Structures with lots of symmetry
(one of the things Bob likes to do when not doing semigroup theory)

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Why?

An advertisement

It is an interesting area

- Provides a meeting-point of ideas from combinatorics, model theory, and permutation group theory.
- At a workshop in Leeds on this topic a few weeks ago several speakers mentioned semigroups in their talks...
Outline

Motivation and background
  Homogeneous structures
  Classification results

Weakening homogeneity
  Set-homogeneous structures
  Enomoto’s argument for finite set-homogeneous graphs
  Classifying the finite set-homogeneous digraphs

Semigroup theory connections
### Homogeneous relational structures

#### Definition

A relational structure $M$ is **homogeneous** if every isomorphism between finite substructures of $M$ can be extended to an automorphism of $M$.

#### Relational structures

- **relational structure** consists of a set $A$, and some relations $R_1, \ldots, R_m$ (can be unary, binary, ternary, ...)
- **substructure** is obtained by taking a subset $B \subseteq A$ and keeping only those relations where all entries in the tuple belong to $B$
- **isomorphism** is a “structure preserving” mapping (i.e. a bijection $\phi$ such that $\phi$ and $\phi^{-1}$ are both homomorphisms)

#### Example

A graph $\Gamma$ is a structure $(V\Gamma, \sim)$ where $V\Gamma$ is a set, and $\sim$ is a symmetric irreflexive binary relation on $V\Gamma$. 
Examples of homogeneous structures

$X$ - a pure set
  - automorphism group is the full symmetric group where any partial permutation can be extended to a (full) permutation

$(\mathbb{Q}, \leq)$ - the rationals with their usual ordering
  - the automorphisms are the order-preserving permutations
  - isomorphisms between finite substructures can be extended to automorphisms that are piecewise-linear

Rado’s countable random graph $R$
  - if we choose a countable graph at random (edges independently with probability $\frac{1}{2}$), then with probability 1 it is isomorphic to $R$
Some history

Origins

- The notion of homogeneous structure goes back to the fundamental work of Fraïssé (1953)
- Fraïssé proved a theorem which helps us determine if a countable structure is homogeneous, using the ideas of:
  - age - the finite substructures they embed, and
  - amalgamation property - the way that they can be glued together

Homogeneous structures are nice because they:
- have “lots of” symmetry;
- often have rich and interesting automorphism groups;
- give examples of “nice” $\aleph_0$-categorical structures (precisely those that have quantifier elimination).

($M$ is $\aleph_0$-categorical if all countable models of the first-order theory of $M$ are isomorphic to $M$.)
Classification results

For certain families of relational structure, those members that are homogeneous have been completely determined.

<table>
<thead>
<tr>
<th>Classification results</th>
<th>Finite</th>
<th>Countably infinite</th>
</tr>
</thead>
<tbody>
<tr>
<td>Posets</td>
<td>(trivial)</td>
<td>Schmerl (1979)</td>
</tr>
</tbody>
</table>
Set-homogeneity

**Definition**

A relational structure $M$ is **set-homogeneous** if whenever two finite substructures $U$ and $V$ are isomorphic, there is an automorphism $g \in \text{Aut}(M)$ such that $Ug = V$.

- It is a concept originally due to Fraïssé and Pouzet.
- The permutation group-theoretic weakening

  \[
  \text{homogeneous} \rightsquigarrow \text{set-homogeneous}
  \]

relates to the model-theoretic weakening

\[
\text{elimination of quantifiers} \rightsquigarrow \text{model complete}.
\]

- Droste et al. (1994) - proved a set-homogeneous analogue of Fraïssé’s theorem, where the amalgamation property is replaced by something called the **twisted amalgamation property**.
Set-homogeneity vs homogeneity

- Clearly if $M$ is homogeneous then $M$ is set-homogeneous.
- What about the converse?

