4. State dependent models

4.1 Markov processes

Markov processes can model a system which is considered to be in any one of a discrete set of states \{S_1, S_2, ..., S_k\} at time \( t \) (continuous time).

The fundamental Markov assumptions are

1. The probability that a system will undergo a transition from one state to another state depends only on the current state and not on the previous state history - system is "memoryless".

2. Transition probabilities (instantaneous rates) are constant over time - system is "stationary".

Throughout this section, components are assumed to have the exponential lifetime distribution.

As an example, consider a system consisting of two components each with constant failure rate (i.e. lifetimes are exponential) We may define the system states as follows:

<table>
<thead>
<tr>
<th>State</th>
<th>Component 1</th>
<th>Component 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>S_1</td>
<td>Working</td>
<td>Working</td>
</tr>
<tr>
<td>S_2</td>
<td>Failed</td>
<td>Working</td>
</tr>
<tr>
<td>S_3</td>
<td>Working</td>
<td>Failed</td>
</tr>
<tr>
<td>S_4</td>
<td>Failed</td>
<td>Failed</td>
</tr>
</tbody>
</table>

For a series system, the system failed or "down" states are \( D = \{S_2, S_3, S_4\} \).

For a parallel system the system failed state is \( D = \{S_4\} \).

The corresponding system "up" states are respectively \( U = \{S_1\} \) and \( U = \{S_1, S_2, S_3\} \).

Let 

\[ P_i(t) = \text{Pr (System is in state } S_i \text{ at time } t) \]  \tag{4.1}

System reliability is defined as the probability of being "up", i.e. not being in a failed state

\[ R_S(t) = \sum_{i \in U} P_i(t) \]  \tag{4.2}

Let the instantaneous failure rate for component \( i \) be \( \lambda_i \) \((i = 1, 2)\).

A rate diagram shows the rates of transition between the states:

\[ \begin{align*}
S_1 & \xrightarrow{\lambda_1} & S_2 & \xleftarrow{\lambda_2} & S_3 & \xrightarrow{\lambda_1} & S_4 \\
S_2 & \xleftarrow{\lambda_2} & S_1 & \xrightarrow{\lambda_2} & S_4 & \xleftarrow{\lambda_1} & S_3
\end{align*} \]

\[ \text{Pr [Transition } S_1 \rightarrow S_2 \text{ in } (t, t + \Delta t) \text{]} = \lambda_1 \Delta t \]
\[ \text{Pr [Transition } S_1 \rightarrow S_3 \text{ in } (t, t + \Delta t) \text{]} = \lambda_2 \Delta t \]
\[ \text{Pr [Transition } S_3 \rightarrow S_4 \text{ in } (t, t + \Delta t) \text{]} = \lambda_1 \Delta t \]
\[ \text{Pr [Transition } S_2 \rightarrow S_4 \text{ in } (t, t + \Delta t) \text{]} = \lambda_2 \Delta t \]
In order for the system to be in state \( S_1 \) at time \( t + \Delta t \)

1) the system must be in state \( S_1 \) at time \( t \), and
2) no transition occurs from state \( S_1 \) in time \( (t, t + \Delta t) \)

\[
\Pr \text{[system is in state } S_1 \text{ at time } t + \Delta t] = \Pr \text{[system is in state } S_1 \text{ at time } t] \times \\
\Pr \text{[no transition during } (t, t + \Delta t)]
\]

\[
\frac{P_1 (t + \Delta t) - P_1 (t)}{\Delta t} = P_1 (t) (1 - \lambda_1 \Delta t - \lambda_2 \Delta t)
\]

In the limit as \( \Delta t \to 0 \)

\[
\frac{d}{dt} P_1 (t) = P_1' (t) = - (\lambda_1 + \lambda_2) P_1 (t)
\] (4.3)

For the system to be in state \( S_2 \) at time \( t + \Delta t \), either

1) system is in state \( S_2 \) at time \( t \) and no transition occurs during \( (t, t + \Delta t) \), or
2) system is in state \( S_1 \) at time \( t \) and a transition \( S_1 \to S_2 \) occurs in \( (t, t + \Delta t) \)

\[
\frac{P_2 (t + \Delta t) - P_2 (t)}{\Delta t} = -\lambda_2 P_2 (t) + \lambda_1 P_1 (t)
\]

In the limit as \( \Delta t \to 0 \)

\[
P_2' (t) = -\lambda_2 P_2 (t) + \lambda_1 P_1 (t)
\] (4.4)

By symmetry in the state transition diagram

\[
P_3' (t) = -\lambda_1 P_3 (t) + \lambda_2 P_1 (t)
\] (4.5)

Since the system must be in one of the four possible states

\[
\sum_{i=1}^{4} P_i = 1 \quad \text{hence} \\
P_4 (t) = 1 - [P_1 (t) + P_2 (t) + P_3 (t)]
\]

Note that \( S_4 \) is an absorbing state. We solve the 1st order differential equation (4.3) for \( P_1 (t) \):

\[
\int \frac{dP_1}{P_1} = - \int (\lambda_1 + \lambda_2) dt \\
\ln (P_1 (t)) = - (\lambda_1 + \lambda_2) t + C
\]

