mathcal{K}(t, \sigma)\mathcal{R}(\sigma, s)d\sigma$, $t \geq s$, the solution of (2.24a) satisfies $\mathcal{U}(t, \sigma) = 1 + \int_\sigma^t \mathcal{R}(t, s)ds$, $\quad (t \geq \sigma)$. The solution of (2.21) is then given by

$$\mathbf{x}(t) = \mathbf{x}_0 + \int_{t_0}^{t} \mathcal{R}(t,s)\mathbf{x}_0 ds = \mathcal{U}(t,t_0)\mathbf{x}_0. \tag{2.25}$$

Letting $\mathbf{g}(t) = \mathbf{x}_0 + \int_{t_0}^{t} \widehat{\boldsymbol{\delta}}(\sigma, \mathbf{y}(\sigma), \mathbf{V}\mathbf{y}(\sigma))d\sigma$ the perturbed $VIDE$ (2.22) has the integrated form

$$\mathbf{y}(t) = \mathbf{g}(t) + \int_{t_0}^{t} \mathcal{K}(t,s)\mathbf{y}(s)ds, \qquad t \geq t_0, \tag{2.26}$$

where $\mathcal{K}(t, s)$ is given by (2.23). Thus, the solution of (2.26) leads to the following variation of parameters formula for the linear VIDEs (2.21) and (2.22).

Theorem 2.3 *The solutions of (2.21) and (2.22) are related by the equations*

$$\mathbf{y}(t) = \mathbf{x}(t) + \int_{t_0}^{t} \mathcal{U}(t,s)\widehat{\boldsymbol{\delta}}(s, \mathbf{y}(s), \mathbf{V}\mathbf{y}(s))ds, \tag{2.27}$$

and

$$\mathbf{y}(t) = \mathcal{U}(t,t_0)\mathbf{x}_0 + \int_{t_0}^{t} \mathcal{U}(t,s)\widehat{\boldsymbol{\delta}}(s, \mathbf{y}(s), \mathbf{V}\mathbf{y}(s))ds. \tag{2.28}$$

The relations (2.27) and (2.28), which are essentially equivalent, are examples of variation of parameters formulae for linear Volterra integro-differential equations. A straightforward example of $\widehat{\boldsymbol{\delta}}$ is $\widehat{\boldsymbol{\delta}}(t, \mathbf{y}(t), \mathbf{V}\mathbf{y}(t)) = \mathbf{y}(t) + \int_{t_0}^{t} \mathbf{v}(t, s, \mathbf{y}(s))ds$.

2.3.1 Reduction to a VIE

We observe that we can rewrite (2.21) as

$$\begin{bmatrix} \mathbf{x}(t) \\ \mathbf{z}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{x}_0 \\ \mathbf{0} \end{bmatrix} + \int_{t_0}^{t} \begin{bmatrix} \mathbf{A}(s) & \mathbf{I} \\ \mathbf{0} & \mathbf{K}(t,s) \end{bmatrix} \begin{bmatrix} \mathbf{x}(s) \\ \mathbf{z}(s) \end{bmatrix} ds \quad (t \geq t_0) \tag{2.29}$$

The resolvent kernel corresponding to (2.29) is

$$\begin{bmatrix} \mathbf{R}_{00}(t,s) & \mathbf{R}_{01}(t,s) \\ \mathbf{R}_{10}(t,s) & \mathbf{R}_{11}(t,s) \end{bmatrix} = \begin{bmatrix} \mathbf{A}(s) & \mathbf{I} \\ \mathbf{0} & \mathbf{K}(t,s) \end{bmatrix} + \int_{s}^{t} \begin{bmatrix} \mathbf{A}(u) & \mathbf{I} \\ \mathbf{0} & \mathbf{K}(t,u) \end{bmatrix} \begin{bmatrix} \mathbf{R}_{00}(u,s) & \mathbf{R}_{01}(u,s) \\ \mathbf{R}_{10}(u,s) & \mathbf{R}_{11}(u,s) \end{bmatrix} du.$$

We find

$$\begin{bmatrix} \mathbf{x}(t) \\ \mathbf{z}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{x}_0 \\ \mathbf{0} \end{bmatrix} + \int_{t_0}^{t} \begin{bmatrix} \mathbf{R}_{00}(t,s) & \mathbf{R}_{01}(t,s) \\ \mathbf{R}_{10}(t,s) & \mathbf{R}_{11}(t,s) \end{bmatrix} \begin{bmatrix} \mathbf{x}_0 \\ \mathbf{0} \end{bmatrix} ds,$$

and, since $\mathbf{x}(t) = \mathbf{x}_0 + \int_{t_0}^{t} \mathbf{R}_{00}(t,s)\mathbf{x}_0 ds$, we can thus obtain the result of Theorem 2.3 by a different route.

The preceding comments establish that the results for VIDEs are essentially corollaries of those for VIEs.

2.4 Previous work on nonlinear Volterra equations

Research into VPFs for nonlinear Volterra integral equations of the form

$$\mathbf{y}(t) = \mathbf{g}(t) + \int_{t_0}^{t} \mathbf{k}(t,s,\mathbf{y}(s))ds + \boldsymbol{\delta}(t, \mathbf{V}\mathbf{y}(t)) \quad (t \geq t_0). \tag{2.30}$$

and for nonlinear Volterra integro-differential equations of the form

$$\mathbf{y}'(t) = \mathbf{f}(t,\mathbf{y}(t)) + \int_{t_0}^{t} \mathbf{h}(t,s,\mathbf{y}(s))ds + \widehat{\boldsymbol{\delta}}(t, \mathbf{y}(t), \mathbf{V}\mathbf{y}(t)) \quad (t \geq t_0), \tag{2.31}$$

where $\mathbf{V}\mathbf{y}$ is given by (1.5), has a chequered history. Such VPFs were discussed by Brauer [6], Bernfeld and Lord [5], Beesack [4], and Lakshimikantham [9]. As Baker [3] observed, many of the results in the numerical analysis of Volterra equations (see [8]) rely upon the results published by Brauer [6],.

References to the literature would be incomplete without citing the reviews appearing in Mathematical Reviews for some of the earlier papers.

• John M. Bownds, in a review of the paper by B. G. Pachpatte, *Perturbations of nonlinear systems of Volterra integral equations, J. Mathematical and Physical Sci. 10 (1976), no. 4, 295–305.* wrote:

> The results of this paper compare the stability, boundedness, and asymptotic behavior of the perturbed nonlinear system of Volterra integral equations with that of the corresponding unperturbed system. Although the results of this paper may very well be correct, their proofs depend on a variation of constants formula of another author (referring to Brauer [6]) which, unfortunately, does not appear to be correct. These results, therefore, as of this writing, have not been established.

• In his review of the paper by B. G. Pachpatte, *Stability of Volterra integral equations under a general class of perturbations. Utilitas Math. 10 (1976), 65–75.*, John M. Cushing wrote:

> This paper is concerned with the nature of solutions as $t \to +\infty$ of a system of Volterra integral equations which is a perturbation of a system (in general nonlinear) which is assumed to have some stability property. While the theorems in this paper appear reasonable and may very well be correct, their proofs unfortunately depend fully on a

> "variation of constants formula" of F. Brauer [6] which is invalid. As pointed out by Bernfeld and Lord in a forthcoming paper and privately communicated to the reviewer by them, the error in Brauer's formula occurs in equation (19) of his paper [Math. Systems Theory 6 (1972), 226–234] in which the first appearing $\mathbf{y}(u)$ after the second equality sign should be $\mathbf{y}(s)$. This in general forbids the cancellation of two terms and results in a representation formula as given by the identity immediately following (19) instead of by that appearing in (18). Thus, the results of the present paper under review should be viewed as still open and inconclusive.

