Model Theory 2016, Assessment 2 - Solutions

Q1. For each of the following languages $\mathcal{L}$ and sentences σ_i of $\mathcal{L}$, compute $\lim_{n\to\infty} p(\sigma, n)$.

(i) $\mathcal{L} = \mathcal{L}_0 \vee \{c, f(-)\}$, where c is a constant symbol and f is a 1-ary function symbol; let σ_1 be $\exists x\,(f(x) = c)$.

Solution: How many ways can we put an $\mathcal{L}$-structure on a set of size n? For the interpretation of the constant symbol c there are n choices and, for each of these choices, there are n^n choices for the interpretation of the function symbol f. So $n.n^n$ structures in total. We will count the number of these which satisfy $\neg\sigma_1$ (since that looks easier than directly computing the number of those which satisfy σ_1): this is $(n-1)^n$ (for each element of the set there are $n-1$ choices for the value of the function on it, since we have to avoid $c^{\mathcal{M}}$ as a value). So $p(\sigma_1, n) = 1 - p(\neg\sigma_1, n) = 1 - \dfrac{n(n-1)^n}{nn^n} = 1 - (1 - \frac{1}{n})^n$. Take the limit as $n \to \infty$, to get $p(\sigma_1, n) = 1 - \dfrac{1}{e}$.

(ii) $\mathcal{L} = \mathcal{L}_0 \vee \{c, f(-)\}$, where c is a constant symbol and f is a 1-ary function symbol; let σ_2 be $\exists x, y\,(f(x) = c \wedge f(y) = c \wedge x \neq y)$.

Solution: Again, we'll use an indirect count since counting structures which satisfy $\neg\sigma_2$ looks easier (though some people did count directly, getting a sum of terms reflecting the possible number of elements mapping to $c^{\mathcal{M}}$, then simplifying this sum using the Binomial Theorem). As above, the total number of $\mathcal{L}$-structures on a set of size n is nn^n. Such a structure $\mathcal{M}$ satisfies $\neg\sigma_2$ iff either $f^{\mathcal{M}}$ sends no element to $c^{\mathcal{M}}$ - we counted those in part (i), obtaining $n(n-1)^n$ - or sends exactly one element to $c^{\mathcal{M}}$. To count the latter: there are n choices of $c^{\mathcal{M}}$, then choose one element (out of the n) to be sent to $c^{\mathcal{M}}$, then send the other $n-1$ elements to any elements different from $c^{\mathcal{M}}$. So that's $n.n.(n-1)^n$. Therefore we obtain $p(\sigma_2, n) = 1 - p(\neg\sigma_2, n) = 1 - \dfrac{n(n-1)^n - nn(n-1)^{n-1}}{nn^n} = 1 - (1 - \frac{1}{n})^n - \dfrac{n}{n-1}(1 - \frac{1}{n})^{n-1}$ which, as $n \to \infty$, has limit $1 - \dfrac{1}{e} - 1.\dfrac{1}{e} = 1 - \dfrac{2}{e}$.

(iii) $\mathcal{L} = \mathcal{L}_0 \vee \{c, P(-)\}$, where c is a constant symbol and P is a 1-ary relation symbol; let σ_3 be $P(c)$.

Solution: Total number of $\mathcal{L}$-structures $= n2^n$. If we require that the choice of $c^{\mathcal{M}}$ (n choices) be in $P^{\mathcal{M}}$ (no choice) that leaves a subset of a set of $n-1$ elements to be chosen for the rest of $P^{\mathcal{M}}$ (2^{n-1} choices). So $p(\sigma_3, n) = \dfrac{n2^{n-1}}{n2^n} = \dfrac{1}{2}$. So the limit also is $\dfrac{1}{2}$.

Alternatively, if M is a set of size n, the set of $\mathcal{L}$-structures on M has a bijection defined by taking a structure $(M; a, X)$ to $(M; a, M \setminus X)$. This interchanges the structures which satisfy σ_3 and those which satisfy $\neg\sigma_3$. So $p(\sigma_3, n)$, and hence the limit, is $\dfrac{1}{2}$.

(iv) $\mathcal{L} = \mathcal{L}_0 \vee \{c, f(-), P(-)\}$, where c is a constant symbol, f is a 1-ary function symbol and P is a 1-ary relation symbol; let σ_4 be $P(c)$.

Solution: It seems reasonable that the function symbol should make no difference, so that the answer should be as in part (iii) but some argument is needed.

Either compute top and bottom lines; on each there will be an extra factor n^n (the number of choices for interpretation of the function symbol). These cancel, so we get $\frac{1}{2}$ for the answer.

