Algebras of reduced E-Fountain semigroups - generalizing the right ample identity

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Algebras of semigroups and categories

- Let S be a (finite) semigroup. Let C be a finite category. Let $\mathbb{k}$ be a commutative unital ring.
- The semigroup algebra

$$\mathbb{k}S = \{\sum k_i s_i \mid k_i \in \mathbb{k} \quad s_i \in S\}$$

- The category algebra $\mathbb{k}C$ is the free $\mathbb{k}$-module of all linear combinations

$$\{\sum_{i=1}^{n} k_i m_i \mid k_i \in \mathbb{k},\ m_i \text{ morphism}\}$$

Algebras of inverse semigroups

- Let S be a (finite) inverse semigroup. Let G be its associated inductive groupoid.

Theorem (Steinberg 2006)

Let $\mathbb{k}$ be any commutative unital ring. Then $\mathbb{k}S \simeq \mathbb{k}G$.

Algebras of E-Ehresmann

Theorem (IS 2016)

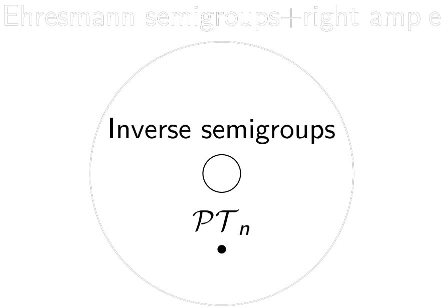
A similar isomorphism holds for the monoid of all partial functions $\mathcal{PT}_n$ and a certain associated category.

- This was useful in computing many invariants of the algebra $\mathbb{C}\mathcal{PT}_n$.
- Let S be a (finite) E-Ehresmann semigroup which satisfies the right ample identity. Let C be its associated Ehresmann category.

Theorem (IS 2017)

Let $\Bbbk$ be any commutative unital ring. Then $\Bbbk S \simeq \Bbbk C$.

Big Picture


Ehresmann semigroups+right ample
Inverse semigroups
$\mathcal{PT}_n$

First Question

- The Catalan monoid - $\mathcal{C}_n$.

$$\{f : \{1, \ldots, n\} \to \{1, \ldots, n\} \mid i \leq f(i), \quad i \leq j \implies f(i) \leq f(j)\}$$

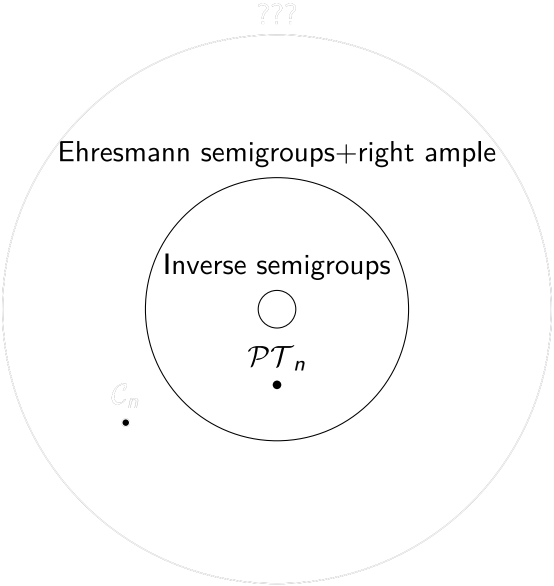
Theorem (Margolis&Steinberg 2018)

A similar isomorphism holds for the Catalan monoid $\mathcal{C}_n$ and a certain associated category.

Question

Can we generalize the theorem on Ehresmann semigroups to include the case of the Catalan monoid??