• Fred Brauer, the author of [6], wrote as follows in his review of the paper by B. G. Pachpatte, *On perturbations of Volterra integro-differential equations, Rev. Roumaine Math. Pures Appl. 22 (1977), no. 6, 831–839.*:

> The author obtains boundedness and asymptotic behavior results for nonlinear perturbations of linear integro-differential equations by combining the variation of constants formula of S. I. Grossman and R. K. Miller [J. Differential Equations 8 (1970), 457–474] and an integral inequality of Gronwall type due to the author [J. Math. Anal. Appl. 44 (1973)]. He also obtains analogous results for perturbations of nonlinear Volterra equations with the aid of a variation of constants formula of the reviewer [Math. Systems Theory 6 (1972), 226–234].
> { Reviewer's remark :The validity of some of the author's earlier results has been questioned [Utilitas Math. 10 (1976), 65–75; J. Mathematical and Physical Sci. 10 (1976), no. 4, 295–305] because of an apparent error in the proof of the reviewer's variation of constants formula [S. R. Bernfeld and M. E. Lord, Appl. Math. Comput. 4 (1978), no. 1, 1–14]. The reviewer claims that the formula is correct and that the apparent error is the result of an ambiguity in his notation.}

• S. R. Bernfeld co-author of [5] wrote as follows in his review of the paper by Paul R. Beesack, *On some variation of parameter methods for integro-differential, integral, and quasi-linear partial integro-differential equations. Appl. Math. Comput. 22 (1987), no. 2-3, 189–215.*:

> The author correctly points out a basic error in the paper of the reviewer and M. E. Lord [same journal 4 (1978), no. 1, 1–14] in their development of the nonlinear variation of parameters formula when the unperturbed system is either an integro-differential system or an integral system. This can be corrected but it leads to a more involved formula.

As seen in the above excerpts, the reviewers in each instance direct attention to some difficulties in the papers. Since the results in [6] have been used in much of the numerical analysis literature, the status of those results requires clarification. We shall follow an approach that, we hope, avoids some of the pitfalls identified in previous research output.

Brauer's theorem for Volterra integral equations (Brauer [6, p. 21]) can be expressed as follows.

Theorem 2.4 *Suppose that* $\mathbf{x}(t)$ *and* $\mathbf{y}(t)$ *are the solutions of (1.1) and (1.3), with*

$$\boldsymbol{\delta}(t, \mathbf{V}\mathbf{y}(t)) = \int_{t_0}^{t} \mathbf{v}(t, s, \mathbf{y}(s))ds,$$

(and $\mathbf{x}_0 = \mathbf{x}(t_0)$*). Let* $\mathbf{U}(t, t_0, \mathbf{x}_0)$ *be the solution of the system*

$$\mathbf{U}(t, t_0, \mathbf{x}_0) = \mathbf{I} + \int_{t_0}^{t} \mathbf{H}(t, s, \mathbf{x}(s))\mathbf{U}(s, t_0, \mathbf{x}_0)ds, \tag{2.32}$$

where

$$\mathbf{H}(t, s, \mathbf{x}(s)) := \frac{\partial}{\partial \mathbf{w}}\mathbf{k}(t, s, \mathbf{w})\Big|_{\mathbf{w}=\mathbf{x}(s)}.$$

Then the solutions $\mathbf{x}(t)$ *and* $\mathbf{y}(t)$ *of (1.1) and (1.3) are related by*

$$\mathbf{y}(t) - \mathbf{x}(t) = \int_{t_0}^{t} \mathbf{U}(t,s,\mathbf{y}(s)) \left\{ \frac{d}{ds}\boldsymbol{\delta}(s, \mathbf{V}y(s)) \right\} ds \tag{2.33}$$

or (equivalently)

$$\mathbf{y}(t) - \mathbf{x}(t) \quad = \quad \int_{t_0}^{t} \mathbf{U}(t,s,\mathbf{y}(s)) \left\{ \frac{d}{ds} \int_{t_0}^{s} \mathbf{v}(s,u,\mathbf{y}(u))du \right\} ds.$$

Observe that

$$\mathbf{H}(t,s,\mathbf{x}(s)) := \mathbf{K}_3(t,s,\mathbf{x}(s)) \equiv \frac{\partial}{\partial \mathbf{w}} \mathbf{k}(t,s,\mathbf{w})\Big|_{\mathbf{w}=\mathbf{x}(s)}.$$

The following corollary is also stated by Brauer.

Corollary 2.1 *If* $\mathbf{v}(t,s,\mathbf{y})$ *is independent of* t*, say* $\mathbf{v}(t,s,\mathbf{y}) = \mathfrak{v}(s,\mathbf{y})$*, then the solutions* $\mathbf{x}(t)$ *and* $\mathbf{y}(t)$ *are related by*

$$\mathbf{y}(t) - \mathbf{x}(t) = \int_{t_0}^{t} \mathbf{U}(t,s,\mathbf{y}(s))\mathfrak{v}(s,\mathbf{y}(s))ds, \qquad \text{for } t \geq t_0.$$

We shall establish the above Theorem (and a corresponding result for integro-differential equations) as a consequence of our results, obtained using a technique that we introduce in the next section.

3 Embedding techniques

The starting point for our analysis is the equation

$$\mathbf{x}(t) = \mathbf{g}(t) + \int_{t_0}^{t} \mathbf{k}(t,s,\mathbf{x}(s))ds, \quad (t \geq t_0), \tag{3.1}$$

and the perturbed equation

$$\mathbf{y}(t) = \mathbf{g}(t) + \int_{t_0}^{t} \mathbf{k}(t,s,\mathbf{y}(s))ds + \boldsymbol{\delta}(t, \mathbf{V}\mathbf{y}(t)) \tag{3.2}$$

where $\mathbf{g} : [t_0, \infty) \to \mathbb{R}^n$, $\mathbf{k} : [t_0, \infty) \times [t_0, \infty) \times \mathbb{R}^n \to \mathbb{R}^n$ and $\boldsymbol{\delta} : [t_0, \infty) \times [t_0, \infty) \times \mathbb{R}^n \to \mathbb{R}^n$ are continuous functions. In addition, we suppose $\mathbf{k}$ possesses continuous partial derivatives with respect to t and $\mathbf{x}$. In our formulation, we shall suppose that the perturbation function satisfy the condition

$$\boldsymbol{\delta}(t_0, \mathbf{V}\mathbf{y}(t_0)) = \mathbf{0}. \tag{3.3}$$

Given the VIEs (3.1) and (3.2), we shall define[3] the functions

$$\mathbf{x}(t,s) = \mathbf{g}(s) + \int_{t_0}^{t} \mathbf{k}(s,u,\mathbf{x}(u))du, \quad s \geq t \geq t_0, \tag{3.4}$$

and

$$\mathbf{y}(t,s) = \mathbf{g}(s) + \int_{t_0}^{t} \mathbf{k}(s,u,\mathbf{y}(u))du + \boldsymbol{\delta}(t, \mathbf{V}\mathbf{y}(t)), \quad s \geq t \geq t_0. \tag{3.5}$$

We can clearly see from (3.4) and (3.5) that

$$\mathbf{x}(s,s) = \mathbf{x}(s) \quad \text{and} \quad \mathbf{y}(s,s) = \mathbf{y}(s). \tag{3.6}$$

[3] We borrow a strategy employed in another context by Pouzet [17].

Therefore, equation (3.4) and equation (3.5) can be written as

$$\mathbf{x}(t,s) = \mathbf{g}(s) + \int_{t_0}^{t} \mathbf{k}(s,u,\mathbf{x}(u,u))du, \quad s \geq t \geq t_0, \tag{3.7}$$

and

$$\mathbf{y}(t,s) = \mathbf{g}(s) + \int_{t_0}^{t} \mathbf{k}(s,u,\mathbf{y}(u,u))du + \boldsymbol{\delta}(t,\mathbf{V}\mathbf{y}(t,t)), \quad s \geq t \geq t_0, \tag{3.8}$$

respectively. In view of (3.6), we say that $\mathbf{x}(s)$ and $\mathbf{y}(s)$ are 'sections' of $\mathbf{x}(t,s)$ and $\mathbf{y}(t,s)$ respectively.