An alternative would be to say that each structure for the language in part (iii) has n^n expansions to the language of this part and the truth value of σ_4 in the expansion is the same as in the restricted structure (since σ_4 involves only symbols from the language in part (iii)), so the ratio of structures satisfying σ_4 is unchanged, therefore $\frac{1}{2}$. In this case, saying this was no shorter than re-computing, but the argument could be made into a generally applicable proposition.

(v) $\mathcal{L} = \mathcal{L}_0 \vee \{c, f(-), P(-)\}$, where c is a constant symbol, f is a 1-ary function symbol and P is a 1-ary relation symbol; let σ_5 be $P(f(c))$.

Solution: Similar to part (iii). The total number of structures is $n.n^n.2^n$.

Then compute the number which satisfy σ_5. Choose the element $a = c^{\mathcal{M}}$ which interprets c (n choices). Then choose $f^{\mathcal{M}}$ (n^n choices). We have to put a into $P^{\mathcal{M}}$, leaving 2^{n-1} choices for the rest of $P^{\mathcal{M}}$ (since it can be any subset of the remaining $n-1$ elements).

So $p(\sigma_5, n) = \dfrac{n.n^n.2^{n-1}}{n.n^n.2^n} = \dfrac{1}{2}$.

Q2. Let $\mathcal{R} = (\mathrm{Ran}; R^{\mathcal{G}})$ be the Random Graph.

(i) Fix a and b to be distinct elements of Ran and set $G = \{z \in \mathrm{Ran} : R^{\mathcal{G}}(z,a)$ holds and $R^{\mathcal{G}}(z,b)$ does not hold$\} \setminus \{a,b\}$. Let $\mathcal{G}$ be the induced subgraph on G. Prove that $\mathcal{G} \simeq \mathcal{R}$.

Solution: We check the property $(*)$ for $\mathcal{G}$. So let U, V be disjoint finite subsets of G. Set $U' = U \cup \{a\}$ and $V' = V \cup \{b\}$; note that U', V' are disjoint. We may apply $(*_{U',V'})$ in the Random Graph $\mathcal{R}$, but the element we get from that might be b. So first apply $(*_{U'\cup V',\emptyset})$ in $\mathcal{R}$ to obtain an element $w \in \mathrm{Ran}$ related to each element of $U' \cup V'$ and note that $w \in \mathrm{Ran} \setminus (U' \cup V')$ (since there are no loops), so $V'' = V' \cup \{w\}$ and U' are disjoint. Then, by $(*_{U',V''})$ in $\mathcal{R}$, there is $z \in \mathrm{Ran}$ related to everything in U' and nothing in V''. In particular z is related to a, not related to b and not equal to b (since it is not related to w), hence $z \in G$. Also z is related to everything in U and to nothing in V. So we do have $(*_{U,V})$ in $\mathcal{G}$. Since $\mathcal{G}$, being a subgraph of the Random Graph, is countable and is, note, non-empty (since $\mathcal{R}$ satisfies $(*)$), we conclude by 8.4 that $\mathcal{G} \simeq \mathcal{R}$.

(ii) Let $\mathcal{H}$ be the "unconnected union" of two copies of the random graph. That is $H = \mathrm{Ran} \times \{0,1\}$ and $R^{\mathcal{H}} = (R^{\mathcal{R}} \times \{0\}) \cup (R^{\mathcal{R}} \times \{1\})$. Is $\mathcal{H} \simeq \mathcal{R}$? Justify your answer.

Solution: No. Take an element $a \in \mathrm{Ran}$ and consider its "copies" $(a,0), (a,1) \in H$. Set $U = \{(a,0),(a,1)\}$. Then U, $V = \emptyset$ are disjoint and finite. If there were $z \in H$ with z connected to each element of U then it would have to have the form $(w,0)$ (since related to $(a,0)$) but also the form $(w,1)$ (since related to $(a,1)$) - contradiction. So $\mathcal{H}$ does not satisfy $(*)$, hence is not isomorphic to $\mathcal{R}$.

Q3. Let $\mathcal{L} = \mathcal{L}_0 \vee \{R(-,-)\}$ where R is a binary relation symbol.

(a) Write down a sentence γ of $\mathcal{L}$ such that an $\mathcal{L}$-structure $\mathcal{G}$ satisfies γ iff $\mathcal{G}$ is a graph, where "graph" means undirected graph with no loops (a **loop** is an element which is joined to itself by an edge).