The constant of integration \( C = 0 \) since \( P_1 (0) = 1 \), hence \( \ln (1) = 0 + C \).

\[
P_1 (t) = e^{-(\lambda_1 + \lambda_2)t}
\] (4.6)

Now substitute (4.6) into (4.4)
\[ P_2'(t) + \lambda_2 P_2(t) = \lambda_1 e^{-(\lambda_1 + \lambda_2)t} \]

An integrating factor is \( e^{\lambda_2 t} \). Then

\[
e^{\lambda_2 t} \left[ P_2'(t) + \lambda_2 P_2(t) \right] = \frac{d}{dt} \left[ e^{\lambda_2 t} P_2(t) \right] = \lambda_1 e^{-\lambda_1 t}
\]

\[
e^{\lambda_2 t} P_2(t) = \lambda_1 \int_0^t e^{-\lambda_1 u} du = (1 - e^{-\lambda_1 t})
\]

using the initial condition \( P_2(0) = 0 \). Therefore

\[
P_2(t) = e^{-\lambda_2 t} \left( 1 - e^{-\lambda_1 t} \right) \quad \text{(4.7)}
\]

By symmetry

\[
P_3(t) = e^{-\lambda_1 t} \left( 1 - e^{-\lambda_2 t} \right) \quad \text{(4.8)}
\]

and \( P_4(t) \) may be found by subtraction.

Inserting these expressions for \( P_i(t) \) into the definition of \( R_S(t) \) given above in (4.2) we can verify the reliability expressions for series and parallel systems:

**Series system**

\[
R_S(t) = \sum_{i \in U} P_i(t) = P_1(t) = e^{-(\lambda_1 + \lambda_2)t} = e^{-\lambda_1 t} e^{-\lambda_2 t} = R_1(t) . R_2(t) \quad \text{as before}
\]

**Parallel system**

\[
R_P(t) = \sum_{i \in U} P_i(t) = P_1(t) + P_2(t) + P_3(t) = e^{-\lambda_1 t} + e^{-\lambda_2 t} - e^{-(\lambda_1 + \lambda_2)t} = 1 - \left( 1 - e^{-\lambda_1 t} \right) \left( 1 - e^{-\lambda_2 t} \right) = 1 - (1 - R_1(t)) (1 - R_2(t)) \quad \text{as before}
\]
4.2 Redundancy and load sharing

Standby systems are a way of increasing reliability. e.g.

- hospital electricity generator
- electrical systems on an aircraft (in triplicate).

Standby systems are also known as systems with redundancy. They may involve a switching process that itself can fail. The standby system(s) may be subject to a small probability of failure through deterioration (even when on standby) and so fail to operate when switched in.

As an alternative to "cold" standby systems or "passive redundancy", reliability is also increased by units in "active redundancy" where units operate in parallel e.g.

- multiple tyres on a lorry
- kidneys in a human body

Such systems are "load-sharing". On failure of the primary unit, the secondary unit operates under an increased load and may have a higher rate of failure as a result.

**Example**

Consider two identical components in active parallel operation. When both are working, the failure rates are $\lambda$ (combined failure rate is $2\lambda$). When one component fails the remaining component has increased rate of failure $\lambda^*$. Find the system reliability and MTBF.

We define a Markov process on a system with three states:

- $S_1$ both units operating
- $S_2$ one failed, one working
- $S_3$ both failed

and transition rates rates as follows

$\begin{bmatrix}
S_1 & 2\lambda & \lambda^* \\
S_2 & 2\lambda & \lambda^* \\
S_3 & & 2\lambda & \lambda^* \\
\end{bmatrix}$

Probability arguments as before lead to the following:

$$P_1 (t + \Delta t) = P_1 (t) [1 - 2\lambda \Delta t]$$
$$P_2 (t + \Delta t) = P_1 (t) . 2\lambda \Delta t + P_2 (t) [1 - \lambda^* \Delta t]$$

Hence

$$P_1' (t) = -2\lambda P_1 (t)$$
$$P_2' (t) = 2\lambda P_1 (t) - \lambda^* P_2 (t)$$

with initial conditions $P_1 (0) = 1$, $P_2 (0) = 0$, $P_3 (0) = 0$. 

As before we obtain

\[ P_1(t) = e^{-2\lambda t} \]  \hspace{1cm} (4.9)

then

\[ P_2'(t) + \lambda^* P_2(t) = 2\lambda e^{-2\lambda t} \]

Use \( e^{\lambda t} \) as integrating factor

\[ \frac{d}{dt} \left[ e^{\lambda t} P_2(t) \right] = 2\lambda e^{(\lambda^*-2\lambda)t} \]

Integrate and use \( P_2(0) = 0 \) to obtain

\[ e^{\lambda t} P_2(t) = \frac{2\lambda}{\lambda^* - 2\lambda} \left\{ e^{(\lambda^*-2\lambda)t} - 1 \right\} \]

assuming that \( \lambda^* \neq 2\lambda \), hence

\[ P_2(t) = \frac{2\lambda}{\lambda^* - 2\lambda} \left\{ e^{-2\lambda t} - e^{-\lambda^* t} \right\} \]  \hspace{1cm} (4.10)

