Positive Imaginaries and Coherent Affine Functors

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1 Overview

The starting point for the ideas in this paper is a result of Kevin Burke [2, Prop 3.2.5] in the model theory of modules. This result establishes a correspondence between syntactically defined "pp-imaginaries" and certain, coherent, functors. Specifically, fix a ring R and let R-**Mod** be the category of left R-modules and R-**mod** be the category of finitely presented left R-modules. The category of additive functors $R\text{-}\mathbf{mod} \to \mathbf{Ab}$ is denoted $(R\text{-}\mathbf{mod}, \mathbf{Ab})$ and is a Grothendieck abelian category. An object X in an abelian category is said to be *coherent* if it is finitely generated (i.e. not a directed union of proper subobjects) and if, whenever Y is another finitely generated object and $f : Y \to X$ is a map, then $\ker f$ is also finitely generated. Burke's result states that the full subcategory of coherent functors $\mathrm{coh}(R\text{-}\mathbf{mod}, \mathbf{Ab})$ is equivalent to Ivo Herzog's category $(R\text{-}\mathbf{Mod})^{\mathrm{eq}+}$ (see [3]) which can be described as follows. The objects of $(R\text{-}\mathbf{Mod})^{\mathrm{eq}+}$ are pairs φ/ψ where φ and ψ are positive primitive (pp) formulas in the language of left R-modules (in the same number of free variables), and where $\psi \to \varphi$ on all modules. The maps $\varphi/\psi \to \varphi'/\psi'$ are given by (logical equivalence classes of) pp-formulas $\rho \leq \varphi \times \varphi'$ which, on any module M, well-define functions $\varphi(M)/\psi(M) \to \varphi'(M)/\psi'(M)$ on cosets. The category $(R\text{-}\mathbf{Mod})^{\mathrm{eq}+}$ is a category of "positive imaginaries" in the sense of model theory. Burke's result may be regarded as a means of translating between the languages of logic and category theory in the context of modules, and has proved to be very useful as a point of interaction between model theory and representation theory (see [6]).

The paper [7] contains a new proof of a topos version of Burke's result which was originally obtained by Makkai and Reyes in [4, p. 269]. Start with a locally finitely presented category $\mathbf{C}$. This is the category of models of a finitary limit theory T in a first order language $\mathcal{L}$ (by [1, 5.9]). The notion of coherence in abelian categories defined above can be generalised to arbitrary categories (with pullbacks) as follows: an object X is *coherent* if it is finitely generated (f.g.) and, whenever Y is a f.g. object and $f : Y \to X$ is a map, the object $Y \times_X Y$ is also f.g. Let us denote the full subcategory of finitely presented objects of $\mathbf{C}$ by $\mathrm{fp}\,\mathbf{C}$ and the category of set-valued functors on $\mathrm{fp}\,\mathbf{C}$ by $(\mathrm{fp}\,\mathbf{C}, \mathbf{Set})$. The Makkai-Reyes result states that the full subcategory of coherent functors $\mathrm{coh}(\mathrm{fp}\,\mathbf{C}, \mathbf{Set})$ is equivalent to a category of "positive imaginaries" $T^{\mathrm{eq}+}$ which can be described as follows. The objects are pairs of positive existential formulas (i.e. formulas built up from atomic formulas using conjunction, disjunction and existential quantification) φ/θ where

$$T \vdash (\theta \subseteq \varphi \times \varphi) \wedge (\theta \text{ defines an equivalence relation on } \varphi)$$

The maps $\varphi/\theta \to \psi/\eta$ are given by (logical equivalence classes of) positive existential formulas ρ such that

$$T \vdash (\rho \subseteq \varphi \times \psi) \wedge (\rho \text{ defines a function } \varphi/\theta \to \psi/\eta)$$

Both Burke's result and the Makkai/Reyes result establish correspondences between syntactic and categorical objects. The former is based in abelian categories and the latter in toposes (the functor category $(\mathrm{fp}\,\mathbf{C}, \mathbf{Set})$ is a topos). For modules, the "right" notion of "formula" is that of pp-formula (such a formula defines an additive functor) whereas for general locally finitely presented categories (which are in general non-additive), the "right" notion of "formula" is that of positive existential formula (which defines a set functor). Both Burke's result and the Makkai/Reyes result positively answer the following vague question: Are the imaginaries formed from the "right" formulas in a particular context equivalent to the coherent functors of an appropriate kind?

Consider the category $R^\gamma/R\text{-}\mathbf{Mod}$ of *γ-pointed modules* where $\gamma \in \mathbb{N}$. The objects of this category are pairs $(M, \boldsymbol{a})$ where M is a module and $\boldsymbol{a} \in M^\gamma$. Maps $f : (M, \boldsymbol{a}) \to (N, \boldsymbol{b})$ are R-linear maps $f : M \to N$ such that $f(\boldsymbol{a}) = \boldsymbol{b}$ pointwise. The objects are models in the language of R-modules with γ extra constants. The "right" formulas in this context are the pp-formulas in this expanded language: any such formula defines an affine functor R^γ/R-**mod** $\to$ **Aff** where **Aff** is the category of affine spaces. Roughly, an affine space is an abelian group without a distinguished identity element. These are typically either $\emptyset$ or cosets of abelian groups. The hom-sets of the category R^γ/R-**Mod** have the structure of affine spaces and an affine functor R^γ/R-**mod** $\to$ **Aff** is one that preserves this affine structure. Any pair φ/ψ of pp-formulas with constants such that $\psi \to \varphi$ will define an affine functor R^γ/R-**mod** $\to$ **Aff** by associating to a pointed module $(M, \boldsymbol{a})$ the quotient $\varphi(M, \boldsymbol{a})/\psi(M, \boldsymbol{a}) \in$ **Aff** (this is a coset of $\varphi(M, \mathbf{0})/\psi(M, \mathbf{0})$ in $M^n/\psi(M, \mathbf{0})$ where n is the number of free variables of φ and ψ). We can now ask the question: can we describe the category of pairs of pp-formulas with constants as the category of coherent affine functors R^γ/R-**mod** $\to$ **Aff**? In this paper we show that the answer is negative. Specifically, we produce a coherent affine functor which is not definable by a pair of pp-formulas with constants. Thus the perfect correspondence between the functorial and the usual model-theoretic approaches in the model theory of modules becomes less than perfect when constants are added to the language.