3.1 Towards a VPF for nonlinear VIEs

If we differentiate (3.7) and (3.8) with respect to t, we obtain

$$\begin{aligned} \frac{\partial}{\partial t}\mathbf{x}(t,s) &= \mathbf{k}(s,t,\mathbf{x}(t,t)), \quad t \geq t_0, \\ \mathbf{x}(t_0,s) &= \mathbf{g}(s), \end{aligned} \tag{3.9}$$

and

$$\begin{aligned} \frac{\partial}{\partial t}\mathbf{y}(t,s) &= \mathbf{k}(s,t,\mathbf{y}(t,t)) + \frac{d}{dt}\boldsymbol{\delta}(t,\mathbf{V}\mathbf{y}(t,t)), \quad t \geq t_0, \\ \mathbf{y}(t_0,s) &= \mathbf{g}(s), \end{aligned} \tag{3.10}$$

respectively. In the case where $\boldsymbol{\delta}(t,\mathbf{V}\mathbf{y}(t)) = \int_{t_0}^{t} \mathbf{v}(t,u,\mathbf{y}(u))du$, $\frac{d}{dt}\boldsymbol{\delta}(t,\mathbf{V}\mathbf{y}(t,t))$ can be expressed as

$$\mathbf{v}(t,t,\mathbf{y}(t,t)) + \int_{t_0}^{t} \mathbf{v}_t(t,u,\mathbf{y}(u,u))du.$$

Related to (3.9) and (3.10) we introduce the equations

$$\frac{\partial}{\partial t}\ \mathbf{x}(t,s,t_0,\gamma) = \mathbf{k}(s,t,\ \mathbf{x}(t,t,t_0,\gamma)), \quad t \geq t_0, \tag{3.11a}$$

$$\mathbf{x}(t_0,s,t_0,\gamma) = \gamma, \tag{3.11b}$$

and

$$\frac{\partial}{\partial t}\ \boldsymbol{y}(t,s,t_0,\gamma) = \mathbf{k}(s,t,\ \boldsymbol{y}(t,t,t_0,\gamma)) + \frac{d}{dt}\boldsymbol{\delta}(t,\mathbf{V}\ \boldsymbol{y}(t,t,t_0,\gamma)), \quad t \geq t_0, \tag{3.12a}$$

$$\boldsymbol{y}(t_0,s,t_0,\gamma) = \gamma, \tag{3.12b}$$

respectively. Thus, of course, when the solutions for an arbitrary fixed $s \geq t_0$ are $\mathbf{x}(t,s,t_0,\gamma)$ and $\boldsymbol{y}(t,s,t_0,\gamma)$ respectively, (i) $\mathbf{x}(t,s,t_0,\mathbf{g}(s))$ is the solution of (3.9) and (ii) $\boldsymbol{y}(t,s,t_0,\mathbf{g}(s))$ is the solution of (3.10). We propose the following lemma for equation (3.11).

Lemma 3.1 *For fixed $s \geq t$, suppose (3.11) is an initial value problem whose solution $\mathbf{x}(t,s,t_0,\gamma)$ exists for $t \geq t_0$. Let $\mathbf{H}(s,t,\ \mathbf{x}(t,t,t_0,\gamma)) = \mathbf{K}_3(s,t,\ \mathbf{x}(t,t,t_0,\gamma))$ be continuous for $t_0 \leq s \leq t$. Then*
(i) $\mathbf{U}(t,s,t_0,\gamma) = \mathbf{X}_4(t,s,t_0,\gamma)$ exists and is solution of

$$\frac{\partial}{\partial t}\mathbf{Z}(t,s,t_0,\gamma) = \mathbf{H}(s,t,\ \mathbf{x}(t,t,t_0,\gamma))\mathbf{Z}(t,t,t_0,\gamma), \tag{3.13}$$

such that $\mathbf{U}(t_0,s,t_0,\gamma) = \mathbf{I}$;
(ii) $\mathbf{v}(t,s,t_0,\gamma) = \ \mathbf{x}_3(t,s,t_0,\gamma)$ exists and is a solution of (3.13) such that

$$\mathbf{v}(t_0,s,t_0,\gamma) = -\mathbf{k}(s,t_0,\ \mathbf{x}(t_0,t_0,t_0,\gamma));$$

(iii) $\mathbf{U}(t,s,t_0,\gamma)$ *and* $\mathbf{v}(t,s,t_0,\gamma)$ *satisfy the relation*

$$\mathbf{v}(t,s,t_0,\gamma) = -\mathbf{U}(t,s,t_0,\gamma)\mathbf{k}(s,t_0,\ \mathbf{x}(t_0,t_0,t_0,\gamma)), \tag{3.14}$$

for $t \geq t_0$.

The reader may compare Lemma 3.1 with Lemma 2.1.

Proof: For fixed s, equation (3.11) can be regarded as an initial value problem, and $\mathbf{x}_3(t,s,t_0,\gamma) = \frac{\partial}{\partial u}\ \mathbf{x}_3(t,s,u,\gamma)$ and $\mathbf{X}_4(t,s,t_0,\gamma) = \frac{\partial}{\partial\gamma}\ \mathbf{x}(t,s,t_0,\gamma)$ exist and can be shown to be continuous. We pause to substantiate the latter conclusions: First, from (3.11),

$$\mathbf{x}(t,s,t_0,\gamma) \quad = \quad \gamma + \int_{t_0}^{t} \mathbf{k}(s,\sigma,\ \mathbf{x}(\sigma,\sigma,t_0,\gamma))d\sigma, \quad (t \geq t_0). \tag{3.15}$$

Differentiating,

$$\frac{\partial}{\partial t_0}\ \mathbf{x}(t,s,t_0,\gamma) = -\mathbf{k}(s,t_0,\ \mathbf{x}(t_0,t_0,t_0,\gamma)) + \int_{t_0}^{t} \mathbf{K}_3(s,\sigma,\ \mathbf{x}(\sigma,\sigma,t_0,\gamma))\frac{\partial}{\partial t_0}\ \mathbf{x}(\sigma,\sigma,t_0,\gamma))d\sigma, \quad (t \geq t_0).$$

Thus, putting $s = t$, $\frac{\partial}{\partial t_0}\ \mathbf{x}(t,s,t_0,\gamma)$ is the solution of a Volterra integral equation of the second kind (with continuous inhomogeneous term) and is continuous and hence $\frac{\partial}{\partial t_0}\ \mathbf{x}(t,s,t_0,\gamma)$ is also continuous. Secondly,

$$\frac{\partial}{\partial\gamma}\ \mathbf{x}(t,s,t_0,\gamma)) \quad = \quad \mathbf{I} + \int_{t_0}^{t} \mathbf{K}_3(s,\sigma,\ \mathbf{x}(\sigma,\sigma,t_0,\gamma))\frac{\partial}{\partial\gamma}\ \mathbf{x}(\sigma,\sigma,t_0,\gamma))d\sigma, \quad (t \geq t_0). \tag{3.16}$$

Picking $s = t$, $\frac{\partial}{\partial\gamma}\ \mathbf{x}(t,t,t_0,\gamma)$ is a solution of a Volterra integral equation of the second kind that is continuous, and hence $\mathbf{X}_4(t,s,t_0,\gamma) = \frac{\partial}{\partial\gamma}\ \mathbf{x}(t,s,t_0,\gamma)$ is also continuous.

In view of the previous remarks, equation (3.11) can be differentiated with respect to t_0 or with respect to γ. To prove part (i) of our Lemma, we differentiate (3.11) with respect to γ, to obtain

$$\frac{\partial}{\partial t}\mathbf{X}_4(t,s,t_0,\gamma) = \mathbf{K}_3(s,t,\ \mathbf{x}(t,t,t_0,\gamma))\mathbf{X}_4(t,t,t_0,\gamma),$$

and $\mathbf{X}_4(t_0,s,t_0,\gamma) = \mathbf{I}$.

