Rees monoids, self-similar groups and fractals

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May 18, 2011
History

- David Rees 1948 studies ideal structure of cancellative monoids
- Perrot 1970’s studies inverse hull
- Cohn and von Karger prove rigid monoids embed in groups
- 1980’s study of automatic groups
- 1990’s study of self-similar groups
- Recently, Alan Cain has studied automaton semigroups
Definition
A monoid $M$ is said to be a *left Rees monoid* (LRM) if the following hold:

1. $M$ is left cancellative: $ab = ac \Rightarrow b = c$ for all $a, b, c \in M$
2. Incomparable principal right ideals are disjoint: $aM \subseteq bM$ or $bM \subseteq aM$ or $aM \cap bM = \emptyset$ for all $a, b \in M$
3. Each principal right ideal is properly contained in only a finite number of principal right ideals

We define *right Rees monoids* analogously: right cancellative monoids with disjoint incomparable principal left ideals and finite inclusion of principal left ideals
Group of units

For a monoid $M$ we will denote by $G(M)$ the group of units of $M$; that is, the elements which are uniquely invertible in the group theoretic sense.
Proposition
Let $M$ be an LRM. Let $X$ be a transversal of the generators of the maximal proper principal right ideals, and denote by $X^*$ the submonoid generated by the set $X$. Then the monoid $X^*$ is free, $M = X^* G(M)$ and every element of $M$ can be written uniquely as a product of an element of $X^*$ and an element of $G(M)$.
Definition
Let $G$ be a group and $X^*$ be the free monoid on $X$. We will say that $G$ and $X^*$ act self-similarly on each other if there exist two maps $G \times X^* \to X^*$, $(g, x) \mapsto g \cdot x$ called the action and $G \times X^* \to G$, $(g, x) \mapsto g|_x$ called the restriction satisfying the following 8 axioms:

(SS1) $1 \cdot x = x$  
(SS2) $(gh) \cdot x = g \cdot (h \cdot x)$
(SS3) $g \cdot 1 = 1$  
(SS4) $g \cdot (xy) = (g \cdot x)(g|_x \cdot y)$
(SS5) $g|_1 = g$  
(SS6) $g|_{xy} = (g|_x)|_y$
(SS7) $1|_x = 1$  
(SS8) $(gh)|_x = g|_{(h \cdot x)}h|_x$

for all $x, y \in X^*$ and $g, h \in G$.  

Proposition
Let $M$ be an LRM. Then $M$ admits a self-similar action.

Proof.
Let $x \in X^*$ and $g \in G(M)$. Since $M = X^* G(M)$ uniquely, we can write $gx$ uniquely as a product of an element of $X^*$ and one of $G(M)$. So define $gx = g \cdot xg|_x$. It is easy to check that this definition satisfies the above axioms.
Definition
Let $G$ be a group and $X^*$ be the free monoid on $X$, such that there is a self-similar action of $G$ on $X^*$. We will define the Zappa-Szép product $X^* \ltimes G$ to be their Cartesian product with the following multiplication:

$$(x, g)(y, h) = (xg \cdot y, g|_{y}h)$$

for $x, y \in X^*$ and $g, h \in G$. 

Zappa-Szép products
Zappa-Szép products

**Theorem**
Every left Rees monoid is isomorphic to a Zappa-Szép product of a free monoid and a group. Conversely every Zappa-Szép product of a free monoid and a group is a left Rees monoid

**Remark**
What this says is that left Rees monoids and self-similar actions are one and the same thing
Green’s $\mathcal{R}$ relation

**Definition**

Let $M$ be a monoid, $s, t \in M$. Then $s \mathcal{R} t$ if $sM = tM$.

**Remark**

The relation $\mathcal{R}$ is an equivalence relation (in fact it is a left congruence).

**Lemma**

Let $M = X^*G$ be an LRM, $x, y \in X^*$, $g, h \in G$. Then $xg \mathcal{R} yh$ if, and only if, $x = y$. 
Rees monoids

Lemma

Let $M$ be a left Rees monoid which is also right cancellative. Then $M$ is also a right Rees monoid.

