Reliability and survival (MATH30018)

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1. Basic concepts

1.1 Books

Wolstenholme, L.C. "Reliability modelling. A statistical approach." Chapman & Hall, 1999.

Ebeling, C. "An introduction to reliability & maintainability engineering." McGraw Hill, 1997.

Leemis, L.M. "Reliability - Probailistic Models & Statistical Methods." Prentice 2^{nd} Ed., 2003.

1.2 Applications

This course is concerned with the study of components and systems that fail in time.
e.g.

Engineering Mechanical and electrical equipment

Medicine Design and analysis of medical trials

1.3 The reliability function

Suppose that an item (*component, device* etc.) is new at time $t = 0$ and fails at time T (a continuous random variable).

- $T > 0$ - only takes positive values
- the item is assumed to be in one of two states - *working* or *failed*.

The reliability function $R(t)$ is defined as

$$R(t) = \Pr(T > t) \tag{1.1}$$

i.e. $R(t)$ is the probability that the item is *still working* at time t

Notice that

$$\begin{aligned} R(t) &= 1 - F(t) \\ R'(t) &= \frac{d}{dt}R(t) = -f(t) \end{aligned} \tag{1.2}$$

where $F(t) = \Pr(T \le t)$ = cumulative distribution function (c.d.f.) of T
and $f(t)$ is the probability density function (p.d.f.) of T.

Some properties of $R(t)$

- $R(t)$ is a *decreasing* function of t. If $t_1 < t_2$ then $R(t_1) \ge R(t_2)$.
- $R(0) = \Pr(T > 0) = 1$

- $R(\infty) = 0$ nothing survives for ever
- $0 \leq R(t) \leq 1$

We will also assume when necessary (for mathematical convenience) that

$$\lim_{t \longrightarrow \infty} tR(t) = 0$$

i.e. $R(t) \longrightarrow 0$ faster than $\frac{1}{t}$.

N.B. In medical survival analysis, $R(t)$ is also known as the *survivor function* $S(t)$.

Example

The failure time of a compressor (in hours) is a random variable with the probability density function (p.d.f.)

$$f(t) = \begin{cases} \frac{.001}{(.001t+1)^2} & t > 0 \\ 0 & \text{otherwise} \end{cases}$$

Find the reliability for a 100 hr. operating life.

Solution

Writing $\alpha = 10^{-3}$ we have

$$\begin{aligned} R(t) &= \int_t^\infty \frac{\alpha}{(\alpha u + 1)^2} du \\ &= \left. \frac{-1}{(\alpha u + 1)} \right|_t^\infty \\ &= \left. \frac{1}{(\alpha u + 1)} \right|_\infty^t \\ &= \frac{1}{(\alpha t + 1)} \end{aligned}$$

Setting $t = 100$ gives $R(100) = \frac{1}{0.1 + 1} = 0.909$.

1.4 Mean time before failure (MTBF)

The mean time before failure is defined as

$$\mu = \mathbb{E}(T) = \int_0^\infty tf(t)\,dt \tag{1.3}$$

where $\mathbb{E}(.)$ denotes mathematical expectation (mean) of the random variable T.

Result. μ is given by the area below the reliability curve.

Proof. Use (1.2) (1.3) and integration by parts

$$\begin{aligned}\mu &= \int_0^\infty tf(t)\,dt = \int_0^\infty -tR'(t)\,dt \\ &= [-tR(t)]_0^\infty + \int_0^\infty R(t)\,dt \\ &= \int_0^\infty R(t)\,dt \qquad (1.4)\end{aligned}$$

noting that $tR(t) = 0$ at $t = 0, \infty$.

The expectation (mean) of a distribution is not always finite. In such cases we can use the median lifetime t_m as an alternative measure of a "typical" lifetime.

The median life t_m satisfies

$$R(t_m) = 0.5 \qquad (1.5)$$

and is the age reached by 50% of the population.

Example

An electronic device has failure time density function (t measured in hours)

$$f(t) = \begin{cases} .002e^{-.002t} & t > 0 \\ 0 & \text{otherwise} \end{cases}$$

Find a) the MTBF, b) the median time to failure of the device.

Solution

Observation: the density $f(t) = \lambda e^{-\lambda t}$, $t > 0$ is known as the exponential distribution.

MTBF:

$$\begin{aligned}R(t) &= \int_t^\infty f(u)\,du \\ &= \int_t^\infty \lambda e^{-\lambda u} du \\ &= e^{-\lambda u}\Big|_\infty^t \\ &= e^{-\lambda t}\end{aligned}$$

$$\begin{aligned}MTBF &= \mathbb{E}(T) = \int_0^\infty R(t)\,dt \\ &= \int_0^\infty e^{-\lambda t} dt \\ &= -\frac{1}{\lambda}\, e^{-\lambda t}\Big|_0^\infty = 500 \text{ hrs.}\end{aligned}$$

Median life:

$$\begin{aligned}R(t_m) &= 0.5 = e^{-\lambda t_m} \\ t_m &= \frac{1}{\lambda}\ln 2 = 500 \ln 2 \\ &= 346.6 \text{ hrs.}\end{aligned}$$

1.5 The hazard function

When an item has reached age t, its lifetime characteristics are usually different from new. The failure time density is $f(t)$. However when $T > t$ we need to condition on this event.