System reliability is

\[ R_S(t) = \sum_{i \in U} P_i(t) \]

\[ = P_1(t) + P_2(t) \]

\[ = e^{-2\lambda t} + \frac{2\lambda}{\lambda^* - 2\lambda} \left\{ e^{-2\lambda t} - e^{-\lambda^* t} \right\} \]

\[ = \frac{1}{\lambda^* - 2\lambda} \left\{ \lambda^* e^{-2\lambda t} - 2\lambda e^{-\lambda^* t} \right\} \]  \hspace{1cm} (4.11)

System mean time before failure is

\[ MTBF = \int_0^\infty R_S(t) \, dt \]

\[ = \frac{1}{\lambda^* - 2\lambda} \left\{ \lambda^* \int_0^\infty e^{-2\lambda t} \, dt - 2\lambda \int_0^\infty e^{-\lambda^* t} \, dt \right\} \]

\[ = \frac{1}{\lambda^* - 2\lambda} \left\{ \lambda^* - \frac{2\lambda}{\lambda^*} \right\} \]

\[ = \frac{1}{2\lambda} + \frac{1}{\lambda^*} \]  \hspace{1cm} (4.12)

recalling that \( \int_0^\infty e^{-at} \, dt = \frac{1}{\alpha} \).

e.g. suppose that

\[ \lambda = .01 \text{ failures per day} \]

\[ \lambda^* = .10 \text{ failures per day} \]

system reliability over a 10 day period is

\[ R_S(10) = \frac{1}{.08} \left\{ .1e^{-0.2} - .02e^{-1} \right\} \]

\[ = .9314 \]

and MTBF is

\[ \frac{1}{.02} + \frac{1}{.10} = 60 \text{ days} \]
Notes on solution

1. The system MTBF has a natural interpretation. Under load sharing the time to first failure of either component is \( \frac{1}{2\lambda} \). n.b. hazard function (failure rate) of a series system is \( 2\lambda \). When either fails the system operates as a single unit with (increased) failure rate \( \lambda^* > 2\lambda \). The system lifetime is the sum of the lifetimes \( T_1, T_2 \) under each phase

\[
(MTTF)_S = \mathbb{E}(T_1) + \mathbb{E}(T_2) = \frac{1}{2\lambda} + \frac{1}{\lambda^*}
\]

2. We can also verify the expression for \( R_S(t) \) by a direct probability argument conditioning on the time to first failure of either component. Note that the event "no system failure by time \( t \)" occurs if and only one of the following occurs:

(a) first failure occurs after time \( t \), or
(b) first failure occurs at time \( u \), then no subsequent failure.

The time to first failure \( T_1 \) is an exponentially distributed with constant failure rate \( 2\lambda \). Therefore the p.d.f. of time to first failure is \( 2\lambda e^{-2\lambda t} \).

\[
\Pr\{\text{system survives to time } t\} = \Pr\{\text{first failure occurs after time } t\} + \lim_{\delta u \to 0} \sum_{u<t} \Pr\{\text{first failure occurs in } (u, u + \delta u)\} \times \Pr\{\text{no subsequent failure in time } t - u\}
\]

Let \( R_1(t) \), \( R_2(t) \) be the reliability function of each phase of system life, \( f_1(t) \) be the p.d.f. of time to first failure.

\[
R_S(t) = R_1(t) + \int_0^t f_1(u) R_2(t-u) \, du
\]

\[
= e^{-2\lambda t} + \int_0^t 2\lambda e^{-2\lambda u} e^{-\lambda^*(t-u)} \, du
\]

\[
= e^{-2\lambda t} + 2\lambda e^{-\lambda^* t} \left[ \int_0^t e^{(\lambda^*-2\lambda)u} \, du \right]
\]

\[
= e^{-2\lambda t} + \frac{2\lambda}{\lambda^* - 2\lambda} e^{-\lambda^* t} \left[ \left( e^{(\lambda^*-2\lambda)u} - 1 \right) \right]
\]

\[
= \frac{1}{\lambda^* - 2\lambda} \left\{ \lambda^* e^{2\lambda t} - 2\lambda e^{-\lambda^* t} \right\}
\]

confirming (4.11).
4.3 Matrix form of Markov Equations

The system of linear differential equations for a load sharing system can be written in matrix-vector form.