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2 Affine spaces

2.1 Definition and notation

An *affine space* is a set X endowed with a ternary operation $t : X^3 \to X$ subject to the following equations:

A1 $\quad t(x, z, z) = x$

A2 $\quad t(x, y, z) = t(z, y, x)$

A3 $\quad t(t(x, u, y), w, z) = t(x, t(u, y, w), z) = t(x, u, t(y, w, z))$

For our purposes it is necessary to include $\emptyset$ as an affine space. This implies that the category **Aff** of all affine spaces and homomorphisms is not an affine category in the sense of [8].

If we fix an element 0 of the affine space X, and define $x + y = t(x, 0, y)$, then X becomes an abelian group with the addition operation $+$ and zero element 0. Moreover, we have

$$t(x,y,z) = x - y + z$$

See [5] for the details.

Given an affine space X we can *define* the expression $x - y + z$ to mean $t(x,y,z)$. Note that we are not implying that the expression $x + y$ should mean anything. We can recursively define

$$\begin{aligned} 0x - 0y + z &= z \\ nx - ny + z &= x - y + ((n-1)x - (n-1)y + z) \quad \text{for } n > 0 \end{aligned}$$

And hence we can define

$$nx + (1-n)y = \begin{cases} nx - ny + y & \text{for } n \geq 0 \\ (-n)y - (-n)x + y & \text{for } n < 0 \end{cases}$$

Now, if $n_1, \dots, n_k$ are integers such that $n_1 + n_2 + \dots n_k = 1$, we can define (by induction on k)

$$\begin{aligned} n_1x_1 + \cdots + n_kx_k &= (n_1x_1 + \cdots + n_{k-2}x_{k-2} + (n_{k-1} + n_k)x_{k-1}) \\ &\quad -(n_kx_{k-1} + (1-n_k)x_k) + x_k \end{aligned}$$

We shall call homomorphisms of affine spaces *affine maps*. It is easy to check that an affine map f satisfies

$$f(n_1x_1 + \cdots + n_kx_k) = n_1f(x_1) + \cdots + n_kf(x_k)$$

whenever $n_1 + \cdots + n_k = 1$.

If $A, B \in \mathbf{Aff}$ then $\mathrm{Hom}(A,B) \in \mathbf{Aff}$ for whenever we have three affine maps $f, g, h : A \to B$, then $f - g + h$ (defined pointwise) is another affine map $A \to B$. Note that any constant map $A \to B$ is an affine map.

Let $x_1, \dots x_k$ be elements of an affine space X. If $n_1, \dots, n_k$ are integers such that $n_1 + \cdots + n_k = 1$ then we say that $n_1x_1 + \cdots + n_kx_k$ is an *affine combination* of $x_1, \dots, x_k$. The subset $S \subseteq X$ generates X as an affine space (in the sense that X is the smallest affine subspace containing S) if and only if every element of X can be written as an affine combination of finitely many elements from S. One can easily check this using the definitions above and the axioms A1-A3. Note also that an affine subspace of an affine space is a subset which is closed under all affine combinations.

An abelian group $(G,+)$ is naturally an affine space with $t(x,y,z) = x - y + z$ so that the above definition of the expression "$x - y + z$" coincides with its usual interpretation in G.

Proposition 2.1. *Any affine subspace of an abelian group G is a coset of a subgroup of G.*

Proof. Let A be an affine subspace of G. Fix an arbitrary element $a_0 \in A$ and consider the set

$$A_0 = \{x \in G : x + a_0 \in A\}$$

Then, since $a = a_0 + (a - a_0)$, we have

$$A = a_0 + A_0$$

We have $0 \in A_0$ since $a_0 \in A$. If $x, y \in A_0$ then $x + a_0, y + a_0 \in A$. So $(x - y) + a_0 = (x + a_0) - (y + a_0) + a_0 \in A$ which implies $x - y \in A_0$. This means A_0 is a subgroup of G. And the result is proved. □

Corollary 2.2. *Any affine subspace A of a finitely generated abelian group G is finitely generated as an affine space.*

Proof. By the Proposition we can write $A = a_0 + A_0$ where A_0 is a subgroup of G. Since G is a finitely generated abelian group, so is A_0. Suppose A_0 is generated as a group by $x_1, \ldots, x_k$. Then A is generated as an affine space by $a_0, a_0 + x_1, \ldots, a_0 + x_k$ since if $a \in A$ then there are integers $\lambda_1, \ldots, \lambda_k$ such that

$$\begin{aligned} a &= a_0 + (\lambda_1 x_1 + \cdots + \lambda_k x_k) \\ &= \lambda_1(a_0 + x_1) + \cdots + \lambda_k(a_0 + x_k) + (1 - \lambda_1 - \cdots - \lambda_k)a_0 \end{aligned}$$

□

2.2 Free objects and coproducts

Proposition 2.3. *The free object on k generators is the affine subspace of $\mathbb{Z}^k$ generated by $\mathbf{e}_1, \ldots, \mathbf{e}_k$. In other words, it is the set*

$$C_k = \{(n_1, \ldots, n_k) \in \mathbb{Z}^k : n_1 + \cdots + n_k = 1\}$$

Proof. Let A be an affine space and suppose that $a_1, \ldots, a_k \in A$. Define the map $f : C_k \to A$ by

$$f(n_1, \ldots, n_k) = n_1 a_1 + \cdots + n_k a_k$$

It is easy to see that f is an affine map and clearly $f(\mathbf{e}_i) = a_i$ for $i = 1, \ldots, k$. □

Note that $\mathbb{Z}^k$ itself is generated as an affine space by the elements $\mathbf{e}_1, \ldots, \mathbf{e}_k$ and $\mathbf{0}$.

Let $A_1, \ldots, A_n$ be nonempty affine spaces and choose for each i a fixed element $0^i \in A_i$. Let A be the set

$$A_1 \times \cdots \times A_n \times C_n$$

with the induced pointwise structure as an affine space. Let $v_i : A_i \to A$ be the map

$$x \longmapsto (0^1, \ldots, 0^{i-1}, x, 0^{i+1}, \ldots, 0^n, \mathbf{e}_i)$$

Proposition 2.4. *$\{v_i : A_i \to A\}_i$ is a coproduct in* **Aff**.