General question
How much stronger is homogeneity than set-homogeneity?
Set-homogeneous finite graphs

<table>
<thead>
<tr>
<th>Ronse (1978)</th>
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<tbody>
<tr>
<td>...proved that for finite graphs <strong>homogeneity and set-homogeneity are equivalent</strong>.</td>
<td></td>
</tr>
<tr>
<td>▶ He did this by classifying the finite set-homogeneous graphs and then observing that they are all, in fact, homogeneous.</td>
<td></td>
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<tr>
<td>▶ This generalised an earlier result of Gardiner, classifying the finite homogeneous graphs.</td>
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<table>
<thead>
<tr>
<th>Enomoto (1981)</th>
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<tbody>
<tr>
<td>...gave a <strong>direct proof</strong> of the fact that for finite graphs set-homogeneous implies homogeneous.</td>
<td></td>
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<tr>
<td>▶ this avoids the need to classify the set-homogeneous graphs</td>
<td></td>
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<tr>
<td>▶ the set-homogeneous classification can then be read off from Gardiner’s result</td>
<td></td>
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</tbody>
</table>
Definition

\( \Gamma = (V_\Gamma, \sim) \) - a graph

So \( \sim \) is a symmetric irreflexive binary relation on \( V_\Gamma \)

- Let \( v \) be a vertex of \( \Gamma \). Then the **neighbourhood** \( \Gamma(v) \) of \( v \) is the set of all vertices adjacent to \( v \). So

\[
\Gamma(v) = \{ w \in V_\Gamma : w \sim v \}
\]

- For \( X \subseteq V_\Gamma \) we define

\[
\Gamma(X) = \{ w \in V_\Gamma : w \sim x \ \forall x \in X \}
\]
Enomoto’s argument

Lemma (Enomoto’s lemma)

Let $\Gamma$ be a finite set-homogeneous graph and let $U$ and $V$ be induced subgraphs of $\Gamma$. If $U \cong V$ then $|\Gamma(U)| = |\Gamma(V)|$.

Proof.

- Let $g \in \text{Aut}(\Gamma)$ such that $Ug = V$.
- Then $(\Gamma(U))g = \Gamma(V)$.
- In particular $|\Gamma(U)| = |\Gamma(V)|$. 
Enomoto’s argument

Γ - finite set-homogeneous graph  X, Y - induced subgraphs
f : X → Y an isomorphism

Claim: The isomorphism f : X → Y is either an automorphism, or extends to an isomorphism f' : X' → Y' where X' ⊇ X and Y' ⊇ Y.

Proof of claim.

▶ Choose a ∈ Γ \ X with |Γ(a) ∩ X| as large as possible.
▶ Choose d ∈ Γ \ Y with |Γ(d) ∩ Y| as large as possible.
▶ Suppose |Γ(a) ∩ X| ≥ |Γ(d) ∩ Y| (the other possibility is dealt with...
Set-homogeneous digraphs

**Question:** Does Enomoto’s argument apply to other kinds of structure?

**Definition (Digraphs)**

A digraph $D$ consists of a set $VD$ of vertices together with an irreflexive antisymmetric binary relation $\rightarrow$ on $VD$.

**Definition (in- and out-neighbours)**

A vertex $v \in VD$ has a set of in-neighbours and a set of out-neighbours

$$D^+(v) = \{ w \in VD : v \rightarrow w \}, \quad D^-(v) = \{ w \in VD : w \rightarrow v \}.$$

A vertex with red in-neighbours and blue out-neighbours
Enomoto’s argument for digraphs

\[ D - \text{finite set-homogeneous digraph} \quad X, Y - \text{induced subdigraphs} \]
\[ f : X \rightarrow Y \text{ an isomorphism} \]

- Follow the same steps but using out-neighbours instead of neighbours.
- Everything works, except the very last step.
- We do not know how \( b \) is related to the vertices in the set \( Y \setminus B \). So \( f' \) may not be an isomorphism.
Enomoto’s argument for digraphs

The key point:

- For graphs, given $u, v \in V\Gamma$ there are 2 possibilities

  $$u \sim v \quad \text{or} \quad u \parallel v$$ (meaning that $u$ & $v$ are unrelated).