Define

\[ \mathbf{p}(t)^T = (P_1(t), P_2(t), P_3(t)) \]

Then

\[ P_1'(t) = -2\lambda P_1(t) \]
\[ P_2'(t) = 2\lambda P_1(t) - \lambda * P_2(t) \]
\[ P_3'(t) = \lambda * P_2(t) \]

may be written

\[ \frac{d}{dt} \mathbf{p}(t)^T = \mathbf{p}(t)^T \mathbf{M} \]  \hspace{1cm} (4.12)

where

\[ \mathbf{M} = \begin{bmatrix} -2\lambda & 2\lambda & 0 \\ 0 & -\lambda & \lambda^* \\ 0 & 0 & 0 \end{bmatrix} \]

Notice that each row of \( \mathbf{M} \) sums to zero. This can be seen by postmultiplying (4.12) by the column vector of ones \( \mathbf{1} = (1, 1, 1)^T \)

\[ \frac{d}{dt} \mathbf{p}(t)^T \mathbf{1} = \mathbf{p}(t)^T \mathbf{M} \mathbf{1} \]  \hspace{1cm} (4.13)

Now \( \mathbf{p}(t)^T \mathbf{1} = \sum_{j=1}^{3} P_j(t) = 1 \) and is constant for all \( t \). So the derivative w.r.t. \( t \) is zero. Therefore \( \mathbf{M} \mathbf{1} = 0 \).

A simplified procedure for calculating MTBF:

We can calculate MTBF (MTTF) without explicitly determining \( R_S(t) \)

\[ R_S(t) = \sum_{j \in U} P_j(t) \]
\[ MTTF = \int_0^\infty R_S(t) \, dt \]
\[ = \int_0^\infty \sum_{j \in U} P_j(t) \, dt \]
\[ = \sum_{j \in U} \int_0^\infty P_j(t) \, dt \]  \hspace{1cm} (4.14)

Let \( q_j = \int_0^\infty P_j(t) \, dt \) then \( MTBF = \sum_{j \in U} q_j \). The vector \( \mathbf{q}^T = (q_1, q_2, q_3) \) can be found by integrating (4.12).

\[ \int_0^\infty \left[ \frac{d}{dt} \mathbf{p}(t)^T \right] \, dt = \int_0^\infty \mathbf{p}(t)^T \mathbf{M} \, dt \]
\[ \mathbf{p}(\infty)^T - \mathbf{p}(0)^T = \mathbf{q}^T \mathbf{M} \]  \hspace{1cm} (4.15)
Now \( \mathbf{p}(0)^T = (1, 0, 0) \) and \( \mathbf{p}(\infty)^T = (0, 0, 1) \) since \( S_1 \) is the initial state at \( t = 0 \) and \( S_3 \) is the failed state at which is achieved with probability 1 as \( t \to \infty \). Therefore

\[
(-1, 0, 1) = (q_1, q_2, q_3) \begin{bmatrix} -2\lambda & 2\lambda & 0 \\ 0 & -\lambda^* & \lambda^* \\ 0 & 0 & 0 \end{bmatrix}
\]

which represents the 3 equations in 2 unknowns \( q_1, q_2 \)

\[
\begin{align*}
-1 &= -2\lambda q_1 \\
0 &= 2\lambda q_1 - \lambda^* q_2 \\
1 &= \lambda^* q_2
\end{align*}
\]

having solution

\[
q_1 = \frac{1}{2\lambda} \quad q_2 = \frac{1}{\lambda^*}
\]

Hence

\[
\text{MTTF} = q_1 + q_2 = \frac{1}{2\lambda} + \frac{1}{\lambda^*}
\]

as we obtained previously.

### 4.4 Further examples

#### 4.4.1 Standby system with two non-identical components

A system consists of a primary unit and a secondary standby unit. The main unit has failure rate \( \lambda_1 \) and the secondary has a failure rate \( \lambda_2 \) when working as the primary. When on standby the secondary has a low probability of failure \( \lambda_2^* \). Find the system MTTF assuming perfect switching and no repairs to either unit.

The system states are

\[
\begin{align*}
S_1 & \quad \text{both working} \\
S_2 & \quad \text{primary failed, secondary working} \\
S_3 & \quad \text{primary working, secondary failed} \\
S_4 & \quad \text{both failed}
\end{align*}
\]

From the rate diagram

\[
\begin{array}{c}
S_1 \\
\lambda_1 \\
S_2 \\
\lambda_2 \\
S_4 \\
\lambda_1 \\
S_3 \\
\lambda_2^*
\end{array}
\]

we obtain the set of differential equations

\[
\begin{align*}
P_1'(t) &= - (\lambda_1 + \lambda_2^*) P_1(t) \\
P_2'(t) &= \lambda_1 P_1(t) - \lambda_2 P_2(t) \\
P_3'(t) &= \lambda_2^* P_1(t) - \lambda_1 P_3(t)
\end{align*}
\]
or \( \frac{d}{dt}p(t)^T = p(t)^T M \) where \( p(t)^T = (P_1(t), P_2(t), P_3(t)) \) is a reduced probability vector (omitting \( P_4(t) \)) and

\[
M = \begin{bmatrix} - (\lambda_1 + \lambda_2^-) & \lambda_1 & \lambda_2^- \\ 0 & -\lambda_2 & 0 \\ 0 & 0 & -\lambda_1 \end{bmatrix}
\]

is the correspondingly reduced matrix.

