Tropical matrix algebra

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The tropical semiring

The **tropical** (or **max-plus**) semiring has elements

$$\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty\}$$

and binary operations

- $x \oplus y = \max(x, y)$; and
- $x \otimes y = x + y$.

Properties

$\overline{\mathbb{R}}$ *is an* **idempotent semifield***:*

- $(\mathbb{R}, \otimes)$ *is an abelian group with identity* 0*;*
- $-\infty$ *is a zero element for* $\otimes$*;*
- $(\overline{\mathbb{R}}, \oplus)$ *is a commutative monoid with identity* $-\infty$*;*
- $\otimes$ *distributes over* $\oplus$*;*
- $x \oplus x = x$

The tropical semiring

Tropical matrix algebra or **max-plus algebra** is linear algebra where the base field is replaced by the tropical semiring.

Applications

Tropical methods have applications in . . .

- *Combinatorial Optimisation*
- *Discrete Event Systems*
- *Control Theory*
- *Formal Languages and Automata*
- *Phylogenetics*
- *Statistical Inference*
- *Geometric Group Theory*
- *Enumerative Algebraic Geometry*

Tropical matrices

We (hope to) study the semigroup $M_n(\overline{\mathbb{R}})$ of all $n \times n$ tropical matrices under multiplication.

Question

What is its abstract algebraic structure?

For example, what are its ...

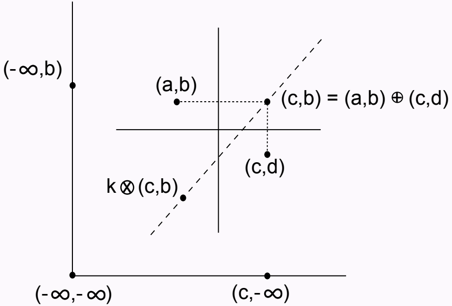
- *Ideals?*
- *Idempotents?*
- *Subgroups?*

Affine tropical n-space

$M_n(\overline{\mathbb{R}})$ comes equipped with a natural action on the space $\overline{\mathbb{R}}^n$ of **tropical** n-**vectors** (**affine tropical** n-**space**).

Example

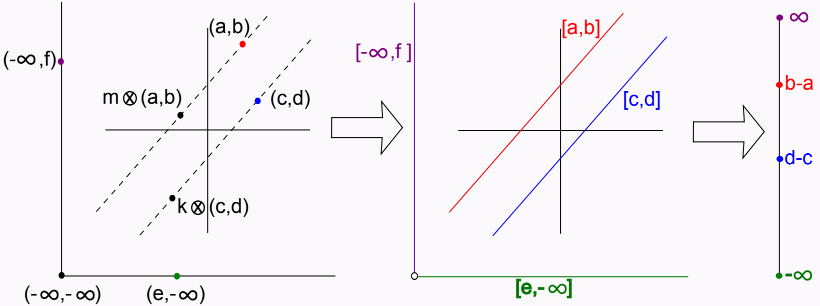
We may think of elements of tropical 2*-space pictorially as follows...*



Projective tropical $(n-1)$-space

From $\overline{\mathbb{R}}^n$ we obtain **projective tropical** $(n-1)$**-space** by discarding the "zero vector" and identifying two vectors which are "tropical scalings" of each other.

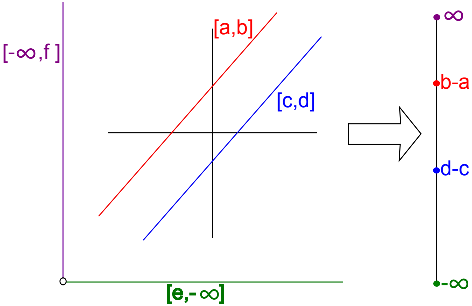
Example



Projective tropical 1-space

Thus we identify projective tropical 1-space with the two-point compactification of the real line $\hat{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$ via the map

$$[x, y] \mapsto y - x.$$



Question

How does the **algebraic** *structure of $M_n(\overline{\mathbb{R}})$ relate to the* **geometric** *structure of affine tropical n-space and projective tropical $(n-1)$-space?*

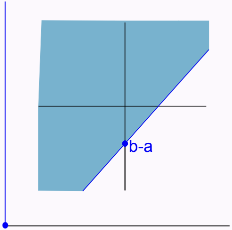
For $A \in M_n(\overline{\mathbb{R}})$ we write

- $C(A)$ for the column span of A (a tropical 'subspace' in $\overline{\mathbb{R}}^n$);
- $R(A)$ for the row span of A (a tropical 'subspace' in $\overline{\mathbb{R}}^n$).

