4. Duality and Sensitivity

For every instance of an LP, there is an associated LP known as the **dual problem**. The original problem is known as the **primal problem**.

There are two definitions of the dual pair of problems, which are equivalent. Duality can also be shown to be a ‘symmetric’ relationship (property of involution) in that “the dual of the dual is the primal problem”.

4.1 Primal Dual Formulations

**CANONICAL FORM**

P1: Minimize \( c^T x \)

subject to \( Ax \geq b \)
\( x \geq 0 \)

D1: Maximize \( y^T b \) (or \( b^T y \))

subject to \( y^T A \leq c^T \) (or \( A^T y \leq c \))
\( y \geq 0 \)

**STANDARD FORM**

P2 Minimize \( c^T x \)

subject to \( Ax = b \)
\( x \geq 0 \)

D2 Maximize \( y^T b \) (or \( b^T y \))

subject to \( y^T A \leq c^T \) (or \( A^T y \leq c \))
\( y \) free variables (u.r.s.)

Note that if the number of variables in the primal problem is \( n \) and the number of constraints is \( m \) then the the number of variables in the dual problem is \( m \) and the number of constraints is \( n \). In fact variables in the dual problem correspond to constraints in the primal problem and vice-versa.

In other words, if \( x \in \mathbb{R}^n \) and \( A \) is \((m \times n)\) then \( y \in \mathbb{R}^m \) and

\[ A^T y \leq c \]
represents a system of \( n \) inequalities (the dual constraints) in \( m \) new unknown dual variables \((y_1, \ldots, y_m)\).

**Example 1**

*Primal:*

Minimize \[ 10x_1 + 5x_2 + 4x_3 \]

subject to \[
3x_1 + 2x_2 - 3x_3 \geq 3 \\
4x_1 + 2x_3 \geq 10 \\
x_1, x_2, x_3 \geq 0
\]

*DUAL:*

Maximize \[ 3y_1 + 10y_2 \]

subject to \[
3y_1 + 4y_2 \leq 10 \\
2y_1 \leq 5 \\
-3y_1 + 2y_2 \leq 4 \\
y_1, y_2 \geq 0
\]

**Example 2**

*Primal:*

Minimize \[ 6x_1 + 8x_2 \]

subject to \[
3x_1 + x_2 - s_1 = 4 \\
5x_1 + 2x_2 - s_2 = 7 \\
x_1, x_2, s_1, s_2 \geq 0
\]

*DUAL:*

Maximize \[ 4w_1 + 7w_2 \]

subject to \[
3w_1 + 5w_2 \leq 6 \\
w_1 + 2w_2 \leq 8 \\
w_1, w_2 \geq 0
\]
The primal in this example is stated in standard form P2. However the variables $s_1, s_2$ play the part of non-negative surplus variables since they do not appear in the objective function and it is easy to write down an equivalent primal problem in the canonical form P1. The dual problem is the same, whether written in the form of D1 or as D2.

**Economic interpretation of the dual**

The dual of the diet problem introduced earlier

$$\begin{align*}
\min_x & \quad c^T x \\
\text{s.t.} & \quad Ax \geq r \\
& \quad x \geq 0
\end{align*}$$

is

$$\begin{align*}
\max_y & \quad y^T r \\
\text{s.t.} & \quad y^T A \leq c^T \\
& \quad y \geq 0
\end{align*}$$

We can provide an interesting economic interpretation of the dual in this case:

Recall that $a_{ij}$ represents the amount of nutrient $i$ contained in unit amount of the $j^{th}$ food.

Consider a pill maker (an alternative supplier of nutrients) who wants to set a price for each nutrient. Let $\pi_i$ be the price per unit of the $i^{th}$ nutrient ($i = 1, \ldots, m$) forming a price vector $\pi$. The pill maker’s price to supply the required amount of each nutrient in the diet is

$$\sum_{i=1}^{m} \pi_i r_i.$$

The price of all nutrients needed to make up a unit amount of the $j^{th}$ food should be competitive, therefore not be more than the price of unit amount of the $j^{th}$ food. Hence

$$\sum_{i=1}^{m} \pi_i a_{ij} \leq c_j \quad j = 1, \ldots, n$$

Identifying $y$ with $\pi$ we see that the dual problem maximizes the income of the pill maker subject to a competitiveness constraint.
4.2 Duality - basic properties

4.2.1 Equivalence of dual forms

The duality relationships P1-D1, P2-D2 are equivalent.