Proof. Let $\{f_i : A_i \to B\}_i$ be a family of affine maps. We need to show that there is a unique map $f : A \to B$ such that $fv_i = f_i$ for each i. Define the map $f : A \to B$ by

$$f(a_1, \ldots, a_n, l_1, \ldots, l_n) = \sum_{i=1}^{n} \Big(f_i(a_i) + (l_i - 1) f_i(0^i) \Big)$$

Note that the coefficients in the right hand summation sum to 1 so it is an affine combination. It is easy to check that f is affine. Clearly $fv_i = f_i$. So we only need to show uniqueness. We can write an arbitrary element $(a_1, \ldots, a_n, l_1, \ldots, l_n) \in A$ as an affine combination

$$\sum_{i=1}^{n} \Big(v_i(a_i) + (l_i - 1) v_i(0^i) \Big)$$

So if $g : A \to B$ is any affine map such that $gv_i = f_i$, then clearly $g = f$. $\square$

2.3 Remarks

The category **Aff** is a finitary variety in the universal algebra sense. It follows that it is locally finitely presented and the forgetful functor to **Set** creates products (see [1, 3.6, 3.7]). So if $\{A_i\}$ is a family of affine spaces then their product is simply the set $\prod_i A_i$ with the induced pointwise affine structure.

3 Pointed modules

3.1 Definition

Let γ be a fixed ordinal number. The category of γ-pointed modules is the comma category $R^{(\gamma)}/R$-**Mod**. In other words, the objects of this category are left R-modules M with a distinguished tuple $\boldsymbol{a} \in M^\gamma$, denoted $(M, \boldsymbol{a})$ or simply $\mathcal{M}$ when there is no need to specify the tuple. A map $f : (M, \boldsymbol{a}) \to (N, \boldsymbol{b})$ is a an R-linear map $f : M \to N$ such that $f(\boldsymbol{a}) = \boldsymbol{b}$.

If $\mathcal{M}, \mathcal{N}$ are two pointed modules, then $\mathrm{Hom}(\mathcal{M}, \mathcal{N})$ is an affine space with $t(f, g, h) = f - g + h$, where the right hand side is defined pointwise. For if f, g, h all take $\boldsymbol{a}$ to $\boldsymbol{b}$, then $f - g + h$ takes $\boldsymbol{a}$ to $\boldsymbol{b} - \boldsymbol{b} + \boldsymbol{b} = \boldsymbol{b}$.

3.2 Coproducts

Let $(A_i, \boldsymbol{a}_i)$ be γ-pointed modules where $i \in I$. Here $\boldsymbol{a}_i = (a_{ik})_{k<\gamma}$. Let $H \leq \bigoplus_{i \in I} A_i$ be the submodule generated by the tuples $\boldsymbol{a}_{mnk} \in \bigoplus_i A_i$, for $m, n \in I$ and $k < \gamma$, where

$$a_{mnk,i} = \begin{cases} a_{mk} & \text{when } i = m \\ -a_{nk} & \text{when } i = n \\ 0 & \text{otherwise} \end{cases}$$

Let $\iota_i : A_i \to \bigoplus_i A_i$ be the canonical ith injection. Define v_i to be the composition of ι_i with the projection

$$\bigoplus_i A_i \to \bigoplus_i A_i/H$$

Then $\forall i, j \in I$, $v_i(\boldsymbol{a}_i) = v_j(\boldsymbol{a}_j)$ since $\iota_i(a_{ik}) - \iota_j(a_{jk}) = \boldsymbol{a}_{ijk}$. So if we put $A = \bigoplus_i A_i/H$ and $\boldsymbol{a} = v_{i_0}(\boldsymbol{a}_{i_0})$ for a fixed $i_0 \in I$, then the maps $v_i : (A_i, \boldsymbol{a}_i) \to (A, \boldsymbol{a})$ are morphisms of pointed modules.

Proposition 3.1. *In the above notation, $\{v_i : (A_i, \boldsymbol{a}_i) \to (A, \boldsymbol{a})\}_i$ is a coproduct of the family $\{(A_i, \boldsymbol{a}_i)\}_i$ of pointed modules.*

Proof. For each $i \in I$, let $f_i : (A_i, \boldsymbol{a}_i) \to (B, \boldsymbol{b})$ be maps of pointed modules. Let $f = \bigoplus_i f_i : \bigoplus_i A_i \to B$ and let $\boldsymbol{x} \in H$. So

$$\boldsymbol{x} = r_1 \boldsymbol{a}_{i_1 j_1 k_1} + \cdots + r_n \boldsymbol{a}_{i_n j_n k_n}$$

and

$$\begin{aligned} f(\boldsymbol{x}) &= r_1 f(\boldsymbol{a}_{i_1 j_1 k_1}) + \cdots + r_n f(\boldsymbol{a}_{i_n j_n k_n}) \\ &= r_1(f_{i_1}(a_{i_1 k_1}) - f_{j_1}(a_{j_1 k_1})) + \cdots + r_n(f_{i_n}(a_{i_n k_n}) - f_{j_n}(a_{j_n k_n})) \\ &= r_1(b_{k_1} - b_{k_1}) + \cdots + r_n(b_{k_n} - b_{k_n}) \\ &= 0 \end{aligned}$$

So f factors uniquely through A as $\overline{f}$ say. We have $\overline{f}(v_i(x)) = f\iota_i(x) = f_i(x)$ and $\overline{f}(\boldsymbol{a}) = \overline{f}(v_0(\boldsymbol{a}_0)) = f_0(\boldsymbol{a}_0) = \boldsymbol{b}$. Moreover, $\overline{f}$ is the unique such map $A \to B$. For suppose $\overline{g}$ is another such map. Then $\overline{g}$ corresponds to a map $g : \bigoplus_i A_i \to B$ such that $H \leq \ker g$ and such that $g\iota_i = f_i$. but this means $g = f$ so that $\overline{g} = \overline{f}$. □

3.3 Finite presentation

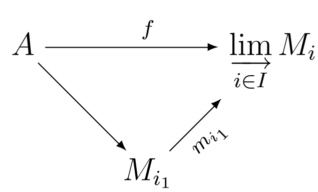
Henceforth $\gamma < \omega$.