Define the *instantaneous failure rate* of the item as

$$h(t) = \lim_{\delta t \to 0} \frac{1}{\delta t} \Pr(\text{Item fails in } (t, t+\delta t) \,|\text{survives to } t) \tag{1.6a}$$

$$= \lim_{\delta t \to 0} \frac{1}{\delta t} \Pr(t < T < t+\delta t \mid T > t) \tag{1.6b}$$

$$\begin{aligned} \Pr(t < T < t+\delta t \mid T > t) &= \frac{\Pr(t < T < t+\delta t)}{\Pr(T > t)} \\ &= \frac{f(t)\ \delta t}{R(t)} \\ &= h(t)\,\delta t \end{aligned}$$

NB.The above makes use of the result $\Pr(A|B) = \dfrac{\Pr(A)}{\Pr(B)}$ when $A \subset B$.

By taking limits we conclude that the *hazard function* defined as

$$h(t) = \frac{f(t)}{R(t)}$$

is the *instantaneous* failure rate (1.6).

NB.

- $h(t)$ is *not* a probability density function, though for *fixed* t_0 we can define the failure time density (lifetime density) *conditional* on survival to time t_0

$$\begin{aligned} f(t|T > t_0) &= \begin{cases} \dfrac{f(t)}{R(t_0)} & t > t_0 \\ 0 & \text{elsewhere} \end{cases} \qquad (1.7) \\ &= \lim_{\delta t \to 0} \frac{1}{\delta t} \Pr(t < T < t+\delta t \mid T > t_0) \end{aligned}$$

  which does satisfy the normalization

$$\int_{t_0}^{\infty} f(t|T > t_0)\, dt = \frac{\int_{t_0}^{\infty} f(t)\, dt}{R(t_0)} = 1$$

- Actuaries use a discrete form of $h(t)$ with $\Delta t = 1$ year e.g.

$$\Pr(\text{person reaching age 69 dies before 70th birthday})$$

  which is known as the "force of mortality"

1.6 Relationship between R(t) and h(t)

$$\begin{aligned} h(t) &= \frac{f(t)}{R(t)} \\ &= \frac{-R'(t)}{R(t)} = -\frac{d}{dt}\ln R(t) \end{aligned}$$

Integrating and using $\ln R(0) = \ln(1) = 0$ we obtain

$$\ln R(t) = -\int_0^t h(u)\,du \qquad (1.8a)$$

$$R(t) = \exp\left\{-\int_0^t h(u)\,du\right\} \qquad (1.8b)$$

$$= \exp\{-H(t)\} \qquad (1.8c)$$

where $H(t)$ is the *cumulative hazard* function.

1.7 Conditional reliability and monotone failure rates

Suppose that an item has survived to time a. Then $U = T - a$ is the *future life* random variable.

Define the conditional reliability function by

$$\begin{aligned} R_U(t|a) &= \Pr(\text{item survives a } \textit{further} \text{ time } t \mid \text{survives to } a) \\ &= \Pr(U > t | T > a) \qquad u > 0 \\ &= \Pr(T > a + t | T > a) \qquad u > 0 \\ &= \frac{\Pr(T > a + t)}{\Pr(T > a)} \\ &= \frac{R(a+t)}{R(a)} \end{aligned} \qquad (1.9)$$

Differentiating $R_U(t|a)$ with respect to t gives the conditional p.d.f. of $U = T - a$:

$$\begin{aligned} f_U(t|a) &= -R'_U(t|a) \\ &= -\frac{R'(a+t)}{R(a)} \\ &= \frac{f(a+t)}{R(a)} \end{aligned}$$

The subscript U has been included to indicate that t is a *future* life, i.e. a value of U. Note that $\int_0^\infty f_U(t|a)\,dt = 1$ as required.

In the following we may drop U to simplify the notation.

An item has monotone failure rate if either $h(t) \uparrow$ or $h(t) \downarrow$ for all t. The item is either IFR or DFR or both according to whether the failure rate $h(t)$ is increasing, decreasing or constant.

Result 1

Conditional reliability improves as a function of a (for all t) if and only if the item is DFR.

Proof

i) Suppose $R(t|a) = \dfrac{R(a+t)}{R(a)}$ is *increasing* as a function of a.

$$\begin{aligned} h(a) &= \lim_{t\to 0} \frac{1}{t} \Pr(\text{fails in } (a, a+t) \,|\text{survives to } a) \\ &= \lim_{t\to 0} \frac{1}{t} [1 - R(t|a)] \end{aligned}$$

Therefore $R(t|a)$ $\uparrow$with $a \Longrightarrow h(a)$ is *decreasing* as a function of a

ii) Conversely, suppose $h(a)$ is DFR, therefore *decreasing* as a function of a.

$$\begin{aligned} R(t|a) &= \frac{\exp(-H(a+t))}{\exp(-H(a))} \\ &= \exp\left(-\int_a^{a+t} h(u)\, du\right) \end{aligned} \quad (1.10)$$

so $R(t|a)$ is *increasing* as a function of a. ■

The above result shows that in certain circumstances aging can be beneficial to reliability.

