An Inverse Monoid Approach to Thompson’s Group V and Generalisations

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Properties

1. $V$ contains every finite group
2. $V$ is simple
3. $V$ is finitely presented
4. $V$ has type $FP_\infty$
5. $V$ has solvable word problem
6. $V$ has solvable conjugacy problem
7. $V$ has a subgroup isomorphic to $F_2 \times F_2$
8. The generalised word problem for $V$ is undecidable
Right ideals of $A^*$

- $A = \{a_1, \ldots, a_k\}$; $u, v \in A^*$. $u$ is a **prefix** of $v$ if $v \in uA^*$.

- **Prefix code** $P$ over $A$: $P \subseteq A^*$ and $uA^* \cap vA^* = \emptyset \ \forall u, v \in P$.

- $P$ is **maximal** if for a prefix code $Q$ over $A$, $P \subseteq Q \Rightarrow P = Q$.

- If $R$ a right ideal of $A^*$, then
  
  (i) $R = PA^*$ for a uniquely determined prefix code $P$;
  
  (ii) $P$ is the unique minimal set of generators for $R$.

- $R$ is **essential** if $R \cap I \neq \emptyset$ for every right ideal $I$ of $A^*$.

- $R = PA^*$ is essential if and only if $P$ is a maximal prefix code.
Thompson-Higman Groups $V_{k,1}$

- $R^e_f(A^*) := \text{set of all } A^*-\text{isomorphisms between finitely generated essential right ideals of } A^*$. 

- It is an inverse submonoid of $\mathcal{I}_{A^*}$. 

- It is an $F$-inverse monoid, i.e., every $\sigma$-class contains a maximum element. 

- $V_{k,1} := R^e_f(A^*)/\sigma$. 

Note An $A^*$-isomorphism $\varphi : P_1 A^* \to P_2 A^* $ ($P_1, P_2$ prefix codes) restricts to a bijection from $P_1$ to $P_2$. **
Generalisation I

$C$ is a **right LCM monoid** if $C$ is left cancellative and for $a, b \in C$, $aC \cap bC = \emptyset$ or is principal.

Artin monoids (in particular, free monoids), Garside monoids.

A projective right ideal of $C$ is a disjoint union of principal right ideals. If

$$P = a_1C \sqcup \cdots \sqcup a_tC$$

is a projective right ideal, say $\{a_1, \ldots, a_t\}$ is a **basis** for $P$.

**Assumption** $C$ has finitely generated essential projective right ideals. Holds if $C$ is a finitely generated monoid.

$R_{fp}^e(C)$:= set of all $C$-isomorphisms between finitely generated essential projective right ideals of $C$. It is an inverse submonoid of $\mathcal{I}_C$.

What about $R_{fp}^e(C)/\sigma$?
Generalisation II

$C$ is still a right LCM monoid.

- If $C$ is right Ore and cancellative, then $R_{fp}^e(C)/\sigma$ is the group of right fractions of $C$.

- If $C$ is a left Rees monoid with finitely generated free part, then $C \cong A^* \rtimes G$ (see next slide) where $A = \{a_1, \ldots, a_k\}$ and $G$ is an appropriate group.

  \[ R_{fp}^e(C)/\sigma \cong V_k(G) \] (introduced by Nekrashevych).

- If $C = A^* \times \cdots \times A^*$ ($n$ factors), then $R_{fp}^e(C)/\sigma \cong nV_{k,1}$ (introduced by Brin).

- Brown-Stein groups???
Left Rees Monoids

A left cancellative monoid $C$ is a left Rees monoid if all its right ideals are projective, and each principal right ideal is contained in only finitely many principal right ideals.

$G, C$ monoids. Actions: $G$ on $C$: $(g, c) \mapsto g \cdot c$; $C$ on $G$: $(g, c) \mapsto g|_c$.

On $C \times G$ define

$$(c, g)(d, h) = (c(g \cdot d), g|_d h).$$

With appropriate conditions on the actions, get a monoid $C \bowtie G$, the Zappa-Szép product of $C$ and $G$.

**Theorem** (Lawson).

A monoid $M$ is a left Rees monoid if and only if $M \cong A^* \bowtie G$ for some set $A$ and group $G$.

