

Sheaves of rings of definable scalars

Mike Prest
Department of Mathematics
University of Manchester
Manchester M13 9PL
UK
mprest@maths.man.ac.uk

November 2, 2006

1. Rings of definable scalars
2. Definable subcategories
3. The Gabriel-Zariski topology
4. Definable categories and exact functors
5. The presheaf of definable scalars
6. Varieties and “abelian spaces”

Rings of definable scalars

Fix R and $M \in \text{Mod-}R$.

Rings of definable scalars

Fix R and $M \in \text{Mod-}R$.

A **definable scalar** of M_R is a biendomorphism whose action is **pp-definable**.

Rings of definable scalars

Fix R and $M \in \text{Mod-}R$.

A **definable scalar** of M_R is a biendomorphism whose action is **pp-definable**.

Let $\bar{x}H = 0$ (*) be a system of R -linear equations.

Rings of definable scalars

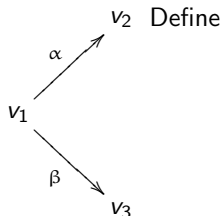
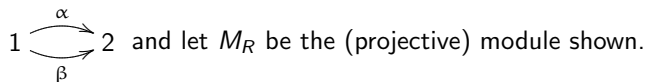
Fix R and $M \in \text{Mod-}R$.

A **definable scalar** of M_R is a biendomorphism whose action is **pp-definable**.

Let $\bar{x}H = 0$ (*) be a system of R -linear equations.

Existentially quantify out all but, say, the first two variables to get the **pp condition/formula** $\exists x_3, \dots, x_n \bar{x}H = 0$. Write, say, ϕ or $\phi(x_1, x_2)$ for this formula/condition.

Example. Let R be the path algebra over a field k of the Kronecker quiver



the k -linear map $M \rightarrow M$ by $v_1 \mapsto 0$, $v_2 \mapsto v_1$, $v_3 \mapsto v_1 + v_2$. Then this map is pp-definable, by the system $\exists x_3, \dots, x_7 (x_1, \dots, x_7)H = 0$ where H is the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 - e_1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & -\alpha & 0 \\ 0 & -1 - \alpha & 0 & 0 & -\beta \end{pmatrix}$$

Theorem

(Burke+Prest) Suppose that M_R is a module of finite endolength. Then every biendomorphism of M is pp-definable. The same conclusion holds if M is finitely presented and is finitely generated over its endomorphism ring.

Theorem

(Burke+Prest) Suppose that M_R is a module of finite endolength. Then every biendomorphism of M is pp-definable. The same conclusion holds if M is finitely presented and is finitely generated over its endomorphism ring.

Theorem

(Prest) Suppose that the injective module E_R cogenerates a torsion theory of finite type in $\text{Mod-}R$. Then the ring of definable scalars of E is the localisation of R at this torsion theory.

Theorem

(Burke+Prest) Suppose that M_R is a module of finite endolength. Then every biendomorphism of M is pp-definable. The same conclusion holds if M is finitely presented and is finitely generated over its endomorphism ring.

Theorem

(Prest) Suppose that the injective module E_R cogenerates a torsion theory of finite type in $\text{Mod-}R$. Then the ring of definable scalars of E is the localisation of R at this torsion theory.

Theorem

(Prest) Suppose that $f : R \rightarrow S$ is an epimorphism of rings. Then the ring of definable scalars of S regarded as an R -module via f is S .

Theorem

(Burke+Prest) Suppose that M_R is a module of finite endolength. Then every biendomorphism of M is pp-definable. The same conclusion holds if M is finitely presented and is finitely generated over its endomorphism ring.

Theorem

(Prest) Suppose that the injective module E_R cogenerates a torsion theory of finite type in $\text{Mod-}R$. Then the ring of definable scalars of E is the localisation of R at this torsion theory.

Theorem

(Prest) Suppose that $f : R \rightarrow S$ is an epimorphism of rings. Then the ring of definable scalars of S regarded as an R -module via f is S .

Theorem

(Krause, Prest) For any module M the ring of definable scalars of M is the biendomorphism ring of any pure-injective module N which generates the same definable subcategory as M and which is cyclic over its endomorphism ring.

Definable subcategories

A subcategory of $\text{Mod-}R$ is **definable** if it is closed under products, direct limits and pure submodules.

Definable subcategories

A subcategory of $\text{Mod-}R$ is **definable** if it is closed under products, direct limits and pure submodules.