- Certain metals increase in strength as they are work-hardened.
- New items can be prone to failure through manufacturing defects. c.f. "infant mortality"

Aging more commonly has an adverse effect on reliability, corresponding to a situation of increasing hazard (IFR). The following result can be proved in the same way as above.

Result 2
Conditional reliability decreases as a function of a (for all t) if and only if the item is IFR.

1.8 The Bathtub curve

The bathtub curve represents the failure rate of a product during its lifecycle. The hazard function comprises three parts:

1. Early failures or infant mortality - DFR period.
2. Useful life - constant (CFR) failure rate period. Failures are "rare" events modelled by a Poisson process.
3. Old age - IFR period, failures due to wear-out.

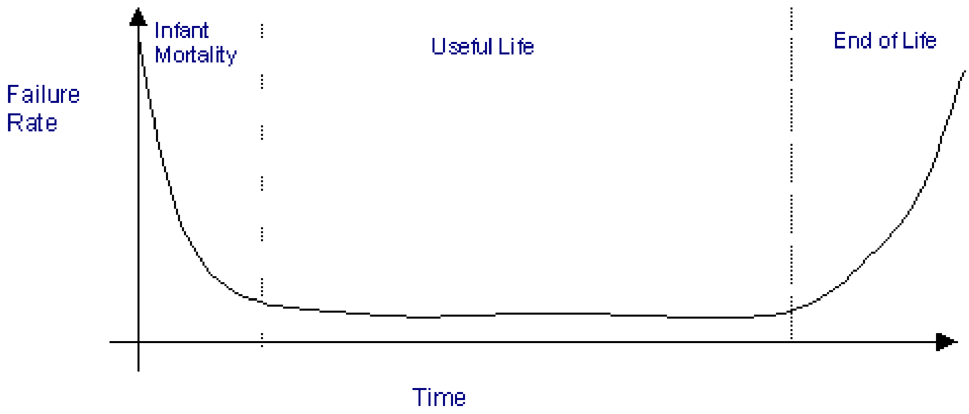

Figure 1: Typical behaviour of $h(t)$ throughout life cycle

1.9 Expected future life

If the hazard function $h(t)$ is *increasing*, then we would expect the *expected future life* conditional on reaching age a to *decrease* with a and conversely. It turns out to be true. We show this by linking the conditional mean future life (i.e. *life expectancy*) to the conditional reliability function.

Result
Define the (future) life expectancy at age a by

$$L(a) = \mathbb{E}(U = T - a | T > a) \tag{1.11}$$
$$= \int_0^\infty t f_U(t|a)\, dt$$
$$= -\int_0^\infty t R'_U(t|a)\, dt \tag{1}$$

Integrating by parts gives

$$L(a) = \left[-tR_U(t|a)\right]_0^\infty + \int_0^\infty R_U(t|a)\, dt$$
$$= 0 + \frac{1}{R(a)} \int_0^\infty R(a+t)\, dt \tag{1.12a}$$
$$= \frac{1}{R(a)} \int_a^\infty R(t)\, dt \tag{1.12b}$$

assuming $\lim_{t\to\infty} tR(t) = 0$. We see that *future* life expectancy is the area below the *conditional* reliability function $R(t|a)$.

Example
Show that the reliability function

$$R(t) = \begin{cases} \dfrac{b-t}{b} & 0 < t < b \\ & \text{elsewhere} \end{cases}$$

is IFR and find the residual mean life (MTTF) at age t_0.

Solution

(i)

$$\begin{aligned} h(t) &= -\frac{d}{dt}(\ln R(t)) \\ &= \frac{1}{b-t} \qquad ,0<t<b \end{aligned}$$

which is an increasing function of t.

(ii) At any age a the conditional life expectancy at age t_0 is

$$\begin{aligned} MTTF(t_0) &= L(t_0) \\ &= \int_{t_0}^{b} \frac{R(t)}{R(t_0)} dt \\ &= \frac{b}{b-t_0} \int_{t_0}^{b} \frac{b-t}{b} dt \\ &= \frac{1}{b-t_0} \left[\frac{1}{2}(b-t)^2 \right]_{b}^{t_0} \\ &= \frac{1}{2}(b-t_0) \end{aligned}$$

which is a decreasing function of t_0. In fact the expected life is half the time interval from t_0 to b.

1.10 Summary of reliability basics

1. Knowing any one of the following functions uniquely characterizes a lifetime distribution:

   - lifetime density $f(t)$
   - reliability (survival) function $R(t)$
   - hazard function $h(t)$

   and we can go from one to another using formulae presented above.

2. The bathtub curve is an important concept characterizing a product life cycle or human lifespan (if we are lucky!).

   BUT we have not yet considered the possibility of repairs and maintenance.

3. The hazard function $h(t)$ is also known as the failure rate or force of mortality.

   Certain $h(t)$ representing monotone or constant failure rates can be used to model different sections of the bathtub curve. Such $h(t)$ also imply monotone (future) life expectancies.

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