- For digraphs, given $u, v \in VD$ there are 3 possibilities

  $$u \rightarrow v \quad \text{or} \quad v \rightarrow u \quad \text{or} \quad u \parallel v.$$

However, the argument does work for tournaments:

Definition

A tournament is a digraph where for any pair of vertices $u, v$ either $u \rightarrow v$ or $v \rightarrow u$.

Corollary

Let $T$ be a finite tournament. Then $T$ is homogeneous if and only if $T$ is set-homogeneous.
A non-homogeneous example

Example

Let \( D_n \) denote the digraph with vertex set \( \{0, \ldots, n - 1\} \) and just with arcs \( i \to i + 1 \pmod{n} \).

The digraph \( D_5 \) is set-homogeneous but is not homogeneous.

\[
\begin{align*}
\begin{array}{c}
\text{(a, c)} \mapsto (a, d) \\
\text{gives an isomorphism between induced subdigraphs that}
\end{array}
\end{align*}
\]

\[
\begin{align*}
\begin{array}{c}
\text{does not extend to an automorphism}
\end{array}
\end{align*}
\]

\[
\begin{align*}
\begin{array}{c}
\text{However, there is an automorphism sending} \{a, c\} \to \{a, d\}.
\end{array}
\end{align*}
\]
Finite set-homogeneous digraphs

**Question**
How much bigger is the class of set-homogeneous digraphs than the class of homogeneous digraphs?

**Theorem (RG, Macpherson, Praeger, Royle (2011))**

Let $D$ be a finite set-homogeneous digraph. Then either $D$ is homogeneous or it is isomorphic to $D_5$.

**Proof.**

- Carry out the classification of finite set-homogeneous digraphs.
- By inspection note that $D_5$ is the only non-homogeneous example.
**Symmetric-digraphs (s-digraphs)**
A common generalisation of graphs and digraphs

**Definition (s-digraph)**

- An s-digraph is the same as a digraph except that we allow pairs of vertices to have **arcs in both directions**.
- So for any pair of vertices \( u, v \) exactly one of the following holds:

\[
    u \rightarrow v, \quad v \rightarrow u, \quad u \leftrightarrow v, \quad u \parallel v.
\]

- Formally we can think of an s-digraph as a structure \( M \) with two binary relations \( \rightarrow \) and \( \sim \) where
  - \( \sim \) is irreflexive and symmetric (and corresponds to \( \leftrightarrow \) above)
  - \( \rightarrow \) is irreflexive and antisymmetric
  - \( \sim \) and \( \rightarrow \) are disjoint
- A graph is an s-digraph (where there are no \( \rightarrow \)-related vertices)
- A digraph is an s-digraph (where there are no \( \sim \)-related vertices)
Classifying the finite homogeneous s-digraphs

- Lachlan (1982) classified the finite homogeneous s-digraphs

To state his result we need the notions of
- complement
- compositional product
Finite homogeneous s-digraphs

**Definition (Complement)**

If $M$ is an s-digraph, then $\bar{M}$, the complement, is the s-digraph with the same vertex set, such that $u \sim v$ in $\bar{M}$ if and only if they are unrelated in $M$, and $u \rightarrow v$ in $\bar{M}$ if and only if $v \rightarrow u$ in $M$.