If M, N generate the same definable subcategory then $R_N = R_M$. If N belongs to the definable subcategory generated by M then there is a canonical homomorphism $R_M \rightarrow R_N$.

Definable subcategories

A subcategory of $\text{Mod-}R$ is **definable** if it is closed under products, direct limits and pure submodules.

If M, N generate the same definable subcategory then $R_N = R_M$. If N belongs to the definable subcategory generated by M then there is a canonical homomorphism $R_M \rightarrow R_N$.

Theorem

(Ziegler) Every definable subcategory of $\text{Mod-}R$ is determined by the indecomposable pure-injective modules in it.

Definable subcategories

A subcategory of $\text{Mod-}R$ is **definable** if it is closed under products, direct limits and pure submodules.

If M, N generate the same definable subcategory then $R_N = R_M$. If N belongs to the definable subcategory generated by M then there is a canonical homomorphism $R_M \rightarrow R_N$.

Theorem

(Ziegler) Every definable subcategory of $\text{Mod-}R$ is determined by the indecomposable pure-injective modules in it.

Let pinj_R denote the set of isomorphism types of indecomposable pure-injective R -modules.

Definable subcategories

A subcategory of $\text{Mod-}R$ is **definable** if it is closed under products, direct limits and pure submodules.

If M, N generate the same definable subcategory then $R_N = R_M$. If N belongs to the definable subcategory generated by M then there is a canonical homomorphism $R_M \rightarrow R_N$.

Theorem

(Ziegler) Every definable subcategory of $\text{Mod-}R$ is determined by the indecomposable pure-injective modules in it.

Let pinj_R denote the set of isomorphism types of indecomposable pure-injective R -modules.

Theorem

*(Ziegler) The definable subcategories of M , by intersection with pinj_R , induce a topology (now called the **Ziegler topology**) on pinj_R in which these (intersections) are exactly the closed subsets.*

Rings of definable scalars are localisations

Rings of definable scalars are localisations
but one representation level up.

Rings of definable scalars are localisations

but one representation level up.

Set $\text{Fun-}R = (\text{mod-}R, \mathbf{Ab})$ and $\text{fun-}R = (\text{mod-}R, \mathbf{Ab})^{\text{fp}}$. Set

$\mathcal{A}_M = \{F \in \text{fun-}R : FM = 0\}$ (rather $\overrightarrow{F}M = 0$) - a Serre subcategory of $\text{fun-}R$, the “Serre annihilator” of M .

Rings of definable scalars are localisations

but one representation level up.

Set $\text{Fun-}R = (\text{mod-}R, \mathbf{Ab})$ and $\text{fun-}R = (\text{mod-}R, \mathbf{Ab})^{\text{fp}}$. Set

$\mathcal{A}_M = \{F \in \text{fun-}R : FM = 0\}$ (rather $\overrightarrow{F}M = 0$) - a Serre subcategory of $\text{fun-}R$, the “Serre annihilator” of M .

Theorem

(Burke+Prest) The ring of definable scalars of a module M_R is isomorphic to the endomorphism ring of the image of $(R, -)$ in the quotient category $\text{fun-}R/\mathcal{A}_M$.

Theorem

There is a natural bijection between definable subcategories of $\text{Mod-}R$ and Serre subcategories of (hence localisations=abelian quotient categories of) $\text{fun-}R$.

Theorem

There is a natural bijection between definable subcategories of $\text{Mod-}R$ and Serre subcategories of (hence localisations=abelian quotient categories of) $\text{fun-}R$.

If \mathcal{D} is a definable subcategory of $\text{Mod-}R$ then set $\text{fun}(\mathcal{D}) = \text{fun-}R/\mathcal{A}_{\mathcal{D}}$ where $\mathcal{A}_{\mathcal{D}} = \{F \in \text{fun-}R : F\mathcal{D} = 0\}$. Call this the **(fp) functor category** of \mathcal{D} . There is a well-defined action of objects $G \in \text{fun}(\mathcal{D})$ on \mathcal{D} , namely the restriction to \mathcal{D} of any pre-image of G in $\text{fun-}R$.