In this case, the action of $G$ on $A^*$ is a self-similar action, i.e.,

$\forall g \in G, a \in A, \exists$ unique $b \in A, h \in G$ such that

$g \cdot (aw) = b(h \cdot w)$ for all $w \in A^*$. ($b = g \cdot a$ and $h = g|_a.$)
Alternative view of $R^e_{fp}(C)$: Inverse Hulls

$C$ left cancellative. For $a \in C$, the mapping $\lambda_a$ defined by

$$\lambda_a(c) = ac.$$ 

is one-one with domain $C$. $IH(C) = \text{Inv}\langle \lambda_a : a \in C \rangle$ is the inverse hull of $C$.

$$IH^0(C) = \begin{cases} IH(C) & \text{if } 0 \in IH(C) \\ IH(C) \cup \{0\} & \text{otherwise.} \end{cases}$$

Theorem (McAlister; also Nivat/Perrot)

The following are equivalent:

1. $IH^0(C)$ is 0-bsimple;
2. every non-zero element of $IH^0(C)$ can be written as $\lambda_a \lambda_b^{-1}$ for some $a, b \in C$;
3. the domain of each non-zero element of $IH^0(C)$ is a principal right ideal;
4. $C$ is a right LCM monoid.
Alternative view of $R_{fp}^e(C)$: Orthogonal Completions

$S$ inverse semigroup with zero. $a, b \in S$ are orthogonal ($a \perp b$) if

$$a^{-1}b = 0 = ab^{-1}.$$

Clearly, $a \perp b$ iff $aa^{-1} \perp bb^{-1}$ and $a^{-1}a \perp b^{-1}b$.

$A \subseteq S$ is orthogonal if $a \perp b$ for all distinct $a, b \in A$.

$S$ is orthogonally complete if it satisfies:

1. \{a_1, \ldots, a_n\} orthogonal implies $a_1 \lor \cdots \lor a_n$ exists (natural po), and
2. multiplication distributes over joins of finite orthogonal sets.

Examples

1. Symmetric inverse monoids.
2. $IH^0(C)$ where $C$ is a right Ore and right LCM monoid.
$S$ inverse semigroup with zero.

\[ D(S) = \{ A \subseteq S : 0 \in A, |A| < \infty, A \text{ is orthogonal} \}. \]

**Theorem (Lawson)**

1. $D(S)$ is an inverse subsemigroup of $P(S)$; it is a monoid if $S$ is a monoid.
2. $\iota : S \rightarrow D(S)$ given by $a \mapsto \{0, a\}$ embeds $S$ in $D(S)$
3. $D(S)$ is orthogonally complete.
4. If $\theta : S \rightarrow T$ is a homomorphism to an orthogonally complete inverse semigroup $T$, then there is a unique join preserving homomorphism $\varphi : D(S) \rightarrow T$ such that $\varphi \iota = \theta$.

Say $D(S)$ is the **orthogonal completion** of $S$. 
C is a right LCM monoid. $R_f(C)$ (resp. $R_{fp}(C)$) is the set of $C$-isomorphisms between finitely generated (resp. finitely generated projective) right ideals of $C$.

$R_{fp}(C) \subseteq R_f(C)$ are inverse submonoids of the symmetric inverse monoid on $C$ and $R_{e,fp}(C) \subseteq R_{fp}(C)$.

The polycyclic monoid $P_n$ on $A = \{a_1, \ldots, a_n\}$ is $IH^0(A^*)$ and a presentation for it is:

$$\langle A \cup A^{-1} \mid aa^{-1} = 1; ab^{-1} = 0 \text{ if } a \neq b \rangle.$$  

**Theorem (Lawson)**

$D(P_n) \cong R_f(A^*) = R_{fp}(A^*)$.  

Recall that

\[ IH^0(C) = \{ \lambda_c \lambda_d^{-1} : c, d \in C \} \cup \{0\}. \]

Product:

\[
(\lambda_a \lambda_b^{-1})(\lambda_c \lambda_d^{-1}) = \begin{cases} 
\lambda_{as} \lambda_{dt}^{-1} & \text{if } bC \cap cC = mC \text{ with } m = bs = ct \\
0 & \text{if } bC \cap cC = \emptyset.
\end{cases}
\]