To prove part (ii), we differentiate (3.11) with respect to t_0, to obtain

$$\frac{\partial}{\partial t}\ \mathbf{x}_3(t,s,t_0,\gamma) = \mathbf{K}_3(s,t,\ \mathbf{x}(t,t,t_0,\gamma))\ \mathbf{x}_3(t,t,t_0,\gamma).$$

Differentiating $\mathbf{x}(t_0,s,t_0,\gamma) = \gamma$ with respect to t_0, we obtain $\mathbf{x}_1(t_0,s,t_0,\gamma) + \mathbf{x}_3(t_0,s,t_0,\gamma) = 0$, implying that $\mathbf{x}_1(t_0,s,t_0,\gamma) = -\ \mathbf{x}_3(t_0,s,t_0,\gamma)$. From (3.11) we have

$$\mathbf{x}_1(t_0,s,t_0,\gamma) = \mathbf{k}(s,t_0,\ \mathbf{x}(t_0,t_0,t_0,\gamma)).$$

Hence $\mathbf{v}(t_0,s,t_0,\gamma) = -\mathbf{k}(s,t_0,\ \mathbf{x}(t_0,t_0,t_0,\gamma))$.

Finally, we prove part (iii) by observing that

$$\mathbf{x}(t,s,t_0,\gamma) = \ \mathbf{x}(t,s,u,\ \mathbf{x}(u,s,t_0,\gamma)), \quad t \geq u. \tag{3.17}$$

If we differentiate (3.17) with respect to the variable t_0, we obtain

$$\mathbf{x}_3(t,s,t_0,\gamma) = \mathbf{X}_4(t,s,u,\ \mathbf{x}(u,s,t_0,\gamma))\ \mathbf{x}_3(u,s,t_0,\gamma),$$

which, when we substitute $u = t_0$, yields

$$\mathbf{x}_3(t,s,t_0,\gamma) = \mathbf{X}_4(t,s,t_0,\ \mathbf{x}(t_0,s,t_0,\gamma))\ \mathbf{x}_3(t_0,s,t_0,\gamma).$$

Consequently, (3.14) is established, i.e.

$$\mathbf{v}(t,s,t_0,\gamma) = -\mathbf{U}(t,s,t_0,\gamma)\mathbf{k}(s,t_0,\ \mathbf{x}(t_0,t_0,t_0,\gamma)),$$

completing the proof. □.

3.2 An Alekseev-type result

Based on Lemma 3.1, we next provide a VPF for the nonlinear equations

$$\frac{\partial}{\partial t}\mathbf{x}(t,s) \quad = \quad \mathbf{k}(s,t,\mathbf{x}(t,t)), \quad t \geq t_0, \tag{3.18}$$

and

$$\frac{\partial}{\partial t}\mathbf{y}(t,s) \quad = \quad \mathbf{k}(s,t,\mathbf{y}(t,t)) + \frac{d}{dt}\boldsymbol{\delta}(t,\mathbf{V}\mathbf{y}(t,t)), \quad t \geq t_0, \tag{3.19}$$

with the same initial condition $\mathbf{x}(t_0,s) = \mathbf{y}(t_0,s) = \mathbf{g}(s)$, assuming that $\frac{d}{du}\boldsymbol{\delta}(u,\mathbf{V}\mathbf{y}(u,u))$ exists.

Theorem 3.1 *Let $\mathbf{x}(t,s)$ be the unique solution of (3.18) satisfying $\mathbf{x}(t_0,s) = \mathbf{g}(s)$ for $s \geq t \geq t_0$. If $\mathbf{y}(t,s)$ is the unique solution of (3.19) satisfying $\mathbf{y}(t_0,s) = \mathbf{g}(s)$ for $s \geq t \geq t_0$, then*

$$\mathbf{y}(t,s) = \mathbf{x}(t,s) + \int_{t_0}^{t} \mathbf{U}(t,s,u,\mathbf{y}(u,s))\left\{\frac{d}{du}\boldsymbol{\delta}(u,\mathbf{V}\mathbf{y}(u,u))\right\}du,$$

where $\mathbf{U}(t,s,t_0,\mathbf{g}(s)) = \mathbf{X}_4(t,s,t_0,\boldsymbol{\gamma})\rfloor_{\boldsymbol{\gamma}=\mathbf{g}(s)}$.

This result is an immediate consequence (on setting $\boldsymbol{\gamma} = \mathbf{g}(s)$) of the following result.

Theorem 3.2 *For fixed $s \geq t$, suppose that the solution $\mathbf{x}(t,s,t_0,\boldsymbol{\gamma})$ of (3.11) with $\mathbf{x}(t_0,s,t_0,\boldsymbol{\gamma}) = \boldsymbol{\gamma}$ exists for $t \geq t_0$. In addition, suppose that Lemma 3.1 holds for (3.11). Then any solution $\boldsymbol{y}(t,s,t_0,\boldsymbol{\gamma})$ of (3.12), such that $\boldsymbol{y}(t_0,s,t_0,\boldsymbol{\gamma}) = \boldsymbol{\gamma}$, for $t \geq t_0$ satisfies*

$$\boldsymbol{y}(t,s,t_0,\boldsymbol{\gamma}) = \mathbf{x}(t,s,t_0,\boldsymbol{\gamma}) + \int_{t_0}^{t} \mathbf{U}(t,s,u,\ \boldsymbol{y}(u,s,t_0,\boldsymbol{\gamma}))\left\{\frac{d}{du}\boldsymbol{\delta}(u,\mathbf{V}\ \boldsymbol{y}(u,u,t_0,\boldsymbol{\gamma}))\right\}du, \tag{3.20}$$

where $\mathbf{U}(t,s,t_0,\mathbf{g}(s)) = \mathbf{X}_4(t,s,t_0,\boldsymbol{\gamma})$.

The reader should compare Theorem 3.2 with Theorem 2.1.
Proof: For fixed s, and for fixed t such that $s \geq t$, define

$$\mathbf{p}(u) = \mathbf{x}(t,s,u,\ \boldsymbol{y}(u,s,t_0,\boldsymbol{\gamma})).$$

Then

$$\mathbf{p}'(u) = \mathbf{x}_3(t,s,u,\ \boldsymbol{y}(u,s,t_0,\boldsymbol{\gamma})) + \mathbf{X}_4(t,s,u,\ \boldsymbol{y}(u,s,t_0,\boldsymbol{\gamma}))\ \boldsymbol{y}_1(u,s,t_0,\boldsymbol{\gamma}).$$

Using Lemma 3.1, we obtain

$$\mathbf{p}'(u) = -\mathbf{U}(t,s,u,\ \boldsymbol{y}(u,s,t_0,\boldsymbol{\gamma}))\mathbf{k}(s,u,\ \mathbf{x}(u,u,t_0,\boldsymbol{\gamma})) + \mathbf{U}(t,s,u,\ \boldsymbol{y}(u,s,t_0,\boldsymbol{\gamma}))\ \boldsymbol{y}_1(u,s,t_0,\boldsymbol{\gamma}).$$

Using (3.19), $\mathbf{p}'(u)$ reduces to

$$\mathbf{p}'(u) = \mathbf{U}(t,s,u,\ \boldsymbol{y}(u,s,t_0,\boldsymbol{\gamma}))\left\{\frac{d}{du}\boldsymbol{\delta}(u,\mathbf{V}\ \boldsymbol{y}(u,u,t_0,\boldsymbol{\gamma}))\right\}. \tag{3.21}$$

Integrating (3.21) with respect to u and observing that $\mathbf{p}(t) = \boldsymbol{y}(t,s,t_0,\boldsymbol{\gamma}))$ and $\mathbf{p}(t_0) = \mathbf{x}(t,s,t_0,\boldsymbol{\gamma})$, we obtain the relation (3.20). This completes the proof. □

4 A VPF for Volterra integral equations

The results in the last section were VPFs for the embedding problem expressed in terms of $\mathbf{U}(t,s,u,\ \boldsymbol{y}(u,s,t_0,\boldsymbol{\gamma}))$. In this section, we shall provide a VPF for VIEs, obtained from Theorem 3.1; this new result is expressed in terms of a function $\mathbf{U}(t,t_0,\boldsymbol{\alpha})$ that (for $\boldsymbol{\alpha} \in \mathbb{R}^m$) satisfies the equation

$$\mathbf{U}(t,t_0,\boldsymbol{\alpha}) = \mathbf{I} + \int_{t_0}^{t} \mathbf{K}_3(t,u,\mathbf{x}(u,t_0,\boldsymbol{\alpha}))\mathbf{U}(u,t_0,\boldsymbol{\alpha})du \quad (t \geq t_0). \tag{4.1}$$

The equation (4.1) can be related directly to the VIE (3.1) (for details see for example [1] or [6]). The following lemma provides the equivalence between $\mathbf{U}(t,t,t_0,\boldsymbol{\gamma})$ and $\mathbf{U}(t,t_0,\boldsymbol{\alpha})$ needed to deduce our result.