Proposition 3.2. *The finitely presented objects in* R^γ/R-**Mod** *are precisely the objects in* R^γ/R-**mod**.

Proof. Suppose $(A, \boldsymbol{a})$ is f.p. in R^γ/R-**Mod**. Let $f : A \to \varinjlim_{i\in I} M_i$ be a map in R-**Mod** where I is a directed poset and the colimit maps $M_j \to \varinjlim_i M_i$ are denoted by m_j and the diagram maps $M_i \to M_j$ by m_{ij} when $i \leq j$. Since $f(\boldsymbol{a})$ is a finite tuple, there is an $i_0 \in I$ such that $f(\boldsymbol{a})$ is represented in M_{i_0}. Let $I' = \{i \in I : i_0 \leq i\}$. Then $\varinjlim_{i\in I} M_i \cong \varinjlim_{i\in I'} M_i$. So f induces a map

$$f : (A, \boldsymbol{a}) \longrightarrow \varinjlim_{i\in I'} (M_i, m_{i_0 i}(f(\boldsymbol{a})))$$

which, by assumption, factors essentially uniquely through some $i_1 \in I'$. This induces an essentially unique factorisation



Hence A is a f.p. module.

For the converse, suppose A is f.p. in R-**Mod**. Let $\boldsymbol{a}$ be a γ-tuple in A and

$$f : (A, \boldsymbol{a}) \longrightarrow \varinjlim (M_i, \boldsymbol{x}_i)$$

a map to a directed colimit in R^γ/R-**Mod**. Then $f : A \to \varinjlim M_i$ is a map in R-**Mod** which must factor essentially uniquely through some M_i. Let $k \geq i$ be an index such that the tuple $f(\boldsymbol{a})$ is represented by $\boldsymbol{x}_k$ in M_k. Then f also factors through M_k which clearly induces an essentially unique factorisation

$$\begin{array}{ccc} (A, \boldsymbol{a}) & \xrightarrow{f} & \varinjlim_i (M_i, \boldsymbol{x}_i) \\ & \searrow \quad \nearrow & \\ & (M_k, \boldsymbol{x}_k) & \end{array}$$

So $(A, \boldsymbol{a})$ is f.p. as required. □

Proposition 3.3. *The category R^γ/R-**Mod** is locally finitely presented.*

Proof. Using Theorem 1.11 from [1], it is sufficient to show that R^γ/R-**Mod** is cocomplete and has a strong generator consisting of f.p. objects. It is easy to see that the pointed module

$$(R^{\gamma+1}, (\mathbf{e}_1, \ldots, \mathbf{e}_\gamma))$$

is a strong generator. For cocompleteness we need to show that the category has an initial object, coequalisers and arbitrary coproducts. We have already shown the existence of arbitrary coproducts. So consider two arrows

$$(M, \boldsymbol{a}) \underset{g}{\overset{f}{\rightrightarrows}} (N, \boldsymbol{b})$$

and let π be the projection $N \to N/\mathrm{im}(f-g)$. The coequaliser of f and g is simply $(N/\mathrm{im}(f-g), \pi(\boldsymbol{b}))$. The initial object of the category is easily seen to be

$$(R^\gamma, (\mathbf{e}_1, \ldots, \mathbf{e}_\gamma))$$

which proves the result. □

4 Affine functors

If $\mathcal{M}$ and $\mathcal{N}$ are two pointed modules, then $\mathrm{Hom}(\mathcal{M}, \mathcal{N})$ is an affine space. So we can consider *affine functors* R^γ/R-**mod** $\to$ **Aff**. These are functors F such that for any f.p. pointed modules $\mathcal{M}, \mathcal{N}$ the induced map

$$\mathrm{Hom}(\mathcal{M}, \mathcal{N}) \xrightarrow{F} \mathrm{Hom}(F\mathcal{M}, F\mathcal{N})$$

is a map of affine spaces. The category of these functors will be denoted $(R^\gamma/R\text{-}\mathbf{mod}, \mathbf{Aff})$. Note that for two affine functors F, G, the set $\mathrm{Nat}(F, G)$ has the structure of an affine space via

$$(\theta - \theta' + \theta'')_{\mathcal{M}}(x) = \theta_{\mathcal{M}}(x) - \theta'_{\mathcal{M}}(x) + \theta''_{\mathcal{M}}(x)$$

for $\theta, \theta', \theta'' \in \mathrm{Nat}(F, G)$ and $\mathcal{M} \in R^\gamma/R\text{-}\mathbf{mod}$. We have an affine version of the Yoneda lemma (which is easily proved).

Lemma 4.1. *The map* $\Phi : \mathrm{Nat}(\mathrm{Hom}(\mathcal{M}, -), F) \to F\mathcal{M}$ *defined by*

$$\theta \longmapsto \theta_{\mathcal{M}}(\mathrm{id})$$

induces an isomorphism of affine spaces.

The category $(R^\gamma/R\text{-}\mathbf{mod}, \mathbf{Aff})$ is a many sorted variety with sorts $\{s_{\mathcal{M}} : \mathcal{M} \in R^\gamma/R\text{-}\mathbf{mod}\}$. Its operations and equations are listed below.

- For each sort $s_{\mathcal{M}}$ a ternary operation $t_{\mathcal{M}} : s_{\mathcal{M}} \times s_{\mathcal{M}} \times s_{\mathcal{M}} \to s_{\mathcal{M}}$ satisfying A1-A3.

- For each arrow $f : \mathcal{M} \to \mathcal{N}$ in $R^\gamma/R\text{-}\mathbf{mod}$, a unary operation $f_* : s_{\mathcal{M}} \to s_{\mathcal{N}}$ such that whenever $f, f', f'' \in \mathrm{Hom}(\mathcal{M}, \mathcal{N})$ and $g : \mathcal{N} \to \mathcal{N}'$ the corresponding operations satisfy the following equations.

$$f_*(t_{\mathcal{M}}(x, , y, z)) = t_{\mathcal{N}}(f_*(x), f_*(y), f_*(z))$$
$$\mathrm{id}_*(x) = x$$
$$(gf)_*(x) = g_* f_*(x)$$
$$(f - f' + f'')_*(x) = t_{\mathcal{N}}(f_*(x), f'_*(x), f''_*(x))$$

As a finitary variety, the category $(R^\gamma/R\text{-}\mathbf{mod}, \mathbf{Aff})$ is locally finitely presented. The representable functors form a strong generating set of f.p. objects. It is natural to ask whether every coherent functor is defined by a quotient of pp-formulas in the language of left R-modules with γ additional constants. We will show that this is not the case.