Big Picture



???
Ehresmann semigroups+right ample
Inverse semigroups
$\mathcal{PT}_n$
$\mathcal{C}_n$

Related results

- Semilattices (Solomon 1967)
- Ample semigroups (Guo&Chen 2012)
- P-Ehresmann semigroups (Wang 2017)
- Strict right ample semigroups (Guo&Guo 2018)

Preliminaries - generalized Green's relations

- Let S be a semigroup and let $E \subseteq S$ be a subset of idempotents.
- $a\tilde{\mathcal{L}}_E b \iff \forall e \in E \quad ae = a \iff be = b$
- $a\tilde{\mathcal{R}}_E b \iff \forall e \in E \quad ea = a \iff eb = b.$
- $\mathcal{L} \subseteq \tilde{\mathcal{L}}_E \qquad \mathcal{R} \subseteq \tilde{\mathcal{R}}_E$

Preliminaries - examples

- S inverse semigroup, $E = E(S)$. $\tilde{\mathcal{L}}_E = \mathcal{L}$, $\tilde{\mathcal{R}}_E = \mathcal{R}$
- S monoid, $E = \{1\}$. $\tilde{\mathcal{L}}_E = \tilde{\mathcal{R}}_E = S \times S$
- $S = \mathcal{B}_n$ (binary relations)\ $S = \mathcal{PT}_n$ (partial functions).
  E =partial identities.
  $f \tilde{\mathcal{L}}_E g \iff \text{dom}(f) = \text{dom}(g), \quad f \tilde{\mathcal{R}}_E g \iff \text{im}(f) = \text{im}(g)$.
  (composition right to left)
- $S = \mathcal{C}_n$ (Catalan monoid), $E = E(\mathcal{C}_n)$.
  $f \tilde{\mathcal{L}}_E g \iff \ker(f) = \ker(g), \quad f \tilde{\mathcal{R}}_E g \iff \text{im}(f) = \text{im}(g)$.

Preliminaries - reduced E-Fountain semigroups

- S is called E-**Fountain** if every $\tilde{\mathcal{L}}_E$ -class and every $\tilde{\mathcal{R}}_E$-class contains an idempotent from E.
- S is called **reduced** E-Fountain if in addition

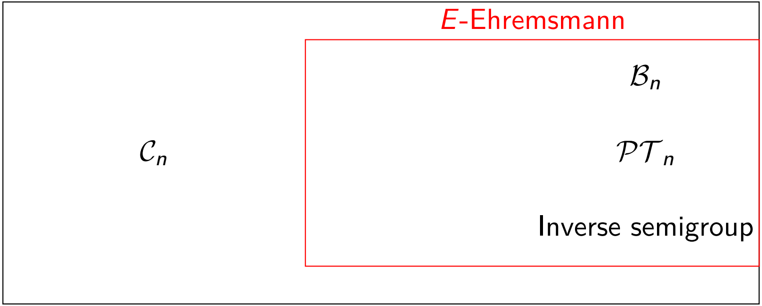
$$\forall e, f \in E \quad ef = e \iff fe = e$$

  - » In this case, every $\tilde{\mathcal{L}}_E$ -class and every $\tilde{\mathcal{R}}_E$-class contains a unique idempotent from E (denoted a^* and a^+ for $a \in S$).
  - » a^* (a^+) is the minimal right (resp. left) identity of a from E (with respect to the natural partial order on idempotents).
- S satisfies the congruence condition if $\tilde{\mathcal{L}}_E$ is a right congruence and $\tilde{\mathcal{R}}_E$ is a left congruence.
  Equivalently: $(ab)^* = (a^*b)^*$ and $(ab)^+ = (ab^+)^+$.
  - » In this case we can associate with S a category $C(S)$.
    Objects: E. Morphisms: S. $\text{dom}(a) = a^*, \text{range}(a) = a^+$.

Preliminaries - reduced E-Fountain semigroups

- We consider the class of reduced E - Fountain semigroups which satisfy the congruence condition.
- Subclass: If E is a subsemilattice (commutative subsemigroup) then S is called E-Ehresmann.
- Inverse semigroups, $\mathcal{B}_n$ and $\mathcal{PT}_n$ are Ehresmann. The Catalan monoid $\mathcal{C}_n$ is not

reduced E - Fountain + congruence condition



Right ample identity

- Let S be a reduced E-Fountain semigroup + congruence condition. S satisfies the **right ample identity** if $ea = a(ea)^*$ for every $e \in E$ and $a \in S$.