Lemma 4.1 *Suppose that the solution* $\mathbf{x}(t,s,t_0,\boldsymbol{\gamma})$ *of (3.11) exists and is unique for* $t \geq t_0$*. Let* $\mathbf{U}(t,s,t_0,\boldsymbol{\gamma}) = \mathbf{X}_4(t,s,t_0,\boldsymbol{\gamma})$ *exist for* $t \geq t_0$*. If* $\mathbf{U}(s,t_0,\boldsymbol{\alpha})$ *is the solution of the equation (4.1) then for* $\boldsymbol{\gamma} = \mathbf{g}(s)$ *and* $\boldsymbol{\alpha} = \mathbf{g}(t_0)$*, we obtain*

$$\mathbf{U}(s,s,t_0,\mathbf{g}(s)) \equiv \mathbf{U}(s,t_0,\mathbf{g}(t_0)). \tag{4.2}$$

Proof: If $\mathbf{U}(s,t_0,\boldsymbol{\alpha})$ is the solution of the equation (4.1) then for $\boldsymbol{\alpha} = \mathbf{g}(t_0)$, we obtain

$$\mathbf{U}(s,t_0,\mathbf{g}(t_0)) = \mathbf{I} + \int_{t_0}^{s} \mathbf{K}_3(s,u,\mathbf{x}(u))\mathbf{U}(u,t_0,\mathbf{g}(t_0))du, \quad (s \geq t_0). \tag{4.3}$$

¿From Lemma 3.1, $\mathbf{U}(t,s,t_0,\boldsymbol{\gamma})$ exists and satisfies the equation

$$\frac{\partial}{\partial t}\mathbf{U}(t,s,t_0,\boldsymbol{\gamma}) = \mathbf{K}_3(s,t,\ \mathbf{x}(t,t,t_0,\boldsymbol{\gamma}))\mathbf{U}(t,t,t_0,\boldsymbol{\gamma}), \tag{4.4}$$

and $\mathbf{U}(t_0,s,t_0,\boldsymbol{\gamma}) = \mathbf{I}$. Integrating (4.4) with respect to t from t_0 to t, we obtain

$$\mathbf{U}(t,s,t_0,\boldsymbol{\gamma}) = \mathbf{I} + \int_{t_0}^{t} \mathbf{K}_3(s,u,\ \mathbf{x}(u,u,t_0,\boldsymbol{\gamma}))\mathbf{U}(u,u,t_0,\boldsymbol{\gamma})du.$$

For $t = s$ and $\boldsymbol{\gamma} = \mathbf{g}(s)$ we obtain, since $\mathbf{x}(u) = \ \mathbf{x}(u,u,t_0,\mathbf{g}(s))$,

$$\mathbf{U}(s,s,t_0,\mathbf{g}(s)) = \mathbf{I} + \int_{t_0}^{s} \mathbf{K}_3(s,u,\mathbf{x}(u))\mathbf{U}(u,u,t_0,\mathbf{g}(s))du. \tag{4.5}$$

From (4.3) and (4.5), we see that $\mathbf{U}(s,s,t_0,\mathbf{g}(s)) \equiv \mathbf{U}(s,t_0,\mathbf{g}(t_0))$. This completes the proof. □

Remark 4.1 *The preceding Lemma states that*

$$\mathbf{U}(s,s,t_0,\mathbf{x}(t_0,s)) \equiv \mathbf{U}(s,t_0,\mathbf{x}(t_0)). \tag{4.6}$$

in the notation of (3.9).

We are now in a position to provide the main VPF for the VIEs

$$\mathbf{x}(t) = \mathbf{g}(t) + \int_{t_0}^{t} \mathbf{k}(t,s,\mathbf{x}(s))ds, \quad (t \geq t_0), \tag{4.7}$$

and

$$\mathbf{y}(t) = \mathbf{g}(t) + \int_{t_0}^{t} \mathbf{k}(t,s,\mathbf{y}(s))ds + \boldsymbol{\delta}(t,\mathbf{V}\mathbf{y}(t)), \quad (t \geq t_0). \tag{4.8}$$

where $\boldsymbol{\delta}$ satisfy the condition (3.3). We now deduce from Theorem 3.1 a result in terms of $\mathbf{U}$ rather than $\mathbf{U}$.

Theorem 4.1 *Suppose that* $\mathbf{x}(t)$ *is the solution of the nonlinear VIE (4.7) for* $t \geq t_0$ *and let* $\mathbf{U}(t,t_0,\boldsymbol{\alpha})$ *be the solution of the equation (4.1). Then any solution* $\mathbf{y}(t)$ *of the perturbed VIE (4.8) for* $(t \geq t_0)$ *satisfies the relation*

$$\mathbf{y}(t) = \mathbf{x}(t) + \int_{t_0}^{t} \mathbf{U}(t,u,\mathbf{y}(u)) \left\{ \frac{d}{du}\boldsymbol{\delta}(u,\mathbf{V}\mathbf{y}(u)) \right\} du \quad (t \geq t_0) \tag{4.9}$$

or

$$\mathbf{y}(t) = \mathbf{x}(t) + \boldsymbol{\delta}(t,\mathbf{y}(t)) - \int_{t_0}^{t} \frac{\partial}{\partial u}\mathbf{U}(t,u,\mathbf{y}(u))\boldsymbol{\delta}(u,\mathbf{V}\mathbf{y}(u))du \quad (t \geq t_0). \tag{4.10}$$

Remark 4.2 *The reader should compare Theorem 4.1 with Theorem 2.4.*

Proof: Suppose that the hypothesis of Theorem 3.1 holds, then for $t = s$ we obtain

$$\mathbf{y}(s,s) = \mathbf{x}(s,s) + \int_{t_0}^{s} \mathbf{U}(s,s,u,\mathbf{y}(u,s)) \left\{ \frac{d}{du}\boldsymbol{\delta}(u,\mathbf{V}\mathbf{y}(u,u)) \right\} du. \tag{4.11}$$

We observe from the definition of the embedded functions $\mathbf{y}(t,s)$ and $\mathbf{x}(t,s)$ that $\mathbf{y}(s,s) = \mathbf{y}(s)$ and $\mathbf{x}(s,s) = \mathbf{x}(s)$ respectively. From Lemma 4.1, it is clear that $\mathbf{U}(s,s,u,\mathbf{y}(u,s)) \equiv \mathbf{U}(s,u,\mathbf{y}(u))$. Equation (4.11) becomes

$$\mathbf{y}(s) = \mathbf{x}(s) + \int_{t_0}^{s} \mathbf{U}(s,u,\mathbf{y}(u)) \left\{ \frac{d}{du}\boldsymbol{\delta}(u,\mathbf{V}\mathbf{y}(u)) \right\} du \quad (s \geq t_0). \tag{4.12}$$