In order to understand coherent functors, we must first understand finitely generated functors. It is easy to see that these are precisely the quotients of finite coproducts of representable functors by considering functors as many-sorted algebras as above and using the algebraic notion of finite generation together with the Yoneda lemma. So what do finite coproducts of representables look like?

Fix f.p. pointed modules $\mathcal{A}_1, \ldots, \mathcal{A}_n$. For each f.p. pointed module $\mathcal{X}$ define $I(\mathcal{X}) = \{i : \mathrm{Hom}(\mathcal{A}_i, \mathcal{X}) \neq \emptyset\}$. Choose for each $\mathcal{X}$ and each $i \in I(\mathcal{X})$ a fixed map $0^i_{\mathcal{X}} : \mathcal{A}_i \to \mathcal{X}$. We shall now define a functor F by specifying that

$$F(\mathcal{X}) = \left(\prod_{i \in I(\mathcal{X})} \mathrm{Hom}(\mathcal{A}_i, \mathcal{X}) \right) \times T(I(\mathcal{X}))$$

where for an arbitrary subset $I \subseteq \{1, \ldots, n\}$, $T(I)$ is the affine subspace of $\mathbb{Z}^n$ defined by

$$T(I) = \{(l_1, \ldots, l_n) \in C_n : l_i = 0 \quad \forall i \notin I\}$$

Furthermore, if $f : \mathcal{X} \to \mathcal{Y}$ is a map in R^γ/R-**mod**, then $I(\mathcal{X}) \subseteq I(\mathcal{Y})$ and we define the map f_* induced by F (or $F(f)$) as that which sends the tuple $((g_i)_{i \in I(\mathcal{X})}, l_1, \ldots, l_n)$ to the tuple $((g'_i)_{i \in I(\mathcal{Y})}, l_1, \ldots, l_n)$ where

$$g'_i = \begin{cases} f g_i + (l_i - 1) f 0^i_{\mathcal{X}} + (1 - l_i) 0^i_{\mathcal{Y}} & \text{when } i \in I(\mathcal{X}) \\ 0^i_{\mathcal{Y}} & \text{when } i \in I(\mathcal{Y}) \setminus I(\mathcal{X}) \end{cases}$$

The need for f_* to be defined in this way will become apparent in the proof of Proposition 4.2.

Now define, for each $i = 1, \ldots, n$, the element $\iota_i = ((a_j)_{j \in I(\mathcal{A}_i)}, \mathbf{e}_i) \in F(\mathcal{A}_i)$ by putting $a_i = \mathrm{id}_{\mathcal{A}_i}$ and $a_j = 0^j_{\mathcal{A}_i}$ for $j \neq i$. Then, by the Yoneda lemma, ι_i defines a natural transformation $\mathrm{Hom}(\mathcal{A}_i, -) \to F$.

Proposition 4.2. *In the above notation, $\{\iota_i : \mathrm{Hom}(\mathcal{A}_i, -) \to F\}_{i=1,\ldots,n}$ is a coproduct diagram in the functor category.*

Proof. Let G be an affine functor and fix elements $\eta_i \in G(\mathcal{A}_i)$ (corresponding to natural transformations $\mathrm{Hom}(\mathcal{A}_i, -) \to G$). Define for each $\mathcal{X}$ the map $\Phi_{\mathcal{X}} : F(\mathcal{X}) \to G(\mathcal{X})$ of affine spaces which acts thus

$$((g_i)_{i \in I(\mathcal{X})}, l_1, \ldots, l_n) \longmapsto \sum_{i \in I(\mathcal{X})} \Big(G(g_i)(\eta_i) + (l_i - 1) G(0^i_{\mathcal{X}})(\eta_i) \Big)$$

(It is easily checked that this is indeed an affine map since G is an affine functor and $\sum_{i \in I(\mathcal{X})} l_i = 1$.) We claim that the family of maps $\{\Phi_{\mathcal{X}}\}_{\mathcal{X}}$ defines a natural transformation of functors. To see this, let $f : \mathcal{X} \to \mathcal{Y}$ be a map in R^γ/R-**mod**.

$$G(f)\Phi_{\mathcal{X}}((g_i)_{i\in I(\mathcal{X})}, l_1, \ldots, l_n) = G(f)\left(\sum_{i\in I(\mathcal{X})}\Big(G(g_i)(\eta_i) + (l_i - 1)G(0^i_{\mathcal{X}})(\eta_i)\Big)\right)$$

$$\begin{aligned}
&= \sum_{i\in I(\mathcal{X})}\Big(G(fg_i)(\eta_i) + (l_i - 1)G(f0^i_{\mathcal{X}})(\eta_i)\Big) \\
&= \sum_{i\in I(\mathcal{X})}\Big(G(fg_i + (l_i - 1)f0^i_{\mathcal{X}} + (1 - l_i)0^i_{\mathcal{Y}})(\eta_i) + (l_i - 1)G(0^i_{\mathcal{Y}})(\eta_i)\Big) \\
&\quad + \sum_{i\in I(\mathcal{Y})\setminus I(\mathcal{X})}\Big(G(0^i_{\mathcal{Y}})(\eta_i) + (0 - 1)G(0^i_{\mathcal{Y}})(\eta_i)\Big) \\
&= \Phi_{\mathcal{Y}}F(f)((g_i)_{i\in I(\mathcal{X})}, l_1, \ldots, l_n)
\end{aligned}$$

So we indeed have a natural transformation $\Phi : F \to G$. Moreover, we have that $\Phi_{\mathcal{A}_i}(\iota_i) = \eta_i$ for each i. So we now need only to prove that Φ is unique with this property.