\( p(\infty)^T - p(0)^T = (0, 0, 0) - (1, 0, 0) \), hence

\[
(-1, 0, 0) = (q_1, q_2, q_3) \begin{bmatrix} - (\lambda_1 + \lambda_2^-) & \lambda_1 & \lambda_2^- \\ 0 & -\lambda_2 & 0 \\ 0 & 0 & -\lambda_1 \end{bmatrix}
\]

\[
-1 = - (\lambda_1 + \lambda_2^-) q_1 \\
0 = \lambda_1 q_1 - \lambda_2 q_2 \\
0 = \lambda_2^- q_1 - \lambda_1 q_3
\]

Hence

\[
q_1 = \frac{1}{\lambda_1 + \lambda_2^-} \\
q_2 = \frac{\lambda_1}{\lambda_2} q_1 \\
q_3 = \frac{\lambda_2^-}{\lambda_1} q_1
\]

and therefore

\[
MTTF = q_1 + q_2 + q_3
\]

\[
= \frac{1}{\lambda_1 + \lambda_2^-} \left[ 1 + \frac{\lambda_1}{\lambda_2} + \frac{\lambda_2^-}{\lambda_1} \right]
\]

\[
= \frac{1}{\lambda_1 + \lambda_2^-} \left( \frac{\lambda_1}{\lambda_2} + \lambda_2^- \right)
\]

### 4.4.2 Standby system with identical components

A special case of the above arises when \( \lambda_1 = \lambda_2 \). Then the system mean time to failure is

\[
MTTF = \frac{1}{\lambda_1} + \frac{1}{(\lambda_1 + \lambda_2^-)} \quad (4.17)
\]

(see Examples 4, Q1). In this case states \( S_2 \) and \( S_3 \) can be combined as whichever state results, the subsequent rate of transition to \( S_4 \) is \( \lambda_1 \). So the state transition diagram is as follows

\[
\begin{array}{ccc}
S_1 & \lambda_1 + \lambda_2^- & \lambda_1 \\
\Downarrow & & \Downarrow \\
S_2 & & S_3
\end{array}
\]

Notice that the combined rate of transition into \( S_2 \) is \( \lambda_1 + \lambda_2^- \) (the time of transition is that of first failure of either component c.f. a series system) and the corresponding set of differential equations

\[
P_1'(t) = - (\lambda_1 + \lambda_2^-) P_1(t) \\
P_2'(t) = (\lambda_1 + \lambda_2^-) P_1(t) - \lambda_1 P_2(t)
\]
\[
\frac{d}{dt} \mathbf{p}(t)^T = \mathbf{p}(t)^T \mathbf{M}
\] where \( \mathbf{p}(t)^T = (P_1(t), P_2(t)) \) and

\[
\mathbf{M} = \begin{bmatrix}
-(\lambda_1 + \lambda_2) & (\lambda_1 + \lambda_2) \\
0 & -\lambda_1
\end{bmatrix}
\]

\[
\mathbf{p}(\infty)^T - \mathbf{p}(0)^T = q^T \mathbf{M}
\]

\[
(-1, 0) = (q_1, q_2) \begin{bmatrix}
-(\lambda_1 + \lambda_2) & (\lambda_1 + \lambda_2) \\
0 & -\lambda_1
\end{bmatrix}
\]

so

\[
q_1 = \frac{1}{\lambda_1 + \lambda_2} \quad q_2 = \frac{1}{\lambda_1}
\] (4.18)

and hence (4.17) results.

### 4.4.3 Switch subject to failure

Consider a standby system that has a probability \( p (< 1) \) of successful operation when required. The rate diagram is then as follows

\[
p\lambda \quad \xrightarrow{S_1} \quad (1 - p)\lambda
\]

\[
\lambda \quad \xrightarrow{S_2} \quad \lambda \quad \xrightarrow{S_3}
\]

The differential equation system is

\[
P'_1(t) = -\lambda P_1(t)
\]

\[
P'_2(t) = p\lambda P_1(t) - \lambda P_2(t)
\]

Hence we find \( P_1(t) = e^{-\lambda t} \) (using \( P_1(0) = 1 \)) and

\[
P'_2(t) + \lambda P_2(t) = p\lambda e^{-\lambda t}
\]

\[
\frac{d}{dt} \left( P_2(t) e^{\lambda t} \right) = p\lambda
\]

\[
P_2(t) e^{\lambda t} = p\lambda t
\]

\[
P_2(t) = p\lambda t e^{-\lambda t}
\]

using \( P_2(0) = 0 \) so

\[
R_S(t) = P_1(t) + P_2(t)
\]

\[
= \frac{(1 + p\lambda t) e^{-\lambda t}}{1 - p\lambda}
\] (4.19)
4.5 Systems with repair

We consider now the possibility of repairs to a system described by a set of states. To maintain the Markov property we will assume that times to repair follow an exponential distribution. Let $MTTR = \frac{1}{\mu}$ denote the component mean time to repair where $\mu$ is the repair rate. As before let $MTTF = \frac{1}{\lambda}$ be the component mean time to failure.

4.5.1 Availability

The ‘life’ of a repairable system consists of alternating periods of being in working and failed states. The mean time to first failure is now denoted $MTTFF$ and we define the availability of a system at time $t$ by

$$A(t) = \Pr \{\text{System is working at time } t\}$$

For a system without repair $A(t) = R_S(t)$.

