Model Theory 2015, Assessment 2, Solutions

1. Suppose that $\mathcal{M}$ is an $\mathcal{L}$-structure and that $\phi(x)$ is a formula of $\mathcal{L}$ such that $\phi(\mathcal{M})$ is infinite.
(a) Suppose that $\psi_1(x), \ldots, \psi_n(x)$ are formulas such that $\phi(\mathcal{M}) = \psi_1(\mathcal{M}) \cup \cdots \cup \psi_n(\mathcal{M})$. Prove that for every elementary extension $\mathcal{N} \succ \mathcal{M}$ we have $\phi(\mathcal{N}) = \psi_1(\mathcal{N}) \cup \cdots \cup \psi_n(\mathcal{N})$.

Solution: By assumption $\mathcal{M} \models \sigma$ where σ is the sentence $\forall x\ (\phi(x) \leftrightarrow \bigvee_{i=1}^{n} \psi_i(x))$. So $\mathcal{N} \models \sigma$ and hence $\phi(\mathcal{N}) = \psi_1(\mathcal{N}) \cup \cdots \cup \psi_n(\mathcal{N})$.

(b) Suppose that $\{\theta_i(x) : i \geq 1\}$ are infinitely many formulas such that $\phi(\mathcal{M}) = \bigcup_{i\geq 1} \theta_i(\mathcal{M})$ and such that for every $n \geq 2$, $\theta_n(\mathcal{M}) \nsubseteq \theta_1(\mathcal{M}) \cup \ldots \theta_{n-1}(\mathcal{M})$. Prove that there is an elementary extension, $\mathcal{N}$, of $\mathcal{M}$ such that $\phi(\mathcal{N}) \neq \bigcup_{i\geq 1} \theta_i(\mathcal{N})$.

Solution: Let $\Phi(x)$ be the partial type $\{\phi(x)\} \cup \{\neg\theta_i(x) : i \geq 1\}$. This is finitely realised in $\mathcal{M}$ since any finite subset of Φ is contained in a finite subset of the form $\Phi_0 = \{\phi(x)\} \cup \{\neg\theta_i(x) : 1 \leq i \leq n-1\}$ for some n. By assumption there is some element in $\theta_n(\mathcal{M}) \setminus \bigcup_{i=1}^{n-1} \theta_i(\mathcal{M})$, hence in $\phi(\mathcal{M}) \setminus \bigcup_{i=1}^{n-1} \theta_i(\mathcal{M})$, and such an element satisfies all the formulas in Φ_0, as required.

So, by [5.13] (realising partial types in elementary extensions), there is an elementary extension $\mathcal{N}$ of $\mathcal{M}$ containing an element $b \in N$ which satisfies all the formulas in Φ, that is, which is in $\phi(\mathcal{N})$ but in no $\theta_i(\mathcal{N})$.

[Comment: [5.13] is essentially the Compactness Theorem plus the method of diagrams. A number of you essentially re-proved [5.13] in this particular context, which is a good exercise, but not necessary since quoting [5.13] allows you to avoid repeating those details.]

(c) Give an example of a language $\mathcal{L}$, an $\mathcal{L}$-structure $\mathcal{M}$ and formulas $\phi(x)$, $\theta_i(x)$ $(i \geq 1)$ which are as in (b).

Solution: There are many, many examples. You could name all the elements of a countably infinite set by adding constant symbols (c_a, for $a \in M$) and then take $x = x$ for $\phi(x)$ and take the $\theta_i(x)$ to be of the form $x = c_a$ where a runs through all the elements of M. For another example, take the reals as an ordered field, take $\phi(x)$ to be $x \geq 0$ and $\theta_i(x)$ to be $i \leq x \leq i+1$ for i a natural number.

2. Let $(\mathbb{Z}; 0, 1, +, \times, \leq)$ be the ring of integers regarded as a structure for a language $\mathcal{L} = \mathcal{L}_0 \vee \{0, 1, +, \times, \leq\}$ for ordered rings. Use the same notation, $\mathbb{Z}$, for this structure.
(i) Write down a formula $\phi(x)$ of $\mathcal{L}$ such that $\phi(\mathbb{Z})$ is the set $\{2^n : n \geq 1\}$ of positive powers of 2.