- Equivalent: $Ea \subseteq aE$ for every $a \in S$.

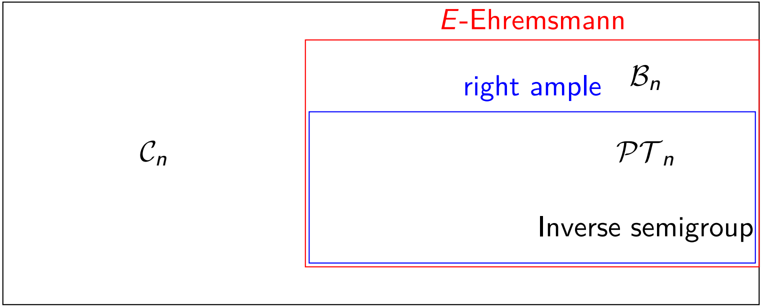
- Equivalent: S can be embedded in $\mathcal{PT}_n$ as an unary semigroup (with $\cdot, *$)

Right ample identity

Fact

If S satisfies the right ample identity then it is E-Ehresmann (aka E-Ehresmann right restriction).

reduced E - Fountain + congruence condition



Generalized right ample identity

- Let S be a reduced E-Fountain semigroup + congruence condition. We say that S satisfies the **generalized right ample identity** if for every $e \in E$ and $a \in S$

$$\left(e\left(a(ea)^{*}\right)^{+}\right)^{*} = \left(a(ea)^{*}\right)^{+}$$

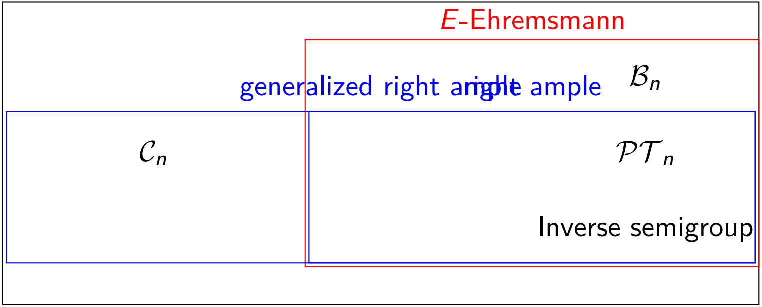
- The Catalan monoid $\mathcal{C}_n$ satisfies the generalized right ample identity (Margolis&Steinberg 2018).

Generalized right ample identity

Claim

If S is Ehresmann then: generalized right ample $\iff$ right ample

reduced E - Fountain + congruence condition



Goal Achieved

- Fix S a finite reduced E-Fountain + congruence condition. Define a relation $a \trianglelefteq_l b$ if $a = be$ for some $e \in E$. (Beware! $\trianglelefteq_l$ is not even a pre-order).
- Let $\varphi : \Bbbk S \to \Bbbk C(S)$ be the linear transformation defined (on basis elements) by

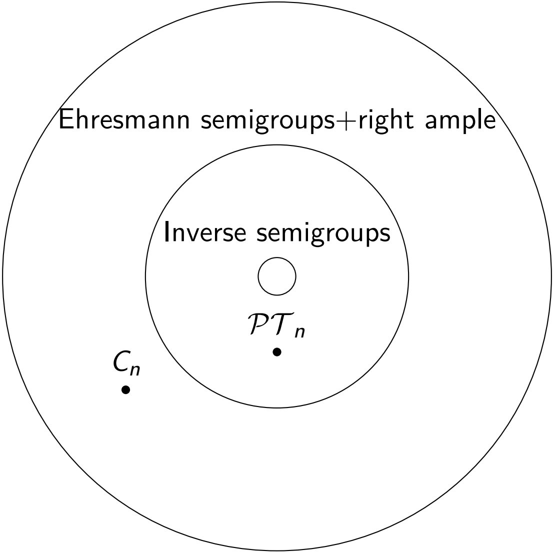
$$\varphi(\alpha) = \sum_{\beta \trianglelefteq_l \alpha} \beta$$

Theorem (IS 2022)

φ is a homomorphism of $\Bbbk$ - algebras if and only if S satisfies the generalized right ample identity.
If $\trianglelefteq_l$ is contained in a partial order then φ is an isomorphism.