Example

Let $A = \begin{pmatrix} a & -\infty \\ b & c \end{pmatrix} \in M_2(\overline{\mathbb{R}})$, *where* $a, b, c \in \mathbb{R}$.

$$\text{Then } C(A) = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} : x + b - a \leq y \right\} \subseteq \overline{\mathbb{R}}^2.$$



Projective column and row spaces

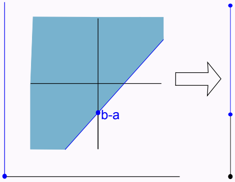
For $A \in M_n(\overline{\mathbb{R}})$ we write

- $PC(A)$ for the image of $C(A)$ in projective space (a convex set);
- $PR(A)$ for the image of $R(A)$ in projective space (a convex set).

Example

In the case $n = 2$, convex sets in $\hat{\mathbb{R}}$ are just intervals.

Consider $A = \begin{pmatrix} a & -\infty \\ b & c \end{pmatrix} \in M_2(\overline{\mathbb{R}})$, *where* $a, b, c \in \mathbb{R}$.



Then $PC(A) = [b - a, \infty] \subseteq \hat{\mathbb{R}}$.

Ideals and Green's relations

We define a pre-order $\leq_{\mathcal{R}}$ on a monoid M by $x \leq_{\mathcal{R}} y \iff xM \subseteq yM$. From this we obtain an equivalence relation

$$x \mathcal{R} y \iff xM = yM \iff x \leq_{\mathcal{R}} y \text{ and } y \leq_{\mathcal{R}} x$$

Similarly ...

- $x \leq_{\mathcal{L}} y \iff Mx \subseteq My, \quad x \mathcal{L} y \iff Mx = My$
- $x \leq_{\mathcal{J}} y \iff MxM \subseteq MyM, \quad x \mathcal{J} y \iff MxM = MyM;$

We also define equivalence relations ...

- $x \mathcal{H} y \iff x \mathcal{R} y$ and $x \mathcal{L} y$;
- $x \mathcal{D} y \iff x \mathcal{R} z$ and $z \mathcal{L} y$ for some $z \in M$;

Note

These relations encapsulate the (left, right and two-sided) ideal structure of M and are fundamental to its structure.

Green's $\mathcal{R}$ relation in $M_n(\overline{\mathbb{R}})$.

Lemma

Let $A, B \in M_n(\overline{\mathbb{R}})$. Then the following are equivalent:

(i) $A \leq_{\mathcal{R}} B$;

(ii) $C(A) \subseteq C(B)$;

(iii) $PC(A) \subseteq PC(B)$.

Corollary

Let $A, B \in M_n(\overline{\mathbb{R}})$. Then the following are equivalent:

(i) $A \mathcal{R} B$;

(ii) $C(A) = C(B)$;

(iii) $PC(A) = PC(B)$.

So $\mathcal{R}$-classes in $M_n(\overline{\mathbb{R}})$ are in 1-1 correspondence with n-generated convex sets in projective tropical $(n-1)$-space.

Green's $\mathcal{L}$ relation in $M_n(\overline{\mathbb{R}})$.

Lemma

Let $A, B \in M_n(\overline{\mathbb{R}})$. Then the following are equivalent:

(i) $A \leq_{\mathcal{L}} B$;

(ii) $R(A) \subseteq R(B)$;

(iii) $PR(A) \subseteq PR(B)$.

Corollary

Let $A, B \in M_n(\overline{\mathbb{R}})$. Then the following are equivalent:

(i) $A \mathcal{L} B$;

(ii) $R(A) = R(B)$;

(iii) $PR(A) = PR(B)$.