Solution: An integer is a positive power of 2 iff it is divisible by 2 and every integer ≥ 2 which divides it is also divisible by 2: an appropriate formula $\phi(x)$ is $2 \mid x \wedge \forall y\, ((y \geq 2 \wedge y \mid x) \rightarrow 2 \mid y)$, where 2 is an abbreviation for $1+1$ and (e.g.) $y \mid x$ is an abbreviation for $\exists z\, (x = yz)$. Alternatively, we can use $\psi(x)$ which is $2 \mid x \wedge \forall y\, ((\pi(y) \wedge y \mid x) \rightarrow (y = 2))$, where $\pi(y)$ is an abbreviation for the formula $\forall u, v\, (y \mid uv \rightarrow (y \mid u \vee y \mid v))$.

(ii) Prove that there is an elementary extension $\mathcal{Z}$ of $\mathbb{Z}$ which contains an element a such that $2^n | a$ for every positive integer n and such that 2 is the only positive prime divisor of a (recall that an element p of a ring is said to be prime if $p|st$ implies that $p|s$ or $p|t$).

Solution: Consider the set of formulas $\Phi(x) = \{2^n \mid x : n \geq 1\} \cup \{\forall y\, (\pi(y) \wedge y \mid x \wedge y \geq 0) \rightarrow (y = 2)\}$ (π as above). Any finite subset of Φ is realised in $\mathbb{Z}$ by choosing x to be a suitably high power of 2 so, by [5.13], there is an elementary extension $\mathcal{Z}$ of $\mathbb{Z}$ containing an element, a, which satisfies all the formulas in Φ, in other words, which is divisible by every 2^n and whose only prime positive divisor is 2, as required.

[Comment: an alternative, which a number of you used, is to apply Q1(b) above in place of [5.13].]

3. Let $\mathcal{L} = \mathcal{L}_0 \vee \{E(-,-)\}$ where E is a binary relation symbol. Suppose that $\mathcal{M}$, $\mathcal{N}$ are $\mathcal{L}$-structures and that in each of $\mathcal{M}$, $\mathcal{N}$ the interpretation of E is an equivalence relation with infinitely many classes and with each equivalence class infinite. Prove that $\mathcal{M} \equiv \mathcal{N}$.

Solution: By Downwards Löwenheim-Skolem there is a countable elementary substructure $\mathcal{M}_0$ of $\mathcal{M}$ and a countable elementary substructure $\mathcal{N}_0$ of $\mathcal{N}$. We will show that $\mathcal{M}_0 \simeq \mathcal{N}_0$ and hence, since then $\mathcal{M} \equiv \mathcal{M}_0 \equiv \mathcal{N}_0 \equiv \mathcal{N}$, that $\mathcal{M} \equiv \mathcal{N}$. So, without loss of generality, assume that both $\mathcal{M}$ and $\mathcal{N}$ are countable. We define an isomorphism from $\mathcal{M}$ to $\mathcal{N}$ as follows.

Enumerate the $E^{\mathcal{M}}$-equivalence classes of M as $E_1, E_2, \ldots, E_n, \ldots$ and those of $E^{\mathcal{N}}$ as $F_1, F_2, \ldots, F_n, \ldots$. Each of those classes is countably infinite, so, for each i, take a bijection $\alpha_i : E_i \rightarrow F_i$. Then define $\alpha : M \rightarrow N$ by $\alpha(a) = \alpha_i(a)$ if $a \in E_i$. Note that this is well-defined since each element of M lies in exactly one E-equivalence class. Furthermore, if $\alpha(a) = \alpha(b)$ then a and b lie in the same E-class, E_i say, (since the images of different α_i - the F_i - are disjoint) and hence $\alpha_i(a) = \alpha_i(b)$, so $a = b$ (since α_i is a monomorphism). Furthermore, each element of N lies in some F_j and hence is the image, under α_j, of some element of E_j, so α is a surjection. Finally, we have to check that α is an isomorphism. The only structure is given by the equivalence relation, so we have to check that, if $a, b \in M$ then $(a, b) \in E^{\mathcal{M}}$ iff $(\alpha(a), \alpha(b)) \in E^{\mathcal{N}}$; that is, two elements of M belong to the same equivalence class iff their images belong to the same equivalence class. But we constructed α exactly to do this. So we do, indeed, have $\mathcal{M} \simeq \mathcal{N}$, as required.

[Comment: using a back-and-forth argument proved to be popular, and is fine, though the argument above, defining α in blocks rather than one element at a time, is quicker.]