Big Picture - revisited

reduced E-Fountain + congruence condition + generalized right ample



But...

- The generalized right ample identity does not look natural. There is only one motivating example (the Catalan monoid).

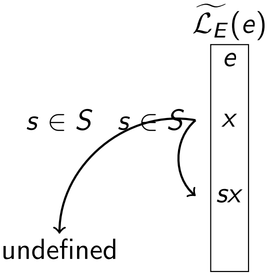
New Goal

Convince that the generalized right ample identity is a reasonable property. Find more examples.

Generalized right ample identity - What does it mean??

Let $\widetilde{\mathcal{L}}_E(e)$ be the $\widetilde{\mathcal{L}}_E$-class of $e \in E$. Then $\widetilde{\mathcal{L}}_E(e)$ is a partial left S-act (= S acts on the left of $\widetilde{\mathcal{L}}_E(e)$ by partial functions)

$$s * x = \begin{cases} sx & sx \in \widetilde{\mathcal{L}}_E(e) \\ \text{undefined} & sx \notin \widetilde{\mathcal{L}}_E(e) \end{cases} \quad (x \in \widetilde{\mathcal{L}}_E(e), \quad s \in S)$$



Generalized right ample identity - What does it mean??

Fact

Every $\alpha \in S$ induces a function $r_\alpha : \widetilde{\mathcal{L}}_E(\alpha^+) \to \widetilde{\mathcal{L}}_E(\alpha^*)$ defined by $r_\alpha(x) = x\alpha$.

Proof.

Assume $x \in \widetilde{\mathcal{L}}_E(\alpha^+) \implies x^* = \alpha^+$. Then $(x\alpha)^* = (x^*\alpha)^* = (\alpha^+\alpha)^* = \alpha^*$ so $x\alpha \in \widetilde{\mathcal{L}}_E(\alpha^*)$. □

Proposition (IS)

S is generalized right ample if and only if for every $\alpha \in S$, r_α is a homomorphism of partial left S-acts.

Corollary

Assume S is E-Ehresmann. S is right ample if and only if for every $\alpha \in S$, r_α is a homomorphism of partial left S-acts.

Generalized right ample identity - What does it mean??

This gives a concrete interpretation of the associated category $C(S)$.

Proposition

Let S be a reduced E-Fountain semigroup + congruence condition + generalized right ample. The category $C(S)^{\mathrm{op}}$ is isomorphic to the category whose objects are left S-acts of the form $\widetilde{\mathcal{L}}_E(e)$ for $e \in E$, and whose morphisms are all homomorphisms of left partial S-acts between them.

Modules and homomorphisms

- Let $\mathbb{k}$ be a field and let $\mathbb{k}S$ be the associated semigroup algebra over some field $\mathbb{k}$.

$$\mathbb{k}S = \{\sum k_i s_i \mid k_i \in \mathbb{k} \quad s_i \in S\}$$

- Let $\widetilde{\mathcal{L}}_E(e)$ be the $\widetilde{\mathcal{L}}_E$-class of $e \in E$. Let $\mathbb{k}\widetilde{\mathcal{L}}_E(e)$ be a $\mathbb{k}$-vector space of formal linear combinations with basis $\widetilde{\mathcal{L}}_E(e)$. Then $\mathbb{k}\widetilde{\mathcal{L}}_E(e)$ is a left $\mathbb{k}S$-module according to

$$s * x = \begin{cases} sx & sx \in \widetilde{\mathcal{L}}_E(e) \\ 0 & sx \notin \widetilde{\mathcal{L}}_E(e) \end{cases} \quad (x \in \widetilde{\mathcal{L}}_E(e), \quad s \in S)$$

- Now, r_α is a linear transformation $r_\alpha : \mathbb{k}\widetilde{\mathcal{L}}_E(\alpha^+) \to \mathbb{k}\widetilde{\mathcal{L}}_E(\alpha^*)$, defined on basis elements $r_\alpha(x) = x\alpha$.