Proof

We first take the dual pair in standard form P2-D2 as the definition and find the dual of a problem in the canonical form P1:

\[
\begin{align*}
\text{Minimize} & \quad c^T x \\
\text{subject to} & \quad Ax \geq b \\
& \quad x \geq 0
\end{align*}
\]

Introduce a vector of \( m \) surplus variables \( s \geq 0 \) so the constraints can be written

\[
Ax - s = b
\]

or

\[
\begin{pmatrix} A \\ -I_m \end{pmatrix} \begin{pmatrix} x \\ s \end{pmatrix} = b
\]

which is of the form \( A'x' = b \) where \( A' \) and \( x' \) are partitioned matrices. Applying the duality result for the primal in standard form P2, the dual problem is

\[
\begin{align*}
\text{Maximize} & \quad b^T y \\
\text{subject to} & \quad \begin{pmatrix} A^T \\ -I_m \end{pmatrix} y \leq \begin{pmatrix} c \\ 0 \end{pmatrix}
\end{align*}
\]

In this dual problem, although the D2 dual variables are not explicitly unrestricted in sign (free variables), the constraints imply that \( A^T y \leq c \) and \( y \geq 0 \), which provides the dual problem D1.

We can also take the dual pair in canonical form P1-D1 as the definition and find the dual of a problem in the canonical form P2. (Exercise) ■

4.2.2 Involution property

The dual of the dual is the primal.

Proof
Consider the dual in canonical form $D_1 \ (y \in \mathbb{R}^m, A(m \times n))$

Maximize $b^T y$
subject to $A^T y \leq c$
$y \geq 0$

We may write this in the form of a canonical $P_1$ primal problem

Minimize $(-b^T) y$
subject to $(-A^T) y \geq -c$
$y \geq 0$

The corresponding canonical dual problem in variables $w \in \mathbb{R}^n$ is

Maximize $-c^T w$
subject to $(-A^T)^T w \leq -b$
$w \geq 0$

which is equivalent to the primal problem $P_1$

Minimize $c^T w$
subject to $Aw \geq b$
$w \geq 0$

4.2.3. Mixed Constraints

The dual of LP problems that are neither in standard form nor in canonical form can be written down using rules summarized in the following table.

<table>
<thead>
<tr>
<th>Min Problem</th>
<th>Max problem</th>
</tr>
</thead>
<tbody>
<tr>
<td>Type of Constraint</td>
<td>Type of Variable</td>
</tr>
<tr>
<td>$a_i^T x \geq b_i$</td>
<td>$y_i \geq 0$</td>
</tr>
<tr>
<td>$a_i^T x \leq b_i$</td>
<td>$y_i \leq 0$</td>
</tr>
<tr>
<td>$a_i^T x = b_i$</td>
<td>$y_i \text{ u.r.s.}$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Type of Variable</th>
<th>Type of Constraint</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_j \geq 0$</td>
<td>$y^T A_j \leq c_j$</td>
</tr>
<tr>
<td>$x_j \leq 0$</td>
<td>$y^T A_j \geq c_j$</td>
</tr>
<tr>
<td>$x_j \text{ u.r.s.}$</td>
<td>$y^T A_j = c_j$</td>
</tr>
</tbody>
</table>

(either case)
Notes:
1. u.r.s. denotes “unrestricted in sign” or free variable
2. \( a_i^T \) denotes \( i^{th} \) primal constraint coefficients
   i.e. the \( i^{th} \) row of \( A \)-matrix
3. \( A_j \) denotes \( j^{th} \) dual constraint coefficients
   i.e. the \( j^{th} \) column of \( A \)-matrix

Example 3 (mixed constraints)

**Primal:**

Maximize \[ z = 5x_1 + 6x_2 \]
subject to
\[ x_1 + 2x_2 = 5 \]
\[ -x_1 + 5x_2 \geq 3 \]
\[ 4x_1 + 7x_2 \leq 8 \]
\[ x_1 \text{ u.r.s.}, \ x_2 \geq 0 \]

**Dual:**

Minimize \[ w = 5y_1 + 3y_2 + 8y_3 \]
subject to
\[ y_1 - y_2 + 4y_3 = 5 \]
\[ 2y_1 + 5y_2 + 7y_3 \geq 6 \]
\[ y_1 \text{ u.r.s.}, \ y_2 \leq 0, \ y_3 \geq 0 \]

Use of the Table:

\( m = 3, n = 2 \) so dual has 3 variables \( y_1, y_2, y_3 \) and 2 constraints

Since primal is a maximization, refer to Table 2nd column for primal properties - read off dual properties from Table 1st column.

Variable \( x_1 \) is u.r.s. \( \Rightarrow \) 1st dual constraint is ‘\( = \)’
Variable \( x_2 \geq 0 \) \( \Rightarrow \) 2nd dual constraint is ‘\( \geq \)’ (for minimization)

1st primal constraint is ‘\( = \)’, \( \Rightarrow \ y_1 \) is u.r.s.
2nd primal constraint is ‘\( \geq \)’ \( \Rightarrow \ y_2 \leq 0 \)
3rd primal constraint is ‘\( \leq \)’ \( \Rightarrow \ y_3 \geq 0 \)
4.3 Duality Theorems

The primal and dual LP problems are so intimately related that solving either problem provides the solution to the other. For the following results we assume the primal is a minimization in standard form P2 with dual D2.