Let Ψ be any other natural transformation $F \to G$ such that $\Psi_{\mathcal{A}_i}(\iota_i) = \eta_i$ for each i. Let $((g_i)_{i\in I(\mathcal{X})}, l_1, \ldots, l_n) \in F(\mathcal{X})$. Then we can write this element as the following affine combination.

$$\sum_{i\in I(\mathcal{X})}\Big(F(g_i)(\iota_i) + (l_i - 1)F(0^i_{\mathcal{X}})(\iota_i)\Big)$$

So, since $\Psi_{\mathcal{X}}$ is an affine map, we have

$$\begin{aligned}
\Psi_{\mathcal{X}}((g_i)_{i\in I(\mathcal{X})}, l_1, \ldots, l_n) &= \sum_{i\in I(\mathcal{X})}\Big(\Psi_{\mathcal{X}}F(g_i)(\iota_i) + (l_i - 1)\Psi_{\mathcal{X}}F(0^i_{\mathcal{X}})(\iota_i)\Big) \\
&= \sum_{i\in I(\mathcal{X})}\Big(G(g_i)\Psi_{\mathcal{A}_i}(\iota_i) + (l_i - 1)G(0^i_{\mathcal{X}})\Psi_{\mathcal{A}_i}(\iota_i)\Big) \\
&= \sum_{i\in I(\mathcal{X})}\Big(G(g_i)(\eta_i) + (l_i - 1)G(0^i_{\mathcal{X}})(\eta_i)\Big)
\end{aligned}$$

In other words $\Psi = \Phi$ and the result is proved. □

Note that a different choice of "constants" $0^i_{\mathcal{X}}$ would give a functor naturally isomorphic to F (by the uniqueness of the coproduct).

Consider now the functor

$$G = \mathrm{Hom}\Big((R^\gamma, (\mathbf{e}_1, \ldots, \mathbf{e}_\gamma)), -\Big) \coprod \mathrm{Hom}\Big((R^\gamma, (\mathbf{e}_1, \ldots, \mathbf{e}_\gamma)), -\Big)$$

Since $\mathrm{Hom}((R^\gamma, (\mathbf{e}_1, \ldots, \mathbf{e}_\gamma)), \mathcal{X})$ is a singleton for each $\mathcal{X}$, it is easy to see that $G(\mathcal{X}) \cong C_2 \cong \mathbb{Z}$ for every $\mathcal{X}$ and that for any map of pointed modules $f : \mathcal{X} \to \mathcal{Y}$, $f_* = \mathrm{id}$. If we extend this functor to a functor on all pointed modules (not just the f.p. ones) then G will still define $\mathbb{Z}$ on all modules (since it will always define a directed colimit of a system with just one object $\mathbb{Z}$ and all identity maps, and this colimit is again $\mathbb{Z}$; see [7, Lemma 3.1] for the definition this extension). Hence G cannot be defined by a quotient of pp-formulas since if that were the case we would be able to get $|G(\mathcal{X})| > \aleph_0$ for a suitably large $\mathcal{X}$ by the upward Löwenheim-Skolem theorem.

We claim that G is a coherent functor. So we must show that whenever F is a f.g. functor and $\theta : F \to G$ is a map, then the pullback $F \times_G F$ is also f.g. It will be sufficient to show this whenever F is a finite coproduct of representables as above. This is because, if $F = \coprod_{i=1}^n \mathrm{Hom}(\mathcal{A}_i, -)$ and there is an epimorphism $F \to F'$, then for any $\theta : F' \to G$, the induced map $F \times_G F \to F' \times_G F'$ will also be an epimorphism (as we shall see).

So let $F = \coprod_{i=1}^n \mathrm{Hom}(\mathcal{A}_i, -)$ and let $\{0^i_{\mathcal{X}} : \mathcal{X} \in R^\gamma/R\text{-}\mathbf{mod},\ i \in I(\mathcal{X})\}$ be a choice of constants. Let $\theta : F \to G$ be a natural transformation. Then $\theta = \coprod_{i=1}^n \theta_i$ where $\theta_i \in \mathrm{Nat}(\mathrm{Hom}(\mathcal{A}_i, -), G) \cong G(\mathcal{A}_i) \cong \mathbb{Z}$.

Proposition 4.3. *We can write the set $(F \times_G F)(\mathcal{X})$ as*

$$\left(\prod_{i \in I(\mathcal{X})} \mathrm{Hom}(\mathcal{A}_i, \mathcal{X})\right)^2 \times S(I(\mathcal{X}))$$

where for an arbitrary subset $I \subseteq \{1, \ldots, n\}$,

$$S(I) = \left\{(l_1, \ldots, l_n, l'_1, \ldots, l'_n) \in T(I)^2 : \sum_{i=1}^n l_i\theta_i = \sum_{i=1}^n l'_i\theta_i\right\}$$

and we can describe the action of $F \times_G F$ on a map $f : \mathcal{X} \to \mathcal{Y}$ as follows. The tuple $((g_i, g'_i)_{i \in I(\mathcal{X})}, l_1, \ldots, l_n, l'_1, \ldots, l'_n)$ gets taken to the tuple

$((h_i, h'_i)_{i\in I(\mathcal{Y})}, l_1, \ldots, l_n, l'_1, \ldots, l'_n)$ where

$$h_i = \begin{cases} fg_i + (l_i - 1)f0^i_{\mathcal{X}} + (1 - l_i)0^i_{\mathcal{Y}} & \text{when } i \in I(\mathcal{X}) \\ 0^i_{\mathcal{Y}} & \text{when } i \in I(\mathcal{Y}) \setminus I(\mathcal{X}) \end{cases}$$

$$h'_i = \begin{cases} fg'_i + (l'_i - 1)f0^i_{\mathcal{X}} + (1 - l'_i)0^i_{\mathcal{Y}} & \text{when } i \in I(\mathcal{X}) \\ 0^i_{\mathcal{Y}} & \text{when } i \in I(\mathcal{Y}) \setminus I(\mathcal{X}) \end{cases}$$

The canonical projections $\pi^1, \pi^2 : F \times_G F \to F$ are given by pointwise projection of the cartesian product.

Proof. We first need to check that $\theta\pi^1 = \theta\pi^2$. For convenience, we write a typical element of $(F \times_G F)(\mathcal{X})$ as $(\boldsymbol{g}, \boldsymbol{l}, \boldsymbol{g}', \boldsymbol{l}')$ where $(\boldsymbol{g}, \boldsymbol{l}), (\boldsymbol{g}', \boldsymbol{l}') \in F(\mathcal{X})$ and the inner product $\langle \boldsymbol{l} - \boldsymbol{l}', \boldsymbol{\theta} \rangle = 0$.