**Example**

(2 state Markov model)

Consider a single component with failure rate $\lambda$ and repair rate $\mu$. The states are $S_1$: working and $S_2$: failed.

\[
\begin{array}{c|c}
S_1 & S_2 \\
\hline
\lambda & \mu \\
\end{array}
\]

Here

\[
\begin{align*}
A(t) &= P_1(t) \\
&= \Pr [\text{System in state } S_1 \text{ at time } t]
\end{align*}
\]

Since $P_2(t) = 1 - P_1(t)$ the system can be described by the single differential equation

\[
\frac{dP_1(t)}{dt} = -\lambda P_1(t) + \mu (1 - P_1(t))
\]

\[
\frac{dP_1(t)}{dt} + (\lambda + \mu) P_1(t) = \mu \quad (4.20)
\]

An integrating factor is $e^{(\mu+\lambda)t}$

\[
\frac{d}{dt} \left[ P_1(t) e^{(\mu+\lambda)t} \right] = \mu e^{(\mu+\lambda)t}
\]

\[
P_1(t) e^{(\mu+\lambda)t} = \frac{\mu}{\mu + \lambda} e^{(\mu+\lambda)t} + C
\]

Using the initial condition $P_1(0) = 1$ we find $C = \frac{\lambda}{\mu + \lambda}$

\[
P_1(t) = \frac{\mu}{\mu + \lambda} + \frac{\lambda}{\mu + \lambda} e^{-(\mu+\lambda)t} \quad (4.21)
\]
This is the availability \( A(t) \). Notice that
\[
A(\infty) = \lim_{t \to \infty} A(t) = \frac{\mu}{\mu + \lambda} \tag{4.22}
\]
\( A(\infty) \) is known as the limiting or ‘steady state’ availability. It may be calculated as
\[
A(\infty) = \frac{1/\lambda}{1/\lambda + 1/\mu} = \frac{MTTF}{MTTF + MTTR}
\]
which is the up-time ratio (UTR).

1. Note the limiting cases:
   - As \( \mu \to \infty \), \( MTTR \to 0 \) and \( A(\infty) \to 1 \) (instant repairs).
   - As \( \mu \to 0 \), \( MTTR \to \infty \) and \( A(\infty) \to 0 \) (slow repair).

2. The steady state availability is also obtained by setting \( \frac{d}{dt} P_1(t) = 0 \) in (4.20).

### 4.5.2 Active parallel redundancy with repair

We consider the effect on the system MTTF of adding a repair facility to an active parallel system of two identical units, each with failure rate \( \lambda \). Define the following states:

- \( S_1 \) : 2 units working
- \( S_2 \) : 1 unit failed under repair
- \( S_3 \) : both units failed \( S_3 \) is system failed state

The rate diagram is as follows:

\[
\begin{array}{c}
S_1 \quad \begin{array}{c}
2\lambda \\
\mu
\end{array} \quad S_2 \\
S_3
\end{array}
\]

Note that \( S_3 \) is an absorbing state. At this stage we assume no return from \( S_3 \). The system differential equations are
\[
\begin{align*}
\frac{dP_1(t)}{dt} &= -2\lambda P_1(t) + \mu P_2(t) \\
\frac{dP_2(t)}{dt} &= 2\lambda P_1(t) - (\mu + \lambda) P_2(t) \\
\frac{dP_3(t)}{dt} &= \lambda P_2(t)
\end{align*}
\tag{4.23}
\]

To determine system MTTF we write the system as
\[
\frac{d}{dt} p(t) = p(t)^T M \text{ where } p(t)^T = (P_1(t), P_2(t), P_3(t)) \text{ and }
\]
\[
M = \begin{bmatrix}
-2\lambda & 2\lambda & 0 \\
\mu & -(\mu + \lambda) & \lambda \\
0 & 0 & 0
\end{bmatrix}
\]

12
Write \( q = \int_0^\infty p(t) \, dt \) then as before by integration we obtain \( p(\infty)^T - p(0)^T = q^T M \)

\[ (-1, 0, 1) = q^T M \]

Solving the system

\[
\begin{align*}
-1 &= -2\lambda q_1 + \mu q_2 \\
0 &= 2\lambda q_1 - (\mu + \lambda) q_2 \\
1 &= \lambda q_2
\end{align*}
\]

gives

\[ q_1 = \frac{\lambda + \mu}{2\lambda^2}, \quad q_2 = \frac{1}{\lambda} \]

Hence

\[ MTTF = q_1 + q_2 = \frac{3}{2\lambda} + \frac{\mu}{2\lambda^2} \quad (4.24) \]

The solution has two components. The MTTF without repair of an active parallel system of two identical components is \( \frac{3}{2\lambda} \). The extra life of the system due to repair is \( \frac{\mu}{2\lambda^2} \).