Integrating (4.12) by parts and noting that $\mathbf{U}(s,s,\mathbf{y}(s)) = \mathbf{I}$, we obtain

$$\mathbf{y}(s) = \mathbf{x}(s) + \boldsymbol{\delta}(s,\mathbf{y}(s)) - \int_{t_0}^{s} \frac{\partial}{\partial u}\mathbf{U}(s,u,\mathbf{y}(u))\boldsymbol{\delta}(u,\mathbf{V}\mathbf{y}(u))du \quad (s \geq t_0).$$

Since s is arbitrary, putting $s = t$ establishes the relations (4.9) and (4.10), thus completing the proof. □

Remark 4.3 *The VPFs for linear VIEs given in (2.19) and (2.20) are special cases of (4.10) and (4.9) respectively.*

5 VPF for nonlinear VIDEs

We now turn to integro-differential equations: consider the general system of nonlinear Volterra integro-differential equations

$$\mathbf{x}'(t) = \mathbf{f}\left(t, \mathbf{x}(t), \int_{t_0}^{t} \mathbf{k}(t,s,\mathbf{x}(s))ds\right); \quad \mathbf{x}(t_0) = \mathbf{x}_0 \quad (t \geq t_0), \tag{5.1}$$

where $\mathbf{f}$ and $\mathbf{k}$ are continuous. The VIDE (5.1) is equivalent to the pair of Volterra integral equations

$$\widehat{\mathbf{x}}(t) = \mathbf{x}_0 + \int_{t_0}^{t} \mathbf{f}(s,\widehat{\mathbf{x}}(s),\widehat{\mathbf{u}}(s))ds \quad (t \geq t_0), \tag{5.2a}$$

and

$$\widehat{\mathbf{u}}(t) = \int_{t_0}^{t} \mathbf{k}(t,s,\widehat{\mathbf{x}}(s))ds \quad (t \geq t_0), \tag{5.2b}$$

where $\widehat{\mathbf{x}}(t) = \mathbf{x}(t)$. Since (5.2) can be express as

$$\begin{bmatrix} \widehat{\mathbf{x}}(t) \\ \widehat{\mathbf{u}}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{x}_0 \\ \mathbf{0} \end{bmatrix} + \int_{t_0}^{t} \begin{bmatrix} \mathbf{f}(s,\widehat{\mathbf{x}}(s),\widehat{\mathbf{u}}(s)) \\ \mathbf{k}(t,s,\widehat{\mathbf{x}}(s)) \end{bmatrix} ds \tag{5.3}$$

our results for Volterra integral equations apply immediately. However, (5.3) is of a special form and hence the sensitivity of a solution $\mathbf{x}(t)$ to $\mathbf{x}_0$ is not immediately transparent by this approach. In this section, we develop a VPF for the problems structured in the form (5.1). We shall also see that, for VIDEs, the perturbation function does not have to be differentiable as was the case with VIEs.

For $s \geq t \geq t_0$, we define the functions $\mathbf{x}^1(t,s)$ and $\mathbf{x}^2(t,s)$ by

$$\mathbf{x}^1(t,s) = \mathbf{x}_0 + \int_{t_0}^{t} \mathbf{f}(\sigma,\widehat{\mathbf{x}}(\sigma),\widehat{\mathbf{u}}(\sigma))d\sigma \tag{5.4a}$$

and

$$\mathbf{x}^2(t,s) = \int_{t_0}^{t} \mathbf{k}(s,\sigma,\widehat{\mathbf{x}}(\sigma))d\sigma \tag{5.4b}$$

where $\widehat{\mathbf{x}}(\sigma), \widehat{\mathbf{u}}(\sigma)$ satisfies the pair of equations (5.2). From (5.4) it is obvious that $\widehat{\mathbf{x}}(\sigma) = \mathbf{x}^1(\sigma,\sigma)$ and $\widehat{\mathbf{u}}(\sigma) = \mathbf{x}^2(\sigma,\sigma)$ and therefore

$$\mathbf{x}^1(t,s) = \mathbf{x}_0 + \int_{t_0}^{t} \mathbf{f}(\sigma,\mathbf{x}^1(\sigma,\sigma),\mathbf{x}^2(\sigma,\sigma)))d\sigma \tag{5.5a}$$

and

$$\mathbf{x}^2(t,s) = \int_{t_0}^{t} \mathbf{k}(s,\sigma,\mathbf{x}^1(\sigma,\sigma))d\sigma, \tag{5.5b}$$

for $t \geq t_0$. The pair of equations (5.5) contains $\left[\widehat{\mathbf{x}}(s), \widehat{\mathbf{u}}(s)\right]^T$ as a 'section' of its solution, that is, $\left[\widehat{\mathbf{x}}(s), \widehat{\mathbf{u}}(s)\right]^T = \left[\mathbf{x}^1(s,s), \mathbf{x}^2(s,s)\right]^T$.

We also consider the perturbed system of nonlinear Volterra integro-differential equations

$$\mathbf{y}'(t) = \widehat{\boldsymbol{\delta}}(t,\mathbf{y}(t),\mathbf{V}\mathbf{y}(t)) + \mathbf{f}\left(t,\mathbf{y}(t),\int_{t_0}^{t}\mathbf{k}(t,s,\mathbf{y}(s))ds\right);\ \mathbf{y}(t_0) = \mathbf{x}_0 \quad (t \geq t_0), \tag{5.6}$$

where $\mathbf{y}(t)$ exists and where $\mathbf{V}$ is a Volterra operator (see (1.4) and (1.5) for examples). Corresponding to the perturbed VIDE (5.6) is the pair of Volterra integral equations

$$\widehat{\mathbf{y}}(t) = \mathbf{x}_0 + \int_{t_0}^{t}\left(\widehat{\boldsymbol{\delta}}(s,\widehat{\mathbf{y}}(s),\mathbf{V}\widehat{\mathbf{y}}(s)) + \mathbf{f}(s,\widehat{\mathbf{y}}(s),\widehat{\mathbf{v}}(s))\right)ds, \quad (t \geq t_0), \tag{5.7a}$$

and

$$\widehat{\mathbf{v}}(t) = \int_{t_0}^{t}\mathbf{k}(t,s,\widehat{\mathbf{y}}(s))ds \quad (t \geq t_0), \tag{5.7b}$$

where $\widehat{\mathbf{y}}(t) = \mathbf{y}(t)$. For the perturbed problem (5.7), we define the functions $\mathbf{y}^1(t,s)$ and $\mathbf{y}^2(t,s)$, for $s \geq t \geq t_0$ by

$$\mathbf{y}^1(t,s) = \mathbf{x}_0 + \int_{t_0}^{t}\left(\widehat{\boldsymbol{\delta}}(\sigma,\widehat{\mathbf{y}}(\sigma),\mathbf{V}\widehat{\mathbf{y}}(\sigma)) + \mathbf{f}(\sigma,\widehat{\mathbf{y}}(\sigma),\widehat{\mathbf{v}}(\sigma))\right)d\sigma \tag{5.8a}$$

and

$$\mathbf{y}^2(t,s) = \int_{t_0}^{t}\mathbf{k}(s,\sigma,\widehat{\mathbf{y}}(\sigma))d\sigma \tag{5.8b}$$

where $\widehat{\mathbf{y}}(\sigma), \widehat{\mathbf{v}}(\sigma)$ satisfies the pair of equations (5.7). Observing that $\widehat{\mathbf{y}}(\sigma) = \mathbf{y}^1(\sigma,\sigma)$ and $\widehat{\mathbf{v}}(\sigma) = \mathbf{y}^2(\sigma,\sigma)$ in (5.8), we have the perturbed pair of systems of functional equations, for $t \geq t_0$,

$$\mathbf{y}^1(t,s) = \mathbf{x}_0 + \int_{t_0}^{t}\left(\widehat{\boldsymbol{\delta}}(\sigma,\mathbf{y}^1(\sigma,\sigma),\mathbf{V}\mathbf{y}^1(\sigma,\sigma)) + \mathbf{f}(\sigma,\mathbf{y}^1(\sigma,\sigma),\mathbf{y}^2(\sigma,\sigma))\right)d\sigma \tag{5.9a}$$

and

$$\mathbf{y}^2(t,s) = \int_{t_0}^{t}\mathbf{k}(s,\sigma,\mathbf{y}^1(\sigma,\sigma))d\sigma. \tag{5.9b}$$

Here, $\left[\widehat{\mathbf{y}}(s), \widehat{\mathbf{v}}(s)\right]^T$ is a 'section' of the solution of the pair of equations (5.9), i.e., $\left[\widehat{\mathbf{y}}(s), \widehat{\mathbf{v}}(s)\right]^T = \left[\mathbf{y}^1(s,s), \mathbf{y}^2(s,s)\right]^T$.