For $(\boldsymbol{g}, \boldsymbol{l}) \in F(\mathcal{X})$ we have

$$\begin{aligned} \theta_{\mathcal{X}}(\boldsymbol{g}, \boldsymbol{l}) &= \theta_{\mathcal{X}}\left(\sum_{i\in I(\mathcal{X})} \Big(F(g_i)(\iota_i) + (l_i - 1)F(0^i_{\mathcal{X}})(\iota_i) \Big) \right) \quad \text{see proof of Prop. 4.2} \\ &= \sum_{i\in I(\mathcal{X})} (\theta_{\mathcal{A}_i}(\iota_i) + (l_i - 1)\theta_{\mathcal{A}_i}(\iota_i)) \\ &= \sum_{i\in I(\mathcal{X})} l_i\theta_i \\ &= \langle \boldsymbol{l}, \boldsymbol{\theta} \rangle \end{aligned}$$

So for $(\boldsymbol{g}, \boldsymbol{l}, \boldsymbol{g}', \boldsymbol{l}') \in (F \times_G F)(\mathcal{X})$,

$$\begin{aligned} \theta_{\mathcal{X}}\pi^1_{\mathcal{X}}(\boldsymbol{g}, \boldsymbol{l}, \boldsymbol{g}', \boldsymbol{l}') &= \theta_{\mathcal{X}}(\boldsymbol{g}, \boldsymbol{l}) \\ &= \langle \boldsymbol{l}, \boldsymbol{\theta} \rangle \\ &= \langle \boldsymbol{l}', \boldsymbol{\theta} \rangle \\ &= \theta_{\mathcal{X}}(\boldsymbol{g}', \boldsymbol{l}') \\ &= \theta_{\mathcal{X}}\pi^2_{\mathcal{X}}(\boldsymbol{g}, \boldsymbol{l}, \boldsymbol{g}', \boldsymbol{l}') \end{aligned}$$

Now suppose that $\eta^1, \eta^2 : H \to F$ are natural transformations from a functor H such that $\theta\eta^1 = \theta\eta^2$. Define the map $\Phi : H \to F \times_G F$ by

$$\Phi_{\mathcal{X}}(h) = (\eta^1_{\mathcal{X}}(h), \eta^2_{\mathcal{X}}(h))$$

This is well defined since if $\eta^1_{\mathcal{X}}(h) = (\boldsymbol{g}, \boldsymbol{l})$ and $\eta^2_{\mathcal{X}}(h) = (\boldsymbol{g}', \boldsymbol{l}')$, then

$$\langle \boldsymbol{l}, \boldsymbol{\theta} \rangle = \theta_{\mathcal{X}}(\boldsymbol{g}, \boldsymbol{l}) = \theta_{\mathcal{X}}(\boldsymbol{g}', \boldsymbol{l}') = \langle \boldsymbol{l}', \boldsymbol{\theta} \rangle$$

It is obvious that Φ is natural and clearly Φ is the unique natural transformation $H \to F \times_G F$ such that $\pi^1\Phi = \eta^1$ and $\pi^2\Phi = \eta^2$. This establishes the result. $\square$

In order to show that the functor $F \times_G F$ is finitely generated, we must first make the following definitions.

For each $I \subseteq \{1, \dots, n\}$, let

$$\mathcal{A}_I = \coprod_{i \in I} \mathcal{A}_i$$

and let $w_i^I : \mathcal{A}_i \to \mathcal{A}_I$ be the canonical injections. Furthermore, define for each i,

$$\mathcal{A}_{ii} = \mathcal{A}_i \coprod \mathcal{A}_i$$

with canonical injections $v_i, v_i' : \mathcal{A}_i \to \mathcal{A}_{ii}$.

For each $I \subseteq \{1, \dots, n\}$, the set $S(I)$ is an affine subspace of $\mathbb{Z}^n \times \mathbb{Z}^n$ and so is finitely generated by Corollary 2.2. Let $X_1^I = (\boldsymbol{x}_{I,1}, \boldsymbol{x}'_{I,1}), \dots, X_{k_I}^I = (\boldsymbol{x}_{I,k_I}, \boldsymbol{x}'_{I,k_I})$ be affine generators for $S(I)$. We fix the notation

$$\begin{aligned} \boldsymbol{x}_{I,r} &= (x_{I,r1}, \dots, x_{I,rn}) \\ \boldsymbol{x}'_{I,r} &= (x'_{I,r1}, \dots, x'_{I,rn}) \end{aligned}$$

For each $i \in I$, let $\boldsymbol{\alpha}_i$ be the element $((g_j, g_j')_{j \in I(\mathcal{A}_{ii})}, \boldsymbol{e}_i, \boldsymbol{e}_i)$ of $(F \times_G F)(\mathcal{A}_{ii})$ defined by

$$g_j = \begin{cases} 0^j_{\mathcal{A}_{ii}} & \text{when } j \neq i \\ v_j & \text{when } j = i \end{cases}$$

$$g_j' = \begin{cases} 0^j_{\mathcal{A}_{ii}} & \text{when } j \neq i \\ v_j' & \text{when } j = i \end{cases}$$

For each $I \subseteq \{1, \dots, n\}$ such that $I = I(\mathcal{A}_I)$ and generator X_r^I for $S(I)$, let $\boldsymbol{\beta}_{X_r^I}$ be the element $((h_i, h_i')_{i \in I}, \boldsymbol{x}_{I,r}, \boldsymbol{x}'_{I,r})$ of $(F \times_G F)(\mathcal{A}_I)$ defined by

$$\begin{aligned} h_i &= x_{I,ri} w_i^I + (1 - x_{I,ri}) 0^i_{\mathcal{A}_I} \\ h_i' &= x'_{I,ri} w_i^I + (1 - x'_{I,ri}) 0^i_{\mathcal{A}_I} \end{aligned}$$

We claim that the elements $\boldsymbol{\alpha}_i$ and $\boldsymbol{\beta}_{X_r^I}$ generate the functor $F \times_G F$, regarded as a many-sorted algebra. Equivalently, the coproduct of these elements, regarded as maps from representables, yields an epimorphism to $F \times_G F$.