The Gabriel-Zariski topology

The Gabriel-Zariski topology

Let \mathcal{C} be a locally coherent abelian category. Denote by $\text{inj}(\mathcal{C})$ the set of isomorphism types of indecomposable injective objects of \mathcal{C} . Define the **Gabriel-Zariski topology** on $\text{inj}(\mathcal{C})$ to have, as a basis of open sets, the $[A] = \{E \in \text{inj}(\mathcal{C}) : (A, E) = 0\}$ where A ranges over \mathcal{C}^{fp} .

The Gabriel-Zariski topology

Let \mathcal{C} be a locally coherent abelian category. Denote by $\text{inj}(\mathcal{C})$ the set of isomorphism types of indecomposable injective objects of \mathcal{C} . Define the **Gabriel-Zariski topology** on $\text{inj}(\mathcal{C})$ to have, as a basis of open sets, the $[A] = \{E \in \text{inj}(\mathcal{C}) : (A, E) = 0\}$ where A ranges over \mathcal{C}^{fp} .

Theorem

(Gruson and Jensen) The functor given on objects by $M \mapsto M \otimes_R -$ from $\text{Mod-}R$ to $\text{Fun-}(R^{\text{op}})$ is a full embedding. It induces a bijection between pinj_R and $\text{inj}(\text{Fun-}(R^{\text{op}}))$.

The Gabriel-Zariski topology

Let \mathcal{C} be a locally coherent abelian category. Denote by $\text{inj}(\mathcal{C})$ the set of isomorphism types of indecomposable injective objects of \mathcal{C} . Define the **Gabriel-Zariski topology** on $\text{inj}(\mathcal{C})$ to have, as a basis of open sets, the $[A] = \{E \in \text{inj}(\mathcal{C}) : (A, E) = 0\}$ where A ranges over \mathcal{C}^{fp} .

Theorem

(Gruson and Jensen) The functor given on objects by $M \mapsto M \otimes_R -$ from $\text{Mod-}R$ to $\text{Fun-}(R^{\text{op}})$ is a full embedding. It induces a bijection between pinj_R and $\text{inj}(\text{Fun-}(R^{\text{op}}))$.

It is the case (Gruson and Jensen, Auslander) that there is a duality $\text{fun-}(R^{\text{op}}) \simeq (\text{fun-}R)^{\text{op}}$. Therefore we also write $\text{fun}^{\text{d}}\text{-}R$ for $\text{fun-}(R^{\text{op}})$ and similarly for Fun . That there is such a duality follows directly from the identification of $\text{fun-}(R^{\text{op}})$ with the free abelian category, $\text{Ab}(R)$, on R .

The Gabriel-Zariski topology

Let \mathcal{C} be a locally coherent abelian category. Denote by $\text{inj}(\mathcal{C})$ the set of isomorphism types of indecomposable injective objects of \mathcal{C} . Define the **Gabriel-Zariski topology** on $\text{inj}(\mathcal{C})$ to have, as a basis of open sets, the $[A] = \{E \in \text{inj}(\mathcal{C}) : (A, E) = 0\}$ where A ranges over \mathcal{C}^{fp} .

Theorem

(Gruson and Jensen) The functor given on objects by $M \mapsto M \otimes_R -$ from $\text{Mod-}R$ to $\text{Fun-}(R^{\text{op}})$ is a full embedding. It induces a bijection between pinj_R and $\text{inj}(\text{Fun-}(R^{\text{op}}))$.

It is the case (Gruson and Jensen, Auslander) that there is a duality $\text{fun-}(R^{\text{op}}) \simeq (\text{fun-}R)^{\text{op}}$. Therefore we also write $\text{fun}^{\text{d}}-R$ for $\text{fun-}(R^{\text{op}})$ and similarly for Fun . That there is such a duality follows directly from the identification of $\text{fun-}(R^{\text{op}})$ with the free abelian category, $\text{Ab}(R)$, on R .

Thus the Gabriel-Zariski topology on $\text{inj}(\text{Fun}^{\text{d}}-R)$ induces one on pinj_R , which we denote by $Z_{\text{ar}}R$ and refer to as the **rep-Zariski spectrum** of R .

Theorem

The dual-Ziegler topology on pinj_R coincides with the rep-Zariski topology on that set.

Theorem

The dual-Ziegler topology on pinj_R coincides with the rep-Zariski topology on that set.

By the duality between the functor categories, the sets $[F] = \{N \in \text{pinj}_R : FN = 0\}$ for $F \in \text{fun-}R$ form a basis of open sets for Zar_R .