Differentiating the systems (5.5) and (5.9) partially with respect to t we obtain

$$\frac{\partial}{\partial t}\mathbf{x}^1(t,s) = \mathbf{f}(t,\mathbf{x}^1(t,t),\mathbf{x}^2(t,t)); \quad \mathbf{x}^1(t_0,s) = \mathbf{x}_0 \tag{5.10a}$$

$$\frac{\partial}{\partial t}\mathbf{x}^2(t,s) = \mathbf{k}(s,t,\mathbf{x}^1(t,t)); \quad \mathbf{x}^2(t_0,s) = \mathbf{0} \tag{5.10b}$$

and

$$\frac{\partial}{\partial t}\mathbf{y}^1(t,s) = \widehat{\boldsymbol{\delta}}(t,\mathbf{y}^1(t,t),\mathbf{V}\mathbf{y}^1(t)) + \mathbf{f}(t,\mathbf{y}^1(t,t),\mathbf{y}^2(t,t)); \quad \mathbf{y}^1(t_0,t) = \mathbf{x}_0 \tag{5.11a}$$

$$\frac{\partial}{\partial t}\mathbf{y}^2(t,s) = \mathbf{k}(s,t,\mathbf{y}^1(t,t)); \quad \mathbf{y}^2(t_0,t) = \mathbf{0} \tag{5.11b}$$

respectively. Analogous to Theorem 3.1, the following theorem provides a VPF for (5.10) and (5.11).

Theorem 5.1 *For $s \geq t$, let $\mathbf{x}^1(t,s)$, $\mathbf{x}^2(t,s)$ be the unique solution of the systems (5.10) satisfying $\mathbf{x}^1(t_0,s) = \mathbf{x}_0$, $\mathbf{x}^2(t_0,s) = \mathbf{0}$ for $t \geq t_0$. If $\mathbf{y}^1(t,s)$, $\mathbf{y}^2(t,s)$ are the unique solution of the systems (5.11) satisfying $\mathbf{y}^1(t_0,s) = \mathbf{x}_0$, $\mathbf{y}^2(t_0,s) = \mathbf{0}$ for $t \geq t_0$, then*

$$\mathbf{y}^1(t,s) = \mathbf{x}^1(t,s) + \int_{t_0}^{t} \mathbf{X}_4^1(t,s,\sigma,\mathbf{y}^1(\sigma,s),\mathbf{y}^2(\sigma,s))\widehat{\boldsymbol{\delta}}(\sigma,\mathbf{y}^1(\sigma,\sigma),\mathbf{V}\mathbf{y}^1(\sigma,\sigma))d\sigma, \tag{5.12a}$$

and

$$\mathbf{y}^2(t,s) = \mathbf{x}^2(t,s) + \int_{t_0}^{t} \mathbf{X}_4^2(t,s,\sigma,\mathbf{y}^1(\sigma,s),\mathbf{y}^2(\sigma,s))\widehat{\boldsymbol{\delta}}(\sigma,\mathbf{y}^1(\sigma,\sigma),\mathbf{V}\mathbf{y}^1(\sigma,\sigma))d\sigma. \tag{5.12b}$$

Proof: Related to equations (5.10) and (5.11) are the equations

$$\frac{\partial}{\partial t}\begin{bmatrix} \mathbf{x}^1(t,s,t_0,\mathbf{x}_0,\boldsymbol{\gamma}) \\ \mathbf{x}^2(t,s,t_0,\mathbf{x}_0,\boldsymbol{\gamma}) \end{bmatrix} = \begin{bmatrix} \mathbf{f}(t,\ \mathbf{x}^1(t,t,t_0,\mathbf{x}_0,\boldsymbol{\gamma}),\ \mathbf{x}^2(t,t,t_0,\mathbf{x}_0,\boldsymbol{\gamma})) \\ \mathbf{k}(s,t,\ \mathbf{x}^1(t,t,t_0,\mathbf{x}_0,\boldsymbol{\gamma})) \end{bmatrix}, \tag{5.13a}$$

$$\begin{bmatrix} \mathbf{x}^1(t_0,s,t_0,\mathbf{x}_0,\boldsymbol{\gamma}) \\ \mathbf{x}^2(t_0,s,t_0,\mathbf{x}_0,\boldsymbol{\gamma}) \end{bmatrix} = \begin{bmatrix} \mathbf{x}_0 \\ \boldsymbol{\gamma} \end{bmatrix}, \tag{5.13b}$$

and

$$\begin{aligned}\frac{\partial}{\partial t}\begin{bmatrix} \boldsymbol{y}^1(t,s,t_0,\mathbf{x}_0,\boldsymbol{\gamma}) \\ \boldsymbol{y}^2(t,s,t_0,\mathbf{x}_0,\boldsymbol{\gamma}) \end{bmatrix} &= \begin{bmatrix} \mathbf{f}(t,\ \boldsymbol{y}^1(t,t,t_0,\mathbf{x}_0,\boldsymbol{\gamma}),\ \boldsymbol{y}^2(t,t,t_0,\mathbf{x}_0,\boldsymbol{\gamma})) \\ \mathbf{k}(s,t,\ \boldsymbol{y}^1(t,t,t_0,\mathbf{x}_0,\boldsymbol{\gamma})) \end{bmatrix} \\ &+ \begin{bmatrix} \widehat{\boldsymbol{\delta}}(t,\ \boldsymbol{y}^1(t,t,t_0,\mathbf{x}_0,\boldsymbol{\gamma}),\mathbf{V}\mathbf{y}^1(t,t,t_0,\mathbf{x}_0,\boldsymbol{\gamma})) \\ \mathbf{0} \end{bmatrix}\end{aligned} \tag{5.14a}$$

$$\begin{bmatrix} \boldsymbol{y}^1(t_0,s,t_0,\mathbf{x}_0,\boldsymbol{\gamma}) \\ \boldsymbol{y}^2(t_0,s,t_0,\mathbf{x}_0,\boldsymbol{\gamma}) \end{bmatrix} = \begin{bmatrix} \mathbf{x}_0 \\ \boldsymbol{\gamma} \end{bmatrix}, \tag{5.14b}$$

By the assumptions of Lemma 3.1 for equation (5.13), the Jacobian matrix $\mathbf{U}(t,s,t_0,\mathbf{x}_0,\boldsymbol{\gamma})$ exists for fixed $s \geq t \geq t_0$, and is given by

$$\mathbf{U}(t,s,t_0,\mathbf{x}_0,\boldsymbol{\gamma}) = \begin{bmatrix} \mathbf{X}_4^1(t,s,t_0,\mathbf{x}_0,\boldsymbol{\gamma}) & \mathbf{x}_5^1(t,s,t_0,\mathbf{x}_0,\boldsymbol{\gamma}) \\ \mathbf{X}_4^2(t,s,t_0,\mathbf{x}_0,\boldsymbol{\gamma}) & \mathbf{x}_5^2(t,s,t_0,\mathbf{x}_0,\boldsymbol{\gamma}) \end{bmatrix}.$$