Because of this lemma we will call right cancellative left Rees monoids Rees monoids.
Restriction map

Definition
For each \( x \in X^* \), define \( \rho_x : G \to G \) by \( g \to g|_x \) and define \( \phi_x : G_x \to G \) to be the restriction of \( \rho_x \) to \( G_x \).

Lemma
An LRM is right cancellative iff \( \phi_x \) is injective for all \( x \in X \).

Definition
An LRM with \( \rho_x \) bijective for all \( x \in X^* \) is called symmetric.
Symmetric Rees monoids

Theorem
An LRM $M$ (which is a Zappa-Szép product of a free monoid $X^*$ and a group $G$) can be extended to the Zappa-Szép product of the free group $FG(X)$ and the group $G$ if, and only if, $M$ is symmetric.

Proof.
$(\Rightarrow)$ Straightforward: uniqueness and existence of restrictions
$(\Leftarrow)$ Define $g|_{x^{-1}} := \rho_x^{-1}(g)$ for $x \in X$ and extend the restriction to $g|_x$ for $x \in FG(X)$ by using rule (SS6):
$g|_{x_1^{\epsilon_1}x_2^{\epsilon_2}\ldots x_n^{\epsilon_n}} = ((g|_{x_1^{\epsilon_1}})|_{x_2^{\epsilon_2}})\ldots|x_n^{\epsilon_n} \quad x_i \in X, \epsilon_i = \pm 1$. For $x \in X^*$, $g \in G$ define $g \cdot x^{-1} := (g|_{x^{-1}} \cdot x)^{-1}$. \hfill \Box
Monoid HNN-extensions

Definition
Let $S$ be a monoid, $T$ a submonoid of $S$ and let $\alpha : T \to S$ be an injective homomorphism. Then $M$ is a monoid HNN-extension of $S$ if $M$ can be defined by the following monoid presentation

$$M = \langle S, t | R(S), \quad ts = \alpha(s)t \quad \forall s \in T \rangle,$$

where $R(S)$ denotes the relations of $S$
Monoid multiple HNN-extensions

Definition
Let $S$ be a monoid, $T_1,\ldots,T_n$ submonoids of $S$ and let $\alpha_i : T_i \to S$ be injective homomorphisms for each $i$. Then $M$ is a monoid multiple HNN-extension of $S$ if $M$ can be defined by the following monoid presentation

$$M = \langle S, t_1,\ldots,t_n | \mathcal{R}(S), \quad t_is = \alpha_i(s)t_i \quad \forall s \in T_i, i = 1,\ldots,n \rangle,$$

where $\mathcal{R}(S)$ denotes the relations of $S$. 
Classification theorem

Theorem
Let $S$ be a group, $T_1, \ldots, T_n$ finite index subgroups of $S$ and let $\alpha_i : T_i \to S$ be injective homomorphisms for each $i$, and let $M$ be the monoid multiple HNN-extension of $S$ as defined above. Then $M$ is a Rees monoid. Furthermore, every Rees monoid can be constructed in this manner.
Generalisation to categories

- Left Rees categories
- Self-similar groupoid actions
- Category HNN-extensions
Sierpinski Gasket
Applying the theorems

- $M$ is the monoid of similarity contractions the Sierpinski gasket
- $R$, $L$ and $T$ be the maps which halve the gasket and translate it, respectively, to the right, left and top of itself
- $\rho$ is rotation by $2\pi/3$ degrees
- $\sigma$ is reflection in the vertical axis
- Group of isometries:
  \[
  G = \langle \rho, \sigma | \rho^3 = \sigma^2 = 1, \rho\sigma = \sigma\rho^2 \rangle
  \]
- $M$ is a left Rees monoid, $X = \{L, R, T\}$, $G$ group of units
- $g|_x = g$ for every $g \in G$, $x \in X$, so symmetric Rees monoid
- $G_T = \{1, \sigma\}$
- Monoid presentation of $M$:
  \[
  M = \langle \rho, \sigma, T | \rho^3 = \sigma^2 = 1, \rho\sigma = \sigma\rho^2, \sigma T = T\sigma \rangle
  \]
Thank you for listening