Definable categories and exact functors

Definable categories and exact functors

Theorem

(Herzog for modules, Krause in general) Let \mathcal{D} be a definable category. Then there is an equivalence $\mathcal{D} \simeq \text{Ex}(\text{fun}(\mathcal{D}), \mathbf{Ab})$

Definable categories and exact functors

Theorem

(Herzog for modules, Krause in general) Let \mathcal{D} be a definable category. Then there is an equivalence $\mathcal{D} \simeq \text{Ex}(\text{fun}(\mathcal{D}), \mathbf{Ab})$ ($\text{fun}(\mathcal{D})$ is indeed well-defined - i.e. independent of representation of \mathcal{D} as a definable subcategory).

Definable categories and exact functors

Theorem

(Herzog for modules, Krause in general) Let \mathcal{D} be a definable category. Then there is an equivalence $\mathcal{D} \simeq \text{Ex}(\text{fun}(\mathcal{D}), \mathbf{Ab})$ ($\text{fun}(\mathcal{D})$ is indeed well-defined - i.e. independent of representation of \mathcal{D} as a definable subcategory).

Therefore, if \mathcal{C}, \mathcal{D} are definable categories and $l_0 : \text{fun}(\mathcal{D}) \rightarrow \text{fun}(\mathcal{C})$ is an exact functor then l_0 induces, by composition, a functor $l^* : \mathcal{C} \rightarrow \mathcal{D}$ which, one may check, commutes with products and direct limits. Conversely, if $l : \mathcal{C} \rightarrow \mathcal{D}$ commutes with direct products and direct limits then there is induced a functor $l_0 : \text{fun}(\mathcal{D}) \rightarrow \text{fun}(\mathcal{C})$ which is exact. That follows from the next result.

Theorem

(Krause for \mathcal{D} a locally finitely presented category, Prest in general) Suppose that \mathcal{D} is a definable category. Then $\text{fun}(\mathcal{D}) \simeq (\mathcal{D}, \mathbf{Ab})^{\Pi \rightarrow}$, the category of functors from \mathcal{D} to \mathbf{Ab} which commute with direct products and direct limits.

Theorem

(Krause for \mathcal{D} a locally finitely presented category, Prest in general) Suppose that \mathcal{D} is a definable category. Then $\text{fun}(\mathcal{D}) \simeq (\mathcal{D}, \mathbf{Ab})^{\Pi \rightarrow}$, the category of functors from \mathcal{D} to \mathbf{Ab} which commute with direct products and direct limits.

Corollary

There is an equivalence between the 2-category of small abelian categories with exact functors and the 2-category of definable categories with functors which commute with direct limits and direct products.

The presheaf of definable scalars

The presheaf of definable scalars

Given a ring R we define a presheaf (defined on a basis) over Zar_R by $[F] = \{N : FN = 0\} \mapsto \text{End}((R, -)/\langle F \rangle)$, where $\langle F \rangle$ is the Serre subcategory of $\text{fun-}R$ generated by F . Equivalently $[F] \mapsto R_M$ where $M = \bigoplus_{N \in [F]} N$.

The presheaf of definable scalars

Given a ring R we define a presheaf (defined on a basis) over Zar_R by $[F] = \{N : FN = 0\} \mapsto \text{End}((R, -)/\langle F \rangle)$, where $\langle F \rangle$ is the Serre subcategory of $\text{fun-}R$ generated by F . Equivalently $[F] \mapsto R_M$ where $M = \bigoplus_{N \in [F]} N$.

If $[F] \supseteq [G]$ then $F \in \langle G \rangle$ so localisation at $\langle G \rangle$ factors through localisation at $\langle F \rangle$: the corresponding map in the presheaf is the map thereby induced from $\text{End}((R, -)/\langle F \rangle)$ to $\text{End}((R, -)/\langle G \rangle)$.

The presheaf of definable scalars

Given a ring R we define a presheaf (defined on a basis) over Zar_R by $[F] = \{N : FN = 0\} \mapsto \text{End}((R, -)/\langle F \rangle)$, where $\langle F \rangle$ is the Serre subcategory of $\text{fun-}R$ generated by F . Equivalently $[F] \mapsto R_M$ where $M = \bigoplus_{N \in [F]} N$.

If $[F] \supseteq [G]$ then $F \in \langle G \rangle$ so localisation at $\langle G \rangle$ factors through localisation at $\langle F \rangle$: the corresponding map in the presheaf is the map thereby induced from $\text{End}((R, -)/\langle F \rangle)$ to $\text{End}((R, -)/\langle G \rangle)$.