\{\lambda_{a_1} \lambda_{b_1}^{-1}, \ldots, \lambda_{a_k} \lambda_{b_k}^{-1}\} \cup \{0\} \text{ is orthogonal}

iff \{a_1, \ldots, a_k\} \text{ and } \{b_1, \ldots, b_k\} \text{ are bases for projective right ideals of } C

iff for all } i, j \text{ with } i \neq j \text{ we have } a_i C \cap a_j C = \emptyset \text{ and } b_i C \cap b_j C = \emptyset.
Theorem
\[ D(IH^0(C)) \cong R_{fp}(C). \]

Idea of proof: Let \( A \in D(IH^0(C)) \), say
\[ A = \{ \lambda_{a_1} \lambda_{b_1}^{-1}, \ldots, \lambda_{a_k} \lambda_{b_k}^{-1} \} \cup \{0\}. \]
Then \( I = \{a_1, \ldots, a_k\}C \) and \( J = \{b_1, \ldots, b_k\}C \) are projective right ideals and

\[ \theta_A : J \to I \text{ given by } (b_i c) \theta_A = a_i c \]

is a \( C \)-isomorphism. Now define

\[ \theta : D(IH^0(C)) \to R_{fp}(C) \text{ by } \theta(A) = \theta_A \]

and verify that \( \theta \) is an isomorphism.
$S$ is an inverse monoid with zero.

$S^e := \{ a \in S : Sa \text{ and } aS \text{ are essential} \}.$

$S^e$ is an inverse submonoid of $S$ called the essential part of $S$.

An idempotent $e$ is essential if $e \in S^e$. This is true if and only if $ef \neq 0$ for all non-zero idempotents $f$ of $S$.

$a \in S^e$ if and only if $aa^{-1}$ and $a^{-1}a$ are essential idempotents.

The isomorphism $\theta$ restricts to an isomorphism

$$D^e(IH^0(C)) \cong R_{fp}^e(C).$$
Let $C$ be right Ore and right LCM. Then every projective right ideal is principal. (Two principal right ideals cannot be disjoint).

So, all orthogonal subsets of $IH^0(C)$ have the form $\{\lambda_a \lambda_b^{-1}, 0\}$; hence the embedding of $IH^0(C)$ into $D(IH^0(C))$ is surjective.

Every nonzero idempotent of $IH^0(C)$ is essential. So

$$D^e(IH^0(C)) = IH(C).$$

Well known that the group of right fractions of $C$ is isomorphic to $IH(C)/\sigma$. 
Let $C$ be a right LCM monoid with trivial group of units, and $G$ be a group. Suppose we have actions so that we can form $D = C \bowtie G$. Then

1. $D$ is left cancellative;
2. $D$ is right LCM;
3. the group of units of $D$ is $\{(1, g) : g \in G\}$;
4. the partially ordered set of principal right ideals of $D$ is order-isomorphic to the partially ordered set of principal right ideals of $C$. 
Remember that for any right LCM monoid $B$,

$$IH^0(B) = \{\lambda_a \lambda_b^{-1} : a, b \in B\}.$$ 

$$\lambda_a \lambda_b^{-1} = \lambda_c \lambda_d^{-1} \iff \exists \text{ unit } u \in B \text{ such that } au = c, bu = d.$$ 

Write elements of $IH^0(B)$ as $\sim$-equivalence classes $[a, b]$ where

$$(a, b) \sim (c, d) \iff \exists \text{ unit } u \in B \text{ such that } au = c, bu = d.$$ 

Now consider $IH^0(D)$ where $D = C \rhd G$. Elements are:

$$[(a, g), (b, h)] = [(a, gh^{-1}), (b, 1)]$$ 

so can represent elements by triples $(a, g, b) \in C \times G \times C$. 
The following are equivalent:

1. \( X = \{(a_1, g_1, b_1), \ldots, (a_t, g_t, b_t)\} \cup \{0\} \) is an orthogonal subset of \( IH^0(D) \);

2. \( \overline{X} = \{\lambda a_1 \lambda_b^{-1}, \ldots, \lambda a_t \lambda_b^{-1}\} \cup \{0\} \) is an orthogonal subset of \( IH^0(C) \);

3. \( A = \{a_1, \ldots, a_t\} \) and \( B = \{b_1, \ldots, b_t\} \) are bases for projective right ideals of \( C \).

Consequently,

\[
D^e(IH^0(D)) = \{X : \overline{X} \in D^e(IH^0(C))\} = \{X : A, B \text{ are bases for essential projective right ideals}\}.
\]