**Example.**

![Diagram](image-url)
Finite homogeneous s-digraphs

**Definition (Composition)**

If $U$ and $V$ are s-digraphs, the compositional product $U[V]$ denotes the s-digraph which is

“$|U|$ copies of $V$”

Vertex set = $U \times V$

→ relations are of form
$(u, v_1) \rightarrow (u, v_2)$ where $v_1 \rightarrow v_2$ in $V$,
or of form $(u_1, v_1) \rightarrow (u_2, v_2)$ where $u_1 \rightarrow u_2$ in $U$,

Similarly for $\sim$. 

\[ K_2 \quad D_3 \quad K_2[D_3] \]
Some finite homogeneous s-digraphs

Sporadic examples

\( \mathcal{L} \) - finite homogeneous graphs, \( \mathcal{A} \) - finite homogeneous digraphs,
\( \mathcal{S} \) - finite homogeneous s-digraphs

\[ C_5 \in \mathcal{L} \]

\[ K_3 \times K_3 \in \mathcal{L} \]
Some finite homogeneous s-digraphs

Sporadic examples

\( \mathcal{L} \) - finite homogeneous graphs, \( \mathcal{A} \) - finite homogeneous digraphs,
\( S \) - finite homogeneous s-digraphs

\( H_0 \in \mathcal{A} \)

\( H_1 \in S \)
Some finite homogeneous s-digraphs
Sporadic examples

\[ H_2 \in S \]
Lachlan’s classification

\( \mathcal{L} \) - finite homogeneous graphs, \( \mathcal{A} \) - finite homogeneous digraphs, 
\( S \) - finite homogeneous s-digraphs

**Theorem (Lachlan (1982))**

Let \( M \) be a finite s-digraph. Then

**Gardiner**

(i) \( M \in \mathcal{L} \iff M \text{ or } \bar{M} \text{ is one of: } C_5, K_3 \times K_3, K_m[\bar{K}_n] \text{ (for } 1 \leq m, n \in \mathbb{N}); \)

**Lachlan**

(ii) \( M \in \mathcal{A} \iff M \text{ is one of: } D_3, D_4, H_0, \bar{K}_n, \bar{K}_n[D_3], \text{ or } D_3[\bar{K}_n], \text{ for some } n \in \mathbb{N} \text{ with } 1 \leq n; \)

(iii) \( M \in S \iff M \text{ or } \bar{M} \text{ is isomorphic to an s-digraph of one of the following forms. } K_n[A], A[K_n], L, D_3[L], L[D_3], H_1, H_2, \text{ where } n \in \mathbb{N} \text{ with } 1 \leq n, A \in \mathcal{A} \text{ and } L \in \mathcal{L}. \)
Set-homogeneous s-digraphs

Theorem (RG, Macpherson, Praeger, Royle (2011))

The finite s-digraphs that are set-homogeneous but not homogeneous are:

Infinite families (with $n \in \mathbb{N}$)

(i) $K_n[D_5]$ or $D_5[K_n]$

(ii) $J_n$

3 Sporadic examples

$E_7$

$F_6$

$J_2$
A monster 27-vertex sporadic example $H_3$
Structure of the proof

<table>
<thead>
<tr>
<th>Part 1: The hunt</th>
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<tbody>
<tr>
<td>Build a catalogue of small examples/families of examples.</td>
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</tbody>
</table>

<table>
<thead>
<tr>
<th>Part 2: The induction</th>
</tr>
</thead>
<tbody>
<tr>
<td>Argue by induction on $</td>
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**Case analysis:** each case leading to either (a) contradiction (b) forces the structure of an example in our list.
Why might a semigroup theorist be interested?

**Inverse semigroups**
- Relationship between partial and global symmetries
  - Factorizable inverse monoids (nice looking survey (Fitzgerald, 2010)).
  - James East says there is a variation of factorizable which gives the analogous class but for set-homogeneity.

**Semigroups (full endomorphism monoids)**
- Maltcev, Mitchell, Péresse, Ruškuc: Bergman property, Sierpiński rank.
- Bodirsky and Pinsker: reducts of the random graph.
- Bonato, Delić, Dolinka, Mašulović, Mudrinski: structural properties.
- Lockett, Truss: generic endomorphisms of homogeneous structures.

**Homomorphism homogeneity**
- Extending homomorphisms to endomorphisms, work of Cameron, Nešetřil, Lockett, Mašulović, Dolinka.