4.3.1 Weak duality lemma

If \( x, y \) are feasible for the primal and dual problems respectively, then:

\[
  c^T x \geq y^T b
\]  

(4.1)

Proof

Let \( x, y \) be feasible for problems P2, D2 respectively. Then

\[
y^T A \leq c^T
\]  

(4.2a)

\[
y^T A - c^T \leq 0
\]  

(4.2b)

\[
\therefore (y^T A - c^T) x \leq 0
\]  

(4.2c)

(4.2c) follows from (4.2b) because \( x \geq 0 \) (feasible for P2) so \( \sum_{i=1}^{n} (y^T A - c^T)_i x_i \leq 0 \). Now \( Ax = b \), again because \( x \) is feasible for P2, so

\[
y^T Ax - c^T x \leq 0
\]

\[
y^T b - c^T x \leq 0
\]

\[
y^T b \leq c^T x
\]

as required.

Corollary 1

If \( x, y \) are feasible for P2, D2 respectively and \( c^T x = y^T b \) then \( x \) and \( y \) are optimal for their respective problems.

i.e. If a pair of feasible solutions for the primal and dual problems have the same objective function value, then they are optimal for their respective problems.

Corollary 2

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If either problem is *unbounded* (optimal solution is at infinity) the other problem is *infeasible*.

[ NB The converse is not true as both problems can be infeasible! ]

**Example**

In Example 2 of Section 4.1 the primal and dual objective functions are

\[
    z = 6x_1 + 8x_2
\]

and

\[
    w = 4y_1 + 7y_2.
\]

It can easily be verified (e.g. graphically) that \( z_{\min} = w_{\max} = \frac{42}{5} \) and these optima are achieved at \( x^* = (\frac{7}{5}, 0)^T \) and \( y^* = (0, \frac{6}{5})^T \).

Let us solve the dual problem through simplex iterations. In standard form constraints are

\[
    3y_1 + 5y_2 + v_1 = 6 \\
    y_1 + 2y_2 + v_2 = 8
\]

A basis of dual slacks is available:

<table>
<thead>
<tr>
<th>Max</th>
<th>( y_1 )</th>
<th>( y_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( v_1 )</td>
<td>3</td>
<td>5</td>
</tr>
<tr>
<td>( v_2 )</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>( z )</td>
<td>-4</td>
<td>-7</td>
</tr>
</tbody>
</table>

After one pivot, the optimal tableau is

<table>
<thead>
<tr>
<th>( y_1 )</th>
<th>( v_1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y_2 )</td>
<td>3/5</td>
</tr>
<tr>
<td>( v_2 )</td>
<td>-1/5</td>
</tr>
<tr>
<td>( z )</td>
<td>1/5</td>
</tr>
</tbody>
</table>

We may confirm the elements of the tableau as follows:

\[
    B = \begin{pmatrix} 5 & 0 \\ 2 & 1 \end{pmatrix}, \quad N = \begin{pmatrix} 3 & 1 \\ 1 & 0 \end{pmatrix}
\]
\[ B^{-1} = \frac{1}{5} \begin{pmatrix} 1 & 0 \\ -2 & 5 \end{pmatrix} \]

\[ B^{-1} N = \frac{1}{5} \begin{pmatrix} 3 & 1 \\ -1 & -2 \end{pmatrix} = \begin{pmatrix} \frac{3}{5} & \frac{1}{5} \\ -\frac{1}{5} & -\frac{2}{5} \end{pmatrix} = (y_{ij}) \]

\[ \bar{b} = (y_{ij}) = \frac{1}{5} \begin{pmatrix} 1 & 0 \\ -2 & 5 \end{pmatrix} \begin{pmatrix} 6 \\ 8 \end{pmatrix} = \begin{pmatrix} \frac{6}{5} \\ \frac{28}{5} \end{pmatrix} \]

\[ (z_j - c_j) = (c_B^T y_j - c_j) = \left( \frac{1}{8}, \frac{7}{8} \right) \]

\[ z_0 = c_B^T \bar{b} = \begin{pmatrix} 7 \\ 0 \end{pmatrix} \begin{pmatrix} \frac{6}{5} \\ \frac{28}{5} \end{pmatrix} = \frac{42}{5} \]

The matrix form of the reduced tableau is

\[
\begin{array}{c|c|c}
\text{Basic vars} & \text{Y = B}^{-1} N = (y_{ij}) & \bar{b} \\
\hline
z & \{z_j - c_j\} & z_0 \\
\end{array}
\]

(4.3)

where \( z_j - c_j \) are the elements of \( c_B^T B^{-1} N - c_N^T \) and \( z_0 = c_B^T \bar{b} \).