To see this, let $\mathcal{Y}$ be a f.p. pointed module and $((f_i, f'_i)_{i\in I(\mathcal{Y})}, l_1, \ldots, l_n, l'_1, \ldots, l'_n)$ be an element of $(F \times_G F)(\mathcal{Y})$. Let $I = I(\mathcal{Y})$. Note that in this case we have $I(\mathcal{A}_I) = I$. The tuple $(l_1, \ldots, l_n, l'_1, \ldots, l'_n)$ is an element of $S(I)$ and so can be expressed as an affine combination

$$\lambda_1 X_1^I + \cdots + \lambda_{k_I} X_{k_I}^I$$

where

$$\lambda_1 + \cdots + \lambda_{k_I} = 1$$

An easy calculation shows the following.

$$(0_{\mathcal{Y}}^i \coprod 0_{\mathcal{Y}}^i)_*(\boldsymbol{\alpha}_i) = ((0_{\mathcal{Y}}^j, 0_{\mathcal{Y}}^j)_{j\in I}, \mathbf{e}_i, \mathbf{e}_i)$$

(by using the explicit description of $F \times_G F$ in Proposition 4.3) and

$$(f_i \coprod f'_i)_*(\boldsymbol{\alpha}_i) = ((p_j, p'_j)_{j\in I}, \mathbf{e}_i, \mathbf{e}_i)$$

where $p_i = f_i$, $p'_i = f'_i$ and $p_j = p'_j = 0_{\mathcal{Y}}^j$ when $j \neq i$. Furthermore,

$$\left(\coprod_{j\in I} 0_{\mathcal{Y}}^j\right)_* \left(\boldsymbol{\beta}_{X_r^I}\right) = ((0_{\mathcal{Y}}^j, 0_{\mathcal{Y}}^j)_{j\in I}, \boldsymbol{x}_{I,r}, \boldsymbol{x}'_{I,r})$$

So we can write

$$((f_i, f'_i)_{i\in I}, l_1, \ldots, l_n, l'_1, \ldots, l'_n) = \left(\sum_{i\in I} (f_i \coprod f'_i)_*(\boldsymbol{\alpha}_i)\right) - \left(\sum_{i\in I} (0_{\mathcal{Y}}^i \coprod 0_{\mathcal{Y}}^i)_*(\boldsymbol{\alpha}_i)\right)$$
$$+\lambda_1 \left(\coprod_{i\in I} 0_{\mathcal{Y}}^i\right)_* \left(\boldsymbol{\beta}_{X_1^I}\right) + \cdots + \lambda_{k_I} \left(\coprod_{i\in I} 0_{\mathcal{Y}}^i\right)_* \left(\boldsymbol{\beta}_{X_{k_I}^I}\right)$$

This is an affine combination since the λ_j's sum to 1 and the result is proved. We state this in a proposition.

Proposition 4.4. *Let G be the functor which is constantly $\mathbb{Z}$ and let F be $\coprod_{i=1}^n \mathrm{Hom}(\mathcal{A}_i, -)$. For any map $\theta : F \to G$ the pullback $F \times_G F$ is finitely generated.*

We must now show that for *any* f.g. functor F' and map $\theta' : F' \to G$, the functor $F' \times_G F'$ is finitely generated. Since F' is f.g. there is a functor $F = \coprod_{i=1}^n \operatorname{Hom}(\mathcal{A}_i, -)$ and an epimorphism

$$\eta = \coprod_{i=1}^n \eta_i : F \twoheadrightarrow F'$$

Let $\theta_i = \theta'(\eta_i(\mathrm{id}_{\mathcal{A}_i})) \in \mathbb{Z}$. Then it is easy to deduce that

$$\theta'(\eta_{\mathcal{X}}((g_i)_{i \in I(\mathcal{X})}, l_1, \ldots, l_n)) = l_1\theta_1 + \cdots + l_n\theta_n$$

because

$$\eta_{\mathcal{X}}((g_i)_{i \in I(\mathcal{X})}, l_1, \ldots, l_n) = \sum_{i \in I(\mathcal{X})} \Big((g_i)_*(\eta_i(\mathrm{id}_{\mathcal{A}_i})) + (l_i - 1)(0^i_{\mathcal{X}})_*(\eta_i(\mathrm{id}_{\mathcal{A}_i})) \Big)$$

and for each $i \in I(\mathcal{X})$,

$$\theta'((g_i)_*(\eta_i(\mathrm{id}_{\mathcal{A}_i}))) = \theta'((0^i_{\mathcal{X}})_*(\eta_i(\mathrm{id}_{\mathcal{A}_i}))) = \theta'(\eta_i(\mathrm{id}_{\mathcal{A}_i}))$$

Let Φ be the induced map $F \times_G F \to F' \times_G F'$. Then the action of $\Phi_{\mathcal{X}}$ is defined by

$$\begin{array}{c} ((g_i, g'_i)_{i \in I(\mathcal{X})}, l_1, \ldots, l_n, l'_1, \ldots, l'_n) \\ \big\downarrow \\ (\eta_{\mathcal{X}}((g_i)_{i \in I(\mathcal{X})}, l_1, \ldots, l_n), \eta_{\mathcal{X}}((g'_i)_{i \in I(\mathcal{X})}, l'_1, \ldots, l'_n)) \end{array}$$

Clearly Φ is an epimorphism since (using the fact that η is epic)

$$\begin{array}{c} (\eta_{\mathcal{X}}((g_i)_{i \in I(\mathcal{X})}, l_1, \ldots, l_n), \eta_{\mathcal{X}}((g'_i)_{i \in I(\mathcal{X})}, l'_1, \ldots, l'_n)) \in F' \times_G F' \\ \Downarrow \\ \sum_i l_i\theta_i = \sum_i l'_i\theta_i \\ \Downarrow \\ ((g_i, g'_i)_{i \in I(\mathcal{X})}, l_1, \ldots, l_n, l'_1, \ldots, l'_n) \in F \times_G F \end{array}$$

This implies that $F' \times_G F'$ is f.g. since $F \times_G F$ is. So G is a coherent functor and we have the following result.

Theorem 4.5. *G is a coherent functor R^γ/R-**mod** $\to$ **Aff** which is not definable by a quotient of pp-formulas in the language of left R-modules with γ additional constants.*

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