Modules and homomorphisms

Proposition

Let S be a reduced E-Fountain semigroup + congruence condition. S is generalized right ample if and only if for every $\alpha \in S$, r_α is a homomorphism of left $\Bbbk S$-modules.

Corollary

In this case, the category $C(S)^{\mathrm{op}}$ is isomorphic to the category whose objects are left $\Bbbk S$-modules of the form $\Bbbk\widetilde{\mathcal{L}}_E(e)$ for $e \in E$, and whose morphisms are homomorphisms of left $\Bbbk S$-modules of the form r_α.

Proposition

If S is finite. The category $\Bbbk C(S)^{\mathrm{op}}$ is isomorphic to the category whose objects are left $\Bbbk S$-modules of the form $\Bbbk\widetilde{\mathcal{L}}_E(e)$ for $e \in E$, and whose morphisms are all homomorphisms of left $\Bbbk S$-modules between them.

Pierce decomposition

- Let A be an algebra and fix a complete set of orthogonal idempotents $E = \{e_1, \ldots, e_n\}$ (not necessarily primitive!).

$$\sum_{i=1}^{n} e_i = 1_A, \quad e_i e_j = 0$$

- We can associate with A a linear category $\mathcal{L}(A)$ whose objects are the pojective modules of the form Ae_i and its morphisms are the A-module homomorphisms between them. The linear category $\mathcal{L}(A)$ is called a Pierce decomposition of A.

Proposition

If S is finite and $\trianglelefteq_l$ is contained in a partial order then modules of the form $\Bbbk\widetilde{\mathcal{L}}_E(e)$ for $e \in E$ are projective modules and $\Bbbk C(S)^{\mathrm{op}}$ is a Pierce decomposition of $\Bbbk S$.

- The category $C(S)^{\mathrm{op}}$ is a "discrete" form of a Pierce decomposition.

Discrete Pierce decomposition

Semigroup Algebra $\longrightarrow$ $\Bbbk$ - linear category

Pierce decomposition

$\uparrow$

Semigroup $\longrightarrow$ Category

$C(S)^{\mathrm{op}}$

More examples

- Let V be a Hilbert space (e.g. $V = \mathbb{R}^n$).
  Let $S = \{T : V \to V \mid T \text{ bounded}\}$.
  For every closed subspace $U \subseteq V$, let P_U be the associated orthogonal projection $P_U : V \to U$.Choose
  $E = \{P_U \mid U \subseteq V, \quad U \text{ closed}\}$. Note that E is not a subsemilattice.

More examples

- The Catalan monoid - $\mathcal{C}_n$.

$$\{f : \{1, \dots, n\} \to \{1, \dots, n\} \mid i \leq f(i), \quad i \leq j \implies f(i) \leq f(j)\}$$

- Order preserving functions with a fixed point - $\mathcal{OPF}_n$.

$$\{f : \{1, \dots, n\} \to \{1, \dots, n\} \mid f(n) = n, \quad i \leq j \implies f(i) \leq f(j)\}$$

$E = E(\mathcal{C}_n)$ (Note that $\mathcal{C}_n \subseteq \mathcal{OPF}_n$).

More examples

- For $S = \mathcal{OPF}_n$ we have

$$f\tilde{\mathcal{L}}_E g \iff f\mathcal{L}g \iff \ker(f) = \ker(g)$$

$$f\tilde{\mathcal{R}}_E g \iff f\mathcal{R}g \iff \operatorname{im}(f) = \operatorname{im}(g).$$

- The algebra $\Bbbk\mathcal{OPF}_n$ is semisimple.

Thank you!