Simplex multipliers (dual variables)

The components of the m-vector

\[ y_0^T = c_B^T B^{-1} \]

(4.4)

known as simplex multipliers are the dual variables associated with a particular basis \( B \). The dual vector is generally infeasible for the dual problem during intermediate tableau iterations. Only when the bottom row satisfies primal optimality does \( y_0^T \) become dual feasible.

We will show below that

1. the dual variables \( y_0 \) corresponding to an optimal basis are feasible
2. the dual objective \( w_0 = y_0^T b \) equals \( z_0 \), the minimum value of the primal.
Therefore by **Corollary 1** to the weak duality lemma we have constructed an optimal dual vector $y_0$.

### 4.3.2 Strong duality theorem

*If either the primal or the dual has a finite optimal solution then so has the other and the corresponding objective function values are equal. If either problem is unbounded the other problem is infeasible.*

**Proof**

Let $y_0^T = c_B^T B^{-1}$ where $B$ is an optimal basis.

\[
y_0^T A = y_0^T \begin{bmatrix} B & N \end{bmatrix} = c_B^T B^{-1} \begin{bmatrix} B & N \end{bmatrix}
\]

\[
= \begin{bmatrix} c_B^T & c_B^T B^{-1} N \end{bmatrix} \leq \begin{bmatrix} c_B^T & c_N^T \end{bmatrix} = c^T
\]

(4.5a)

(4.5b)

so that $y_0^T$ is dual feasible, with objective value

\[
w_0 = y_0^T b
\]

\[
= c_B^T B^{-1} b
\]

\[
= c_B^T x_B
\]

\[
= z_0
\]

Notice that in going from (4.5a) to (4.5b) we have assumed that $c_B^T B^{-1} N \leq c_N^T$.

This follows from the optimality conditions $z_j - c_j \leq 0 \forall j$, which are simply the components of $c_B^T B^{-1} N - c_N^T$.

**Example (continued)**

For the above example

\[
y_0^T = c_B^T B^{-1}
\]

\[
= \frac{1}{5} \begin{pmatrix} 7, & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -2 & 5 \end{pmatrix}
\]

\[
= \begin{pmatrix} \frac{7}{5}, & 0 \end{pmatrix}
\]

\[
= \begin{pmatrix} x_1^*, & x_2^* \end{pmatrix}
\]

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Hence the primal optimal solution is

\[ x^*_1 = \frac{7}{5}, \quad x^*_2 = 0. \]

### 4.3.3 Complementary slackness

The complementary slackness (C-S) conditions, which are a consequence of duality, relate the values of the primal and dual vectors at a common optimum. They can be stated for either the asymmetric \((P2, D2)\) or the symmetric \((P1, D1)\) primal dual pair.

**Theorem (Complementary slackness - asymmetric case)**

Let \(x, y\) be feasible for \(P2, D2\) respectively. A necessary and sufficient condition (NSC) that they both be optimal is that for each \(j (j = 1, \ldots, n)\)

i) If \(x_j > 0\) then \(y^T A_j = c_j\)

ii) If \(y^T A_j < c_j\) then \(x_j = 0\)

**Proof.**

Since \(x, y\) are feasible for their respective problems, we have

\[
Ax = b, \quad x \geq 0 \\
y^T A \leq c^T
\]

Thus \(y^T A + v^T = c^T\) with \(v^T \geq 0\). Postmultiply by \(x\) and apply \(Ax = b\)

\[
y^T Ax + v^T x = c^T x \\
y^T b + v^T x = c^T x \\
c^T x - y^T b = v^T x \quad (4.6)
\]

By corollary 2 to the Weak Duality theorem, \(x, y\) are optimal if and only if r.h.s. = 0.

Since \(v^T x = \sum_{j=1}^n v_j x_j\) with \(v_j, x_j \geq 0\), r.h.s. = 0 if and only if \(v_j x_j = 0\) for each \(j\).

So i) \(x_j > 0 \Rightarrow v_j = 0\) and ii) \(v_j > 0 \Rightarrow x_j = 0\) as required.

[In fact ii) follows from i) and feasibility of \(y\) ]

**Theorem (Complementary slackness - symmetric case)**
Let \( x, y \) be feasible for P1, D1 respectively. A necessary and sufficient condition (NSC) that they both be optimal is that

i) \( x_j v_j = 0 \ \forall j \)

ii) \( s_i y_i = 0 \ \forall i \)

where \( v_j \) is the \( j^{th} \) dual slack (corresponding to \( x_j \)) and \( s_i \) is the \( i^{th} \) primal slack

Comment: By analogy with the asymmetric case, we can write each equation as a pair of implications, so

\[
\begin{align*}
\forall j, \ x_j > 0 & \Rightarrow v_j = 0, \ v_j > 0 & \Rightarrow x_j = 0 & (4.7a) \\
\forall i, \ s_i > 0 & \Rightarrow y_i = 0, \ y_i > 0 & \Rightarrow s_i = 0 & (4.7b)
\end{align*}
\]

Proof. Since \( x, y \) are feasible for their respective problems, we have

\[
\begin{align*}
Ax & \geq b, \quad x \geq 0 \\
y^T A & \leq c^T, \quad y \geq 0
\end{align*}
\]

or, introducing slack variables \( s \geq 0, v^T \geq 0 \)

\[
\begin{align*}
Ax - s & = b, \quad x, s \geq 0 \\
y^T A + v^T & = c^T, \quad y, v \geq 0
\end{align*}
\]

Then

\[
\begin{align*}
y^T Ax - y^T s & = y^T b \\
y^T Ax + v^T x & = c^T x
\end{align*}
\]

so

\[
c^T x - y^T b = v^T x + y^T s \quad (4.8)
\]

The result follows by analogy to the previous (asymmetric) case.

Example 2 (continued)

Recall the primal was

\[
\begin{align*}
\text{Minimize} & \quad 6x_1 + 8x_2 \\
\text{subject to} & \quad 3x_1 + x_2 \geq 4 \\
& \quad 5x_1 + 2x_2 \geq 7 \\
& \quad x_1, x_2 \geq 0
\end{align*}
\]
and the dual

Maximize \[ 4w_1 + 7w_2 \]
subject to \[ 3w_1 + 5w_2 \leq 6 \]

\[ w_1 + 2w_2 \leq 8 \]

\[ w_1, w_2 \geq 0 \]

The optimal solution to the primal is

\[ (x_1^*, x_2^*, s_1^*, s_2^*) = \left( \frac{7}{5}, 0, \frac{1}{5}, 0 \right) \]

From the C-S conditions, the optimal solution to the dual takes the form

\[ (v_1^*, v_2^*, w_1^*, w_2^*) = (0, \alpha, 0, \beta) \]

i.e. the 1st dual variable is zero at the optimum \((w_1^* = 0)\). Hence \(5\beta = 6\) from the 1st dual constraint which is active (satisfied with equality) since \(v_1^* = 0\). Hence

\[ (w_1^*, w_2^*) = \left( 0, \frac{6}{5} \right) \]

4.4 The Dual Simplex Algorithm

Suppose we have a tableau that satisfies the bottom row optimality conditions but which corresponds to an infeasible solution.

i.e. we have

<table>
<thead>
<tr>
<th>(x_B)</th>
<th>non-basic variables</th>
<th>(Y = [y_{ij}])</th>
<th>(\bar{b})</th>
</tr>
</thead>
<tbody>
<tr>
<td>(z)</td>
<td>(z_j - c_j s)</td>
<td>(\leq 0)</td>
<td></td>
</tr>
</tbody>
</table>

with \(z_j - c_j \leq 0\), each \(j\) (for minimization) and \(\bar{b}_i < 0\) for some \(i\).

Such a tableau is said to be dual feasible because the associated dual vector \(y_0^T = c_B^TB^{-1}\) satisfies

\[ y_0^TA = y_0^T[B|N] \leq c^T \]

The dual simplex algorithm is applicable. The rules for selecting a pivot element are:

1. If \(\bar{b} \geq 0\), stop; the current basis is optimal. Otherwise choose pivot row \(p\) by\n
\[ \min_i \{ \bar{b}_i \} = \bar{b}_p \]
(p_j < 0)

2. If \( y_{pj} \geq 0 \ \forall j \) stop; the problem is infeasible (because dual is unbounded).

Otherwise, choose pivot column \( q \) by the minimum ratio rule

\[
\min_j \left\{ \frac{z_j - c_j}{y_{pj}} : y_{pj} < 0 \right\} = \frac{z_q - c_q}{y_{pq}}
\]  \hspace{1cm} (4.9)

3. Pivot on the element \( y_{pq}(< 0) \) according to usual Simplex rules and return to Step 1.

Notes

i) The dual simplex algorithm is in fact the primal simplex algorithm applied to the dual problem.

ii) The dual simplex algorithm can be useful

(a) in avoiding a Phase I calculation when an infeasible but “optimal” tableau is available “by inspection”,

(b) in sensitivity analysis when the current optimal solution becomes infeasible through the addition of an extra constraint or a small change in the right hand side vector \( b \).

in integer programming by Gomory’s cutting planes.

Example (Dual Simplex)

Minimize \( 2x_1 + 3x_2 + 4x_3 \)

subject to \( x_1 + 2x_2 + 3x_3 \geq 3 \)
\( 2x_1 - x_2 + 3x_3 \geq 4 \)
\( x_1, x_2, x_3 \geq 0 \)

An initial (infeasible) tableau using \( s_1, s_2 \) as basic variables is

<table>
<thead>
<tr>
<th></th>
<th>( x_1 )</th>
<th>( x_2 )</th>
<th>( x_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( s_1 )</td>
<td>-1</td>
<td>-2</td>
<td>-1</td>
</tr>
<tr>
<td>( s_2 )</td>
<td>-2</td>
<td>1</td>
<td>-3</td>
</tr>
<tr>
<td></td>
<td>-2</td>
<td>-3</td>
<td>-4</td>
</tr>
</tbody>
</table>

\[\uparrow\]
After 1\textsuperscript{st} dual simplex pivot

<table>
<thead>
<tr>
<th></th>
<th>$s_2$</th>
<th>$x_2$</th>
<th>$x_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s_1$</td>
<td>-1 &amp; $\frac{2}{3}$ &amp; $\frac{1}{2}$</td>
<td>-1</td>
<td></td>
</tr>
<tr>
<td>$x_1$</td>
<td>-1 &amp; $\frac{1}{2}$ &amp; $\frac{3}{2}$</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td></td>
<td>-1 &amp; -4 &amp; -1</td>
<td>4</td>
<td></td>
</tr>
</tbody>
</table>

After 2\textsuperscript{nd} dual simplex pivot

<table>
<thead>
<tr>
<th></th>
<th>$s_1$</th>
<th>$s_2$</th>
<th>$x_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_2$</td>
<td>$\frac{1}{5}$ &amp; $-\frac{2}{5}$ &amp; $-\frac{1}{5}$</td>
<td>$\frac{2}{5}$</td>
<td></td>
</tr>
<tr>
<td>$x_1$</td>
<td>$-\frac{2}{5}$ &amp; $\frac{4}{5}$ &amp; $\frac{7}{5}$</td>
<td>$\frac{11}{5}$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$-\frac{1}{5}$ &amp; $-\frac{8}{5}$ &amp; $-\frac{9}{5}$</td>
<td>$\frac{28}{5}$</td>
<td></td>
</tr>
</tbody>
</table>

Optimal solution is $(x_1^*, x_2^*, x_3^*, s_1^*, s_2^*) = \left(\frac{11}{5}, \frac{2}{5}, 0, 0, 0\right)$

Optimal solution to dual $(w_1^*, w_2^*) = \left(\frac{2}{5}, \frac{1}{5}\right)$ can be confirmed using C-S conditions (Ex.)

N.B.

1. The OF value $z$ is monotonic increasing in successive tableaux (min problem).
2. Since $\bm{I}_m$ appears in the the initial $A$–matrix and $\bm{y}_0^T = \bm{c}_B^T \bm{B}^{-1} = -\bm{c}_B^T \bm{B}^{-1} (-\bm{I}_m)$

   $w_1^*, w_2^*$ also appear negatively as the $z_j - c_j$ values for surplus variables $s_1, s_2$.

4.5 Sensitivity Analysis

How is an optimal solution (of a LP in standard form) influenced by

1. changes to the objective function coefficients $\bm{c}$
2. changes to the rhs. of the constraints $\bm{b}$
3. changes to the constraint matrix $\bm{A}$, in particular
   
   (a) adding a new constraint $\bm{a}_{m+1}^T \bm{x} = b_{m+1}$
   (b) adding a new variable (or activity) $x_{m+1}$?
From a knowledge of the simplex tableau at optimality

\[
\begin{array}{c|c|c}
 x_B & \text{non-basic variables} & \\
 B & Y = B^{-1}N & b = B^{-1}b \\
 z\text{-row} & c_B B^{-1}N - c_N & z_0 = c_B B^{-1}b \\
\end{array}
\]  

(4.10)

we see that

1. small changes to \( c \) affect only the bottom row
2. small changes to \( b \) affect only the rhs.
3. changes to \( A \) can potentially influence all sections of the tableau
   
   (we will only consider in detail adding a new constraint)

**Example (Furniture manufacturer)**

A manufacturer of furniture makes three products: desks, tables and chairs. Each item of furniture made consumes amounts of three resources: timber, finishing time and carpentry time.

To manufacture one desk requires 8 ft. of timber, 4 hours finishing time and 2 hours carpentry time A table requires 6 ft. of timber, 2 hours finishing time and 1.5 hours carpentry time A chair requires 1 ft. of timber, 1.5 hours finishing time and 0.5 hours carpentry time.

The total weekly availability of timber is 48 ft., of finishing time is 20 hrs and of carpentry time is 8 hrs. The unit profit from selling a desk is $60, from a table is $30, from a chair is $20.

The LP formulation to maximize the manufacturers profit subject to the given resource constraints is as follows:

Let \( x_1, x_2, x_3 = \) no. of units of each product (desks, tables and chairs respectively) to make per week.

Maximize

\[
60x_1 + 30x_2 + 20x_3
\]

subject to

\[
\begin{align*}
8x_1 + 6x_2 + x_3 & \leq 48 \\
4x_1 + 2x_2 + 1.5x_3 & \leq 20 \\
2x_1 + 1.5x_2 + 0.5x_3 & \leq 8 \\
x_1, x_2, x_3 & \geq 0
\end{align*}
\]
resulting in the following tableaux

<table>
<thead>
<tr>
<th></th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Initial Tableau</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$s_1$</td>
<td>8</td>
<td>6</td>
<td>1</td>
<td>48</td>
</tr>
<tr>
<td>$s_2$</td>
<td>4</td>
<td>2</td>
<td>$\frac{3}{2}$</td>
<td>20</td>
</tr>
<tr>
<td>$s_3$</td>
<td>2</td>
<td>$\frac{3}{2}$</td>
<td>$\frac{1}{2}$</td>
<td>8</td>
</tr>
<tr>
<td></td>
<td>$-60$</td>
<td>$-30$</td>
<td>$-20$</td>
<td>0</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>$s_3$</th>
<th>$x_2$</th>
<th>$s_2$</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Final Tableau</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$s_1$</td>
<td>$-8$</td>
<td>$-2$</td>
<td>2</td>
<td>24</td>
</tr>
<tr>
<td>$x_3$</td>
<td>$-4$</td>
<td>$-2$</td>
<td>2</td>
<td>8</td>
</tr>
<tr>
<td>$x_1$</td>
<td>$\frac{3}{2}$</td>
<td>$\frac{5}{4}$</td>
<td>$-\frac{1}{2}$</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>5</td>
<td>10</td>
<td>280</td>
</tr>
</tbody>
</table>

The optimal solution is to manufacture 2 desks and 8 chairs resulting in a profit of $280. There is a surplus of 24 ft. of timber, but all available hours of finishing time, carpentry time are used up.

Often the data assumed in the calculation are imprecise or subject to change. We now ask

Q1. What range of variation in unit profit leaves the current solution optimal?

Q2. What is the optimal product mix if the unit profit for a table increases to $40?

Q3. What if the total available finishing time increases to 30 hours?

These questions can be answered through a sensitivity analysis.

1. Changes to $c$

From (4.10) we see these affect the optimality conditions $z_j - c_j \geq 0$ (for max) each $j$, and the optimal value $z_0 = c_B^T \bar{b}$ (though not of course the polytope representing the feasible region).

Consider increasing a single component of $c_B$, say $c_1$ by an amount $\Delta$, i.e. $c_1 \leftarrow c_1 + \Delta$. The modified final tableau is

<table>
<thead>
<tr>
<th></th>
<th>$s_3$</th>
<th>$x_2$</th>
<th>$s_2$</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$s_1$</td>
<td>$-8$</td>
<td>$-2$</td>
<td>2</td>
<td>24</td>
</tr>
<tr>
<td>$x_3$</td>
<td>$-4$</td>
<td>$-2$</td>
<td>2</td>
<td>8</td>
</tr>
<tr>
<td>$x_1$</td>
<td>$\frac{3}{2}$</td>
<td>$\frac{5}{4}$</td>
<td>$-\frac{1}{2}$</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>5</td>
<td>10</td>
<td>280</td>
</tr>
<tr>
<td></td>
<td>$+\frac{3}{2} \Delta$</td>
<td>$+\frac{5}{4} \Delta$</td>
<td>$-\frac{1}{2} \Delta$</td>
<td>$+2 \Delta$</td>
</tr>
</tbody>
</table>
The condition for the current solution to remain optimal is

\[
10 + \frac{3}{2}\Delta \geq 0 \\
5 + \frac{5}{4}\Delta \geq 0 \\
10 - \frac{1}{2}\Delta \geq 0
\]

which need to be satisfied simultaneously. Hence

\[-4 \leq \Delta \leq 20\]

or equivalently

\[56 \leq c_1 \leq 80\]

For example, if \(c_1 = 56\) (\(\Delta = -4\)) we find an alternative optimal manufacturing plan is to stop manufacturing desks and instead to produce 1.6 tables and 11.2 desks per week

\[
\begin{array}{ccc}
  \text{s}_3 & \text{x}_2 & \text{s}_2 \\
  \text{s}_1 & -8 & -2 & 2 & 24 \\
  \text{x}_3 & -4 & -2 & 2 & 8 \\
  \text{x}_1 & 3 & \frac{5}{4} & -\frac{1}{2} & 2 \\
  & 4 & 0 & 12 & 272 \\
\end{array}
\quad
\begin{array}{c}
  \text{s}_3 \\
  \text{s}_1 & 136 \\
  \text{x}_3 & 56 \\
  \text{x}_2 & \frac{8}{5} \\
  & 4 \\
\end{array}
\]

NB Non-integer production values can still have meaning in a continuous production setting, although integer programming (IP) techniques (see Section 5) will provide ways of finding optimal integer solutions. Similarly we can establish ranges for \(c_2\) and \(c_3\) (leaving the other components unchanged) that ensure the current solution remains optimal.

2. Suppose \(c_2 = 40\) (\(\Delta = 10\) in \(c_2\)). As \(x_2\) is nonbasic in the optimal tableau, only that column’s \(z_j - c_j\) value is changed. It becomes negative (-5) and after one pivot optimality is restored

\[
\begin{array}{ccc}
  \text{s}_3 & \text{x}_2 & \text{s}_2 \\
  \text{s}_1 & -8 & -2 & 2 & 24 \\
  \text{x}_3 & -4 & -2 & 2 & 8 \\
  \text{x}_1 & 3 & \frac{5}{4} & -\frac{1}{2} & 2 \\
  & 10 & -5 & 10 & 280 \\
\end{array}
\quad
\begin{array}{cc}
  \text{s}_3 & \text{s}_1 \\
  \text{s}_1 & \frac{136}{5} \\
  \text{x}_3 & \frac{56}{5} \\
  \text{x}_2 & \frac{8}{5} \\
  & 16 \\
\end{array}
\]

18
This is the same BFS as before.

3. Changes to \( b \)

These affect the rhs. of an optimal tableau (4.10). Consider an increase of \( \Delta \) to the second resource limitation \( b_2, b_2 \leftarrow b_2 + \Delta \).

We consider the effect on \( \bar{b} \) of the rhs. becoming \( b' = b + \delta b \) with \( \delta b = (0, \Delta, 0)^T = \Delta e_2 \) where \( e_2 = (0, 1, 0) \), the 2nd column of the identity matrix \( I_3 \).

\[
\begin{align*}
\bar{b} &= B^{-1}b \\
\bar{b}' &= B^{-1}(b + \delta b) \\
&= \bar{b} + B^{-1} \delta b \\
&= \bar{b} + \Delta B^{-1}e_2
\end{align*}
\]

Notice that the product \( B^{-1}e_2 \) picks out the second column of \( B^{-1} \) which appears in the final tableau under \( s_2 \). In the original \( A - matrix \) the column \( A_j \) corresponding to \( s_2 \) is \( e_2 \). Hence \( B^{-1}A_j = B^{-1}e_2 \) appears in the body of the final tableau under \( s_2 \).

Hence

\[
\delta \bar{b} = \begin{bmatrix}
2\Delta \\
2\Delta \\
-\frac{1}{2}\Delta
\end{bmatrix}
\]

The rhs. remains feasible while

\[
24 + 2\Delta \geq 0, \ 8 + 2\Delta \geq 0, \ 2 - \frac{1}{2}\Delta \geq 0
\]

i.e. while

\[-4 \leq \Delta \leq 4 \iff 16 \leq b_2 \leq 24\]

E.g. if \( b_2 = 30 (\Delta = 10) \) the final tableau becomes infeasible. In this example feasibility (hence the optimal solution) is obtained after one iteration of the dual simplex algorithm.

<table>
<thead>
<tr>
<th>( s_3 )</th>
<th>( x_2 )</th>
<th>( s_2 )</th>
<th>( s_3 )</th>
<th>( x_2 )</th>
<th>( x_1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( s_1 )</td>
<td>(-8)</td>
<td>(-2)</td>
<td>(2)</td>
<td>(44)</td>
<td>(s_1)</td>
</tr>
<tr>
<td>( x_3 )</td>
<td>(-4)</td>
<td>(-2)</td>
<td>(2)</td>
<td>(28)</td>
<td>(x_3)</td>
</tr>
<tr>
<td>( x_1 )</td>
<td>(\frac{3}{2})</td>
<td>(\frac{5}{4})</td>
<td>(\frac{-1}{2})</td>
<td>(-3)</td>
<td>(s_2)</td>
</tr>
<tr>
<td>(10)</td>
<td>(5)</td>
<td>(10)</td>
<td>(380)</td>
<td>(40)</td>
<td>(30)</td>
</tr>
</tbody>
</table>