So $\mathcal{L}$-classes in $M_n(\overline{\mathbb{R}})$ are in 1-1 correspondence with n-generated convex sets in projective tropical $(n-1)$-space.

Green's $\mathcal{R}$ relation in $M_2(\overline{\mathbb{R}})$.

Corollary

Let $A, B \in M_2(\overline{\mathbb{R}})$. Then the following are equivalent:

(i) $A \mathcal{R} B$;

(ii) $C(A) = C(B)$;

(iii) $PC(A) = PC(B)$.

Corollary

The lattice of principal right ideals in $M_2(\overline{\mathbb{R}})$ is isomorphic to the intersection lattice generated by closed subintervals of the closed unit interval.

Isometries in projective tropical 1-space

We can define a "metric" on $\hat{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$ by

$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ \infty & \text{if } x = -\infty \neq y \text{ or } x = \infty \neq y \\ |y - x| & \text{otherwise.} \end{cases}$$

This gives a natural notion of **isometry** (denoted by $\cong$).

Proposition

Let $A \in M_2(\overline{\mathbb{R}})$. Then $PC(A) \cong PR(A)$.

Green's $\mathcal{J}$ relation in $M_2(\overline{\mathbb{R}})$

Proposition

Let $A, B \in M_2(\overline{\mathbb{R}})$. Then $A \leq_{\mathcal{J}} B$ if and only if $PC(A)$ embeds isometrically in $PC(B)$.

Theorem

Let $A, B \in M_2(\overline{\mathbb{R}})$. Then the following are equivalent

(i) $A \mathcal{J} B$;

(ii) $A \mathcal{D} B$;

(iii) $PC(A) \cong PC(B)$

(iv) $PR(A) \cong PR(B)$

Corollary

The lattice of principal two-sided ideals in $M_2(\overline{\mathbb{R}})$ is isomorphic to the lattice of isometry types of closed convex subsets of $\hat{\mathbb{R}}$.

Idempotents and regularity

The idempotents in $M_2(\overline{\mathbb{R}})$ are

$$\begin{pmatrix} 0 & x \\ y & x+y \end{pmatrix}, \begin{pmatrix} 0 & x \\ y & 0 \end{pmatrix}, \begin{pmatrix} x+y & x \\ y & 0 \end{pmatrix} \text{ and } \begin{pmatrix} -\infty & -\infty \\ -\infty & -\infty \end{pmatrix}$$

where $x, y \in \overline{\mathbb{R}}$ with $x + y \leq 0$.

Fact

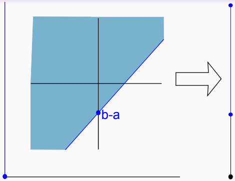
For every 2-generated convex subset X of $\hat{\mathbb{R}}$, there is an idempotent $E \in M_2(\overline{\mathbb{R}})$ with $PC(E) = X$. Thus $M_2(\overline{\mathbb{R}})$ is **regular**.

Example

Consider $X = [b - a, \infty] \subseteq \hat{\mathbb{R}}$.
Then we can choose

$$E = \begin{pmatrix} 0 & -\infty \\ b-a & 0 \end{pmatrix} \in M_2(\overline{\mathbb{R}})$$

such that $PC(E) = X$



Groups of 2×2 tropical matrices

Let S be a semigroup. It is well known that the maximal subgroups of S are exactly the $\mathcal{H}$-classes of idempotents and that any two maximal subgroups in the same $\mathcal{D}$-class are isomorphic.

Theorem

Let $M \subseteq \hat{\mathbb{R}}$ be a closed convex subset. The maximal subgroups in the $\mathcal{D}$-class corresponding to M are isomorphic to:

- $\{1\}$ *if* $M = \emptyset$;
- $\mathbb{R}$ *if M is a point or an interval with one real endpoint;*
- $\mathbb{R} \times S_2$ *if M is an interval with 2 real endpoints;*
- $\mathbb{R} \wr S_2$ *if* $M = \hat{\mathbb{R}}$.

Corollary

Every group of 2×2 tropical matrices is torsion-free abelian, or has a torsion-free abelian subgroup of index 2.