Solution: $\forall x, y\left(\neg R(x,x) \wedge ((R(x,y) \rightarrow R(y,x))\right)$

(b) By an n-**cycle** we mean a sequence $(a_1, \ldots, a_n)$ of elements in a graph (in the above sense) $\mathcal{G}$ such that there is an edge between a_i and a_j iff $|i-j| = 1$ or $\{i,j\} = \{1,n\}$. Write down a formula $\phi_n(x_1, \ldots, x_n)$ such that, if $\mathcal{G}$ is a graph, regarded as an $\mathcal{L}$-structure, and $a_1, \ldots, a_n \in \mathcal{G}$, then $\mathcal{G} \models \phi_n(a_1, \ldots, a_n)$ iff $(a_1, \ldots, a_n)$ is an n-cycle.

Solution: $\bigwedge_{i=1}^{n-1} R(x_i, x_{i+1}) \wedge R(x_n, x_1) \wedge \bigwedge_{i,j=1, |i-j| \neq 1, \{i,j\} \neq \{1,n\}}^{n} \neg R(x_i, x_j)$

(c) Let $\mathcal{G}$ be a graph which is a bouquet of cycles - one n-cycle for each $n \geq 3$. That is, for each $n \geq 3$, $\mathcal{G}$ has one n-cycle and all these cycles have a single point, a_0 say, in common.

Explicitly, $G = \{a_0\} \cup \{a_{ij} : 1 \leq i \leq j \ (j \geq 2)\}$ and, for each $j \geq 2$, $(a_0, a_{1,j}, \ldots, a_{j,j})$ is a $j+1$-cycle.

(You might like to sketch part of the graph to check you have this clear.)

Suppose that $\mathcal{H}$ is an elementary extension of $\mathcal{G}$.

(i) Prove that every element of $\mathcal{H}$ is directly connected (that is, connected by an edge) to at least two points.

Solution: By inspection, this is true of $\mathcal{G}$. And it can be expressed by a sentence of $\mathcal{L}$: $\forall x \exists y, z\, (y \neq z \wedge R(x,y) \wedge R(x,z))$. Every elementary extension $\mathcal{H}$ must therefore satisfy this sentence so, in $\mathcal{H}$, every element is connected to at least two distinct points.

(ii) Is there an element of $H \setminus G$ which is directly connected to more than two points? Justify your answer.

Solution: No. We can similarly write down a formula $\phi(x)$ which says that x is connected to at least 3 points. $[\exists y, z, w\, (y \neq z \wedge y \neq w \wedge z \neq w \wedge R(x,y) \wedge R(x,z) \wedge R(x,w))]$. Then $\phi(\mathcal{G}) = \{a_0\}$, so $\mathcal{G} \models \forall x, x'\, ((\phi(x) \wedge \phi(x')) \rightarrow x = x')$ - expressing the fact that there is just one solution for ϕ. So $\mathcal{H}$ must also satisfy this. But since $\mathcal{G} \prec \mathcal{H}$, $\mathcal{H} \models \phi(a_0)$, so a_0 is the only solution in $\mathcal{H}$. Therefore there is no element in $H \setminus G$ connected to more than two elements.

(iii) Prove that, for every $n \geq 3$, $\mathcal{H}$ has exactly one n-cycle.

Solution: Clarification: my intention was that if $(a_1, \ldots, a_n)$ is an n-cycle then the same points ordered in reverse and/or in any cyclic permutation are "the same" n-cycle. (Also note that doing, say a 3-cycle, then doing the same 3-cycle again or another cycle does not satisfy the definition of being a cycle because there would be more pairs of elements related than in the definition of a cycle.)

We can see that in $\mathcal{G}$ there is just one n-cycle for each $n \geq 3$. So, if we can express the property of having just one n-cycle by some sentence γ_n, then this will enough because $\mathcal{G} \prec \mathcal{H}$.

One sentence which will do is $\forall x_1, \ldots, x_n \forall y_1, \ldots, y_n \left((\phi_n(x_1, \ldots, x_n) \wedge \phi_n(y_1, \ldots, y_n)) \rightarrow \bigvee_{\eta \in S} \bigwedge_{i=1}^{n} y_i = x_{\eta(i)}\right)$, where S is the set of all permutations of the form $(1\ 2\ \ldots\ n)^k$ for $k \in \{\pm 1, \ldots, \pm n\}$.

There are various alternatives; even $\forall x_1, \ldots, x_n \forall y_1, \ldots, y_n \left((\phi_n(x_1, \ldots, x_n) \wedge \phi_n(y_1, \ldots, y_n)) \rightarrow \bigwedge_{i=1}^{n} \bigvee_{j=1}^{n} y_i = x_j\right)$ (because the fact of being a cycle forces more than this might at first sight imply). Referring to the point a_0 in the sentence and 'working out from there' was also used by a number of people.