**4.5.3 Reliability function calculation**

We can verify this result by solving the differential equation system (4.23) to obtain the reliability function \( R_S(t) \) which we then integrate. Recall the following properties of Laplace transforms:

1. \[
\mathcal{L} \left( \frac{dy}{dt} \right) = \int_0^\infty e^{-st} y'(t) \, dt
\]
   \[
   = [e^{-st} y(t)]_0^\infty + \int_0^\infty se^{-st} y(t) \, dt
   \]
   \[
   = sy(s) - y(0)
   \]

2. \[
\mathcal{L} \left( e^{-at} \right) = \int_0^\infty e^{-(s+a)t} \, dt
\]
   \[
   = \frac{1}{s + a} \quad \text{shift theorem}
\]

Take Laplace transforms of (4.23)

\[
\begin{align*}
sp_1 - 1 &= -2\lambda \tilde{P}_1 + \mu \tilde{P}_2 \\
sp_2 &= 2\lambda \tilde{P}_1 - (\mu + \lambda) \tilde{P}_2 \\
sp_3 &= \lambda \tilde{P}_2 \quad (4.25)
\end{align*}
\]
Since $R_S(t) = 1 - P_3(t)$ we need only solve for $\tilde{P}_3$. We find that

$$\tilde{P}_3 = \frac{2\lambda^2}{s [s^2 + (3\lambda + \mu) s + 2\lambda^2]} \tag{4.26}$$

\begin{align*}
&= \frac{A}{s - \alpha_1} + \frac{B}{s - \alpha_2} + \frac{C}{s - \alpha_2} \quad \text{say}
\end{align*}

where

$$\alpha_1, \alpha_2 = \frac{1}{2} \left[ -(3\lambda + \mu) \pm \sqrt{(3\lambda + \mu)^2 + 6\lambda\mu + 2\lambda^2} \right]$$

are the roots of the quadratic in the denominator of (4.26).

By partial fractions we find

$$A = \frac{2\lambda^2}{\alpha_1 \alpha_2}, \quad B = \frac{2\lambda^2}{\alpha_1 (\alpha_1 - \alpha_2)}, \quad C = \frac{2\lambda^2}{\alpha_2 (\alpha_2 - \alpha_1)}$$

Inverting (4.26)

$$P_3(t) = A + Be^{\alpha_1 t} + Ce^{\alpha_2 t}$$

$$= \frac{2\lambda^2}{\alpha_1 \alpha_2} + \frac{2\lambda^2}{\alpha_1 (\alpha_1 - \alpha_2)} \left[ \frac{e^{\alpha_1 t}}{\alpha_1} - \frac{e^{\alpha_2 t}}{\alpha_2} \right]$$

Observe that $\alpha_1 \alpha_2 = 2\lambda^2$ (product of roots of quadratic)

$$R_S(t) = 1 - P_3(t)$$

$$= \frac{\alpha_1}{\alpha_1 - \alpha_2} e^{\alpha_2 t} - \frac{\alpha_2}{\alpha_1 - \alpha_2} e^{\alpha_1 t} \tag{4.27}$$

Hence

$$MTTF = \int_0^\infty R_S(t) \, dt$$

$$= \int_0^\infty \left[ \frac{\alpha_1}{\alpha_1 - \alpha_2} e^{\alpha_2 t} - \frac{\alpha_2}{\alpha_1 - \alpha_2} e^{\alpha_1 t} \right] \, dt$$

$$= \frac{-1}{\alpha_1 - \alpha_2} \left[ \frac{\alpha_1}{\alpha_2} - \frac{\alpha_2}{\alpha_1} \right]$$

$$= \frac{- (\alpha_1 + \alpha_2)}{\alpha_1 \alpha_2} \frac{3\lambda + \mu}{2\lambda^2}$$

as obtained in (4.24).

### 4.5.4 Steady state availability

Consider now the 2 component active parallel system with the possibility of a repair when the system reaches the failed state $S_3$. The repair restores the system to the state $S_2$ it was in immediately preceding failure. The rate diagram and differential equations are now

\[
\begin{array}{c|c|c}
S_1 & 2\lambda & S_2 \\
\hline
\mu & \lambda & S_3
\end{array}
\]
\[ \frac{dP_1(t)}{dt} = -2\lambda P_1(t) + \mu P_2(t) \]
\[ \frac{dP_2(t)}{dt} = 2\lambda P_1(t) - (\mu + \lambda) P_2(t) + \mu P_3(t) \]
\[ \frac{dP_3(t)}{dt} = \lambda P_2(t) - \mu P_3(t) \]

so the revised \( M \)-matrix is

\[
M = \begin{bmatrix}
-2\lambda & 2\lambda & 0 \\
\mu & - (\mu + \lambda) & \lambda \\
0 & \mu & -\mu
\end{bmatrix}
\]

The equilibrium distribution is found using

\[
\frac{d}{dt} \mathbf{p}^T = \mathbf{p}^T M = 0 \quad (4.28)
\]

\[-2\lambda p_1 + \mu p_2 = 0 \]
\[2\lambda p_1 - (\mu + \lambda) p_2 + \mu p_3 = 0 \]
[\lambda p_2 - \mu p_3 = 0