Clearly, $\mathbf{U}(t_0,s,t_0,\mathbf{x}_0,\boldsymbol{\gamma}) = \mathbf{I}$. An application of the result of Theorem 3.2 to the systems (5.13) and (5.14) yields for $t \geq t_0$

$$\begin{aligned}\begin{bmatrix} \boldsymbol{y}^1(t,s,t_0,\mathbf{x}_0,\boldsymbol{\gamma}) \\ \boldsymbol{y}^2(t,s,t_0,\mathbf{x}_0,\boldsymbol{\gamma}) \end{bmatrix} &= \begin{bmatrix} \mathbf{x}^1(t,s,t_0,\mathbf{x}_0,\boldsymbol{\gamma}) \\ \mathbf{x}^2(t,s,t_0,\mathbf{x}_0,\boldsymbol{\gamma}) \end{bmatrix} \\ &+ \int_{t_0}^{t} \begin{bmatrix} \mathbf{X}_4^1(t,s,\sigma,\ \boldsymbol{y}^1(\sigma,s),\ \boldsymbol{y}^2(\sigma,s))\widehat{\boldsymbol{\delta}}(\sigma,\ \boldsymbol{y}^1(\sigma,\sigma),\mathbf{V}\ \boldsymbol{y}^1(\sigma,\sigma)) \\ \mathbf{X}_4^2(t,s,\sigma,\ \boldsymbol{y}^1(\sigma,s),\ \boldsymbol{y}^2(\sigma,s))\widehat{\boldsymbol{\delta}}(\sigma,\ \boldsymbol{y}^1(\sigma,\sigma),\mathbf{V}\ \boldsymbol{y}^1(\sigma,\sigma)) \end{bmatrix} d\sigma,\end{aligned} \tag{5.15}$$

where $\boldsymbol{y}^i(\sigma,\cdot) = \boldsymbol{y}^i(\sigma,\cdot,\mathbf{x}_0,\boldsymbol{\gamma})$, for $i = 1,2$ under the integral sign. The relations (5.12) immediately follow by substituting for $\boldsymbol{\gamma} = \mathbf{0}$ in (5.15). This completes the proof. □

The following result is our main VPF for the general nonlinear VIDEs

$$\mathbf{x}'(t) = \mathbf{f}\left(t, \mathbf{x}(t), \int_{t_0}^{t} \mathbf{k}(t, s, \mathbf{x}(s))ds\right), \quad (t \geq t_0), \tag{5.16}$$

and

$$\mathbf{y}'(t) = \widehat{\boldsymbol{\delta}}(t, \mathbf{y}(t), \mathbf{V}\mathbf{y}(t)) + \mathbf{f}\left(t, \mathbf{y}(t), \int_{t_0}^{t} \mathbf{k}(t, s, \mathbf{y}(s))ds\right), \quad (t \geq t_0), \tag{5.17}$$

with the same initial conditions.

Theorem 5.2 *Let the solution $\mathbf{x}(t)$ of (5.16) such that $\mathbf{x}(t_0) = \mathbf{x}_0$, exist and be unique for $t \geq t_0$. Suppose that Theorem 5.1 holds for the functional systems (5.10) and (5.11). If $\mathbf{y}(t)$ is the unique solution of (5.6) satisfying $\mathbf{y}(t_0) = \mathbf{x}_0$, for $t \geq t_0$, then*

$$\mathbf{y}(t) = \mathbf{x}(t) + \int_{t_0}^{t} \widehat{\mathbf{X}}_3(t, s, \mathbf{y}(s), \widehat{\mathbf{v}}(s))\widehat{\boldsymbol{\delta}}(s, \mathbf{y}(s), \mathbf{V}\mathbf{y}(s))ds, \tag{5.18}$$

where

$$\widehat{\mathbf{v}}(t) = \widehat{\mathbf{u}}(t) + \int_{t_0}^{t} \widehat{\mathbf{U}}_3(t, s, \mathbf{y}(s), \widehat{\mathbf{v}}(s))\widehat{\boldsymbol{\delta}}(s, \mathbf{y}(s), \mathbf{V}\mathbf{y}(s))ds.$$

Proof. Suppose that Theorem 5.1 holds for the functional systems (5.10) and (5.11). Then letting $t = s$ in (5.12), we obtain

$$\mathbf{y}^1(s,s) = \mathbf{x}^1(s,s) + \int_{t_0}^{s} \mathbf{X}_4^1(s, s, \sigma, \mathbf{y}^1(\sigma, s), \mathbf{y}^2(\sigma, s))\widehat{\boldsymbol{\delta}}(\sigma, \mathbf{y}^1(\sigma,\sigma), \mathbf{V}\mathbf{y}^1(\sigma,\sigma))d\sigma$$

and

$$\mathbf{y}^2(s,s) = \mathbf{x}^2(s,s) + \int_{t_0}^{s} \mathbf{X}_4^2(s, s, \sigma, \mathbf{y}^1(\sigma, s), \mathbf{y}^2(\sigma, s))\widehat{\boldsymbol{\delta}}(\sigma, \mathbf{y}^1(\sigma,\sigma), \mathbf{V}\mathbf{y}^1(\sigma,\sigma))d\sigma.$$

Since $\left[\mathbf{x}^1(s,s), \mathbf{x}^2(s,s)\right]^T = \left[\widehat{\mathbf{x}}(s), \widehat{\mathbf{u}}(s)\right]^T$ and $\left[\mathbf{y}^1(s,s), \mathbf{y}^2(s,s)\right]^T = \left[\widehat{\mathbf{y}}(s), \widehat{\mathbf{v}}(s)\right]^T$ we obtain

$$\widehat{\mathbf{y}}(s) = \widehat{\mathbf{x}}(s) + \int_{t_0}^{t} \mathbf{X}_4^1(s, s, \sigma, \mathbf{y}^1(\sigma, s), \mathbf{y}^2(\sigma, s))\widehat{\boldsymbol{\delta}}(s, \widehat{\mathbf{y}}(s), \mathbf{V}\widehat{\mathbf{y}}(s))ds$$

and

$$\widehat{\mathbf{v}}(s) = \widehat{\mathbf{u}}(s) + \int_{t_0}^{t} \mathbf{X}_4^2(s, s, \sigma, \mathbf{y}^1(\sigma, s), \mathbf{y}^2(\sigma, s))\widehat{\boldsymbol{\delta}}(s, \widehat{\mathbf{y}}(s), \mathbf{V}\widehat{\mathbf{y}}(s))ds.$$

Since $\widehat{\mathbf{x}}(s) = \mathbf{x}(s)$ and $\widehat{\mathbf{y}}(s) = \mathbf{y}(s)$, we obtain

$$\mathbf{y}(s) = \mathbf{x}(s) + \int_{t_0}^{s} \mathbf{X}_4^1(s, s, \sigma, \mathbf{y}^1(\sigma, s), \mathbf{y}^2(\sigma, s))\widehat{\boldsymbol{\delta}}(s, \mathbf{y}(s), \mathbf{V}\mathbf{y}(s))ds$$

and

$$\widehat{\mathbf{v}}(s) = \widehat{\mathbf{u}}(s) + \int_{t_0}^{s} \mathbf{X}_4^2(s, s, \sigma, \mathbf{y}^1(\sigma, s), \mathbf{y}^2(\sigma, s))\widehat{\boldsymbol{\delta}}(s, \mathbf{y}(s), \mathbf{V}\mathbf{y}(s))ds.$$

To complete the proof, we show that $\mathbf{X}_4^1(s, s, \sigma, \mathbf{y}^1(\sigma, s), \mathbf{y}^2(\sigma, s)) = \widehat{\mathbf{X}}_3(t, s, \widehat{\mathbf{y}}(s), \widehat{\mathbf{v}}(s))$ and $\mathbf{X}_4^2(s, s, \sigma, \mathbf{y}^1(\sigma, s), \mathbf{y}^2(\sigma, s)) = \widehat{\mathbf{U}}_3(t, s, \widehat{\mathbf{y}}(s), \widehat{\mathbf{v}}(s))$. This can be easily shown by imitating the proof of Lemma 4.1. We omit the details and this completes the proof. □

6 Conclusion

In view of the number of previous contributions by respected mathematicians, in which incorrect results have appeared, it remains to search diligently for any lack of rigour in the preceding results.

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