This presheaf-on-a-basis, Def_R , is separated (this follows from the interpretation of $\text{End}((R, -)/\langle F \rangle)$ as a ring of certain functions on a certain module) so embeds in its sheafification, which we denote by LDef_R and refer to as the **sheaf of locally definable scalars** (for R -modules).

$$[F] \mapsto \text{fun-}R/\langle F \rangle$$

$$[F] \mapsto \text{fun-}R/\langle F \rangle$$

- a presheaf of small abelian categories (all localisations of $\text{fun-}R$). It is separated and embeds in its sheafification, \mathbf{LDef}_R , the “sheaf of *categories* of locally definable scalars” (or of “locally definable scalars in all sorts”).

$$[F] \mapsto \text{fun-}R/\langle F \rangle$$

- a presheaf of small abelian categories (all localisations of $\text{fun-}R$). It is separated and embeds in its sheafification, \mathbf{LDef}_R , the “sheaf of *categories* of locally definable scalars” (or of “locally definable scalars in all sorts”).

If we apply this definition for $\text{Mod-}\mathcal{A}$ and so for arbitrary definable categories \mathcal{D} , we associate to each definable category \mathcal{D} a “categorized space” $(\text{Zar}(\mathcal{D}), \mathbf{LDef}(\mathcal{D}))$.

$$[F] \mapsto \text{fun-}R/\langle F \rangle$$

- a presheaf of small abelian categories (all localisations of $\text{fun-}R$). It is separated and embeds in its sheafification, \mathbf{LDef}_R , the “sheaf of *categories* of locally definable scalars” (or of “locally definable scalars in all sorts”).

If we apply this definition for $\text{Mod-}\mathcal{A}$ and so for arbitrary definable categories \mathcal{D} , we associate to each definable category \mathcal{D} a “categorized space” $(\text{Zar}(\mathcal{D}), \mathbf{LDef}(\mathcal{D}))$.

Theorem

If $\mathcal{C} \rightarrow \mathcal{D}$ is a functor between definable categories which commutes with products and direct limits then there is induced a morphism of “categorized locales” (“abelian spaces”) $(\text{Zar}(\mathcal{C}), \mathbf{LDef}(\mathcal{C})) \rightarrow (\text{Zar}(\mathcal{D}), \mathbf{LDef}(\mathcal{D}))$.

Varieties and “abelian spaces”

Varieties and “abelian spaces”

R (commutative) $\mapsto \text{Ab}(R^{(\text{op})}) = \mathcal{A}$ (a small abelian category)
 $\mapsto \text{Ex}(\mathcal{A}, \mathbf{Ab}) = \mathcal{D}$ (a definable category, in fact $\text{Mod-}R$) $\mapsto (\text{Zar}(\mathcal{D}), \mathbf{LDef}(\mathcal{D}))$
 (= $(\text{Zar}_R, \mathbf{LDef}_R)$):

Varieties and “abelian spaces”

R (commutative) $\mapsto \text{Ab}(R^{(\text{op})}) = \mathcal{A}$ (a small abelian category)
 $\mapsto \text{Ex}(\mathcal{A}, \mathbf{Ab}) = \mathcal{D}$ (a definable category, in fact $\text{Mod-}R$) $\mapsto (\text{Zar}(\mathcal{D}), \mathbf{LDef}(\mathcal{D}))$
 (= $(\text{Zar}_R, \mathbf{LDef}_R)$): a rather complicated object - a sheaf of categories

Varieties and “abelian spaces”

R (commutative) $\mapsto \text{Ab}(R^{(\text{op})}) = \mathcal{A}$ (a small abelian category)
 $\mapsto \text{Ex}(\mathcal{A}, \mathbf{Ab}) = \mathcal{D}$ (a definable category, in fact $\text{Mod-}R$) $\mapsto (\text{Zar}(\mathcal{D}), \mathbf{LDef}(\mathcal{D}))$
 $(= (\text{Zar}_R, \mathbf{LDef}_R))$: a rather complicated object - a sheaf of categories - but it contains, as a subsheaf, over the subspace of injective points of Zar_R , the thread consisting of localisations of $(R, -)$ and that is exactly $(\text{Spec}(R), \mathcal{O}_{\text{Spec}(R)})$, the usual structure sheaf of R .