Hence letting \( \rho = \frac{\lambda}{\mu} \) and \( \alpha = p_1 \):

\[ p_1 = \alpha, \quad p_2 = 2\rho \alpha, \quad p_3 = 2\rho^2 \alpha \]

Now \( p_1 + p_2 + p_3 = 1 \Rightarrow \alpha = (1 + 2\rho + 2\rho^2)^{-1} \)

\[ p_1 = \frac{1}{(1 + 2\rho + 2\rho^2)} \]
\[ p_2 = \frac{2\rho}{(1 + 2\rho + 2\rho^2)} \]
\[ p_3 = \frac{2\rho^2}{(1 + 2\rho + 2\rho^2)} \]

Therefore

\[ A(\infty) = p_1 + p_2 \]
\[ = \frac{1}{1 + 2\rho} \]
\[ = \frac{1 + 2\rho + 2\rho^2}{(1 + 2\rho + 2\rho^2)} \]
\[ = \frac{\mu^2 + 2\lambda \mu}{\mu^2 + 2\lambda \mu + 2\lambda^2} \quad (4.29) \]

4.5.5 Alternative derivation

We may verify (4.29) using the expression [see (4.22) above]

\[ A(\infty) = \frac{MTTF}{MTTF + MTTR} \quad (4.30) \]
known as the uptime ratio (UTR).

However (4.24) which is $MTTFF$ is not the appropriate $MTTF$ to use because when the system is restored (after failed state $S_3$) it begins a new working cycle starting from state $S_2$ not from the new state $S_1$ as we assumed before!

\[
\begin{array}{ccc}
S_1 & \xrightarrow{2\lambda} & S_2 \\
& \mu & \xrightarrow{\lambda} \\
& & S_3
\end{array}
\]

Therefore $MTTF = q_1 + q_2$ where $q_1, q_2$ satisfy

\[ p(\infty)^T - p(0)^T = q^T M \]

with $p(\infty) = (0, 0)$ and $p(0) = (0, 1)$. Thus

\[
\begin{align*}
0 &= -2\lambda q_1 + \mu q_2 \\
-1 &= 2\lambda q_1 - (\lambda + \mu) q_2
\end{align*}
\]

giving

\[
q_1 = \frac{\mu}{2\lambda^2} = \frac{1}{2\rho} \frac{1}{\lambda}
\]

\[
q_2 = \frac{1}{\lambda}
\]

where, as before, $\rho = \frac{\lambda}{\mu}$ so

\[
MTTF = \left(1 + \frac{1}{2\rho}\right) \frac{1}{\lambda}
\]

\[
MTTR = \frac{1}{\mu} = \frac{1}{\rho} \frac{1}{\lambda}
\]

and

\[
A(\infty) = \frac{MTTF}{MTTF + MTTR} = \frac{1 + 2\rho}{1 + 2\rho + 2\rho^2}
\]

as before.
4.5.6 Further example (2 out of 3 system)

[see Q2. of Examples 5]

Consider a system of 3 identical units of which 2 are in active parallel operation. The 3rd system is either in working order on standby or failed and under repair. If only one unit is working the system is shut down to prevent further failure and is non-operational. The repair rate is \( \mu \) (single repair facility). The system becomes operational again when two units are available.

<table>
<thead>
<tr>
<th>State</th>
<th>Working</th>
<th>Failed</th>
</tr>
</thead>
<tbody>
<tr>
<td>( S_1 )</td>
<td>3</td>
<td>0</td>
</tr>
<tr>
<td>( S_2 )</td>
<td>2</td>
<td>1 (under repair)</td>
</tr>
<tr>
<td>( S_3 )</td>
<td>1</td>
<td>2 (1 under repair)</td>
</tr>
</tbody>
</table>

To find the MTTF \( F \) we assume no return from \( S_3 \) (an absorbing state). Then \( \frac{d}{dt}p(t)^T = p(t)^T M \) where \( p(t)^T = (P_1(t), P_2(t), P_3(t)) \) and

\[
M = \begin{bmatrix} 2\lambda & 2\lambda & 0 \\ \mu & -\mu \lambda - 2\lambda & 2\lambda \\ 0 & 0 & 0 \end{bmatrix}
\]

Write \( q = \int_0^\infty p(t) dt \) then \( p(\infty)^T - p(0)^T = q^T M \) with \( p(\infty) = (0, 0, 1) \) and \( p(0) = (1, 0, 0) \) gives the system

\[
\begin{align*}
-1 & = -2\lambda q_1 + \mu q_2 \\
0 & = 2\lambda q_1 - (\mu + 2\lambda) q_2 \\
1 & = 2\lambda q_2
\end{align*}
\]

We find

\[
q_2 = \frac{1}{2\lambda} \\
q_1 = \frac{\mu + 2\lambda}{4\lambda^2}
\]

and

\[
MTTF = q_1 + q_2 = \frac{1}{\lambda} + \frac{\mu}{4\lambda^2}
\]

System availability can be determined from the steady state condition \( p^T M = 0 \) together with normalization of probabilities

\[
A(\infty) = \frac{1 + 2\rho}{1 + 2\rho + 4\rho^